

## Lê constant families of singular hypersurfaces \*

*R. Callejas-Bedregal*  
*U. Federal Paraíba, Brasil*

*V. H. Jorge Pérez*  
*U. São Paulo, Brasil*

*M. J. Saia*  
*U. Federal São Paulo, Brasil*  
*J. N. Tomazella*

*U. Federal São Carlos, Brasil*

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### Abstract

*We investigate the constancy of the Lê numbers of one parameter deformations  $F : (\mathbf{C} \times \mathbf{C}^n, 0) \rightarrow (\mathbf{C}, 0)$  of holomorphic germs of functions  $f : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}, 0)$  which have singular set with any dimension  $s \geq 1$ . We characterize Lê constant deformations in terms of the non-splitting of the polar varieties and also from the integral closure of the ideal  $J_z(F)$  in  $\mathcal{O}_{n+1}$  generated by the partial derivatives of  $F$  with respect to the variables  $z = (z_1, \dots, z_n)$ .*

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## 1. Introduction

By a celebrated result of Lê and Ramanujam given in 1976, the constancy of a single numerical invariant, the Milnor number  $\mu$ , in members of a given 1-parameter family of hypersurface with isolated singularities, implies topological triviality of the family. More precisely,  $\mu$  is a topological invariant in the following sense, let  $X_t$  be a family of  $n$ -dimensional hypersurfaces in  $\mathbf{C}^{n+1}$ , with  $n \neq 2$ , such that each  $X_t$  has isolated singularity at 0 and  $\mu(X_t)$  is independent of  $t$ , then the local topological type of the hypersurfaces at 0 must be constant [6].

Therefore the characterization of families of isolated singularity hypersurfaces with constant Milnor number has become one of the main question in Singularity Theory and in the 80's several authors worked on the characterization of such families.

Greuel shows in [4, p.161] a full characterization, done by Greuel, Lê, Saito and Teissier, of Milnor constant deformations of germs of hypersurfaces with isolated singularity in terms of the integral closure of the Jacobian ideal and also by a non-splitting condition of the polar curve of the deformation.

Nowadays, the search for equisingularity conditions for families of hypersurfaces with singular set having dimension bigger than zero is one of the main question of interest in Singularity Theory.

For handling such non-isolated hypersurface singularities, there is available the machinery of Lê numbers, developed by Massey in [9]. To any given analytic function with a singular locus of arbitrary dimension  $s$ , there are  $\lambda^0, \dots, \lambda^s$  numbers, which perform together, a role roughly analogous to that of the Milnor number in the case  $s = 0$ .

Therefore, an interesting question is to characterize families of germs of non-isolated hypersurface singularities which are Lê constant, i.e, which have constant Lê numbers in a neighborhood of the origin. As in the result for isolated singularities, we are looking for conditions in terms of the non-splitting of the singular set of the deformation and also in terms of the integral closure of the Jacobian ideal of the deformation.

In the first part of this article we describe the results for the case of isolated singularity, in special the theorem of Greuel, and show some examples. Then we show the basic definitions of Lê numbers and polar multiplicities, these concepts are the necessary machinery to show our main result and were developed by Massey in [9]. The third part is for the main results of the article and in the Corollary 4.2 we show that Greuel's results extend for families of hypersurfaces with singular set of any dimension. We remark

that in order to obtain this generalization we need a new condition which does not appear in the case of isolated singularity.

## 2. Milnor constant deformations

We describe here the results of Greuel in [4] for the constancy of the Milnor number of families of hypersurfaces with isolated singularity.

In a general set-up, let  $F: (\mathbf{C} \times \mathbf{C}^n, 0) \rightarrow (\mathbf{C}, 0)$ ,  $F(x, t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$  be a one parameter deformation of a holomorphic germ  $f: (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}, 0)$  where  $\delta_s: (\mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$  and  $g_s: (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}, 0)$  are holomorphic germs of functions and  $\delta_s \neq 0$ .

Denote by  $J_z(F) = \langle \partial F / \partial z_1, \dots, \partial F / \partial z_n \rangle$ , the ideal in  $\mathcal{O}_{n+1}$  generated by the partial derivatives of  $F$  with respect to the variables  $z_1, \dots, z_n$  and  $J(F)$  denotes the Jacobian ideal of  $F$  in  $\mathcal{O}_{n+1}$ ,  $J(F) = \langle \partial F / \partial t, \partial F / \partial z_1, \dots, \partial F / \partial z_n \rangle$ . As usual  $\sqrt{J_z(F)}$  denotes the radical of  $J_z(F)$ ;  $\overline{J_z(F)}$  denotes the integral closure of  $J_z(F)$  and  $\overline{J_z(F)}^+$  denotes the strict integral closure of  $J_z(F)$ , see [2] or [3] for the definition of strict integral closure of an ideal.

If each germ  $f_t$  has isolated singularity at zero,  $F$  is said to be Milnor constant if the Milnor number  $\mu(f)$  is independent of  $t$  for small values of  $t$ .

**Theorem 2.1.** (Greuel, Kuo, Lê, Saito and Teissier) [4, p.161] Suppose that each germ  $f_t$  has isolated singularity at zero, then following statements are equivalent:

1.  $F$  is a Milnor constant deformation of  $f$ ;
  2.  $\frac{\partial F}{\partial t} \in \overline{J_z(F)}^+$ ;
  3.  $\frac{\partial F}{\partial t} \in \overline{J_z(F)}$ ;
  4.  $\frac{\partial F}{\partial t} \in \sqrt{J_z(F)}$ ,
  5. the polar curve of  $F$  with respect to  $\{t = 0\}$  does not split near  $(0, 0)$ ,
- or

$$V(J_z(F)) = \left\{ (z, t) \in \mathbf{C}^{n+1} \times \mathbf{C} \mid \frac{\partial F}{\partial z_i}(z, t) = 0, \forall i = 1, \dots, n \right\} = \{0\} \times \mathbf{C}.$$

**Example 2.2.** In order to better illustrate this theorem, consider the family  $f_t(x, y) = -x^3 + y^2 - tx^2$ . Here the Milnor number of  $f_0$  at  $(0, 0)$  is

$\mu(f_0(0,0)) = 2$  and the Milnor number of  $f_t$  at  $(0,0)$  is  $\mu(f_t(0,0)) = 1$ , therefore the family  $f_t$  is not Milnor constant in a neighborhood of  $(0,0)$ .

The main difference between the geometry of  $f_0$  and  $f_t$  with  $t \neq 0$  is that the singular set of  $f_0(x,y) = -x^3 + y^2$  has one point  $\Sigma(f_0) = \{(0,0)\}$  and the singular set of each  $f_t(x,y) = -x^3 + y^2 - tx^2$  has two points  $\Sigma(f_t) = \{(0,0) \cup (-2/3t, 0)\}$ . In this case we see clearly that the singular set of  $f_0$  splits in two points for  $f_t$  with  $t \neq 0$ . We remark that at the point  $P = (-2/3t, 0)$  the Milnor number of  $f_t$  is  $\mu(f_t((-2/3t, 0))) = 1$ .

If we consider the deformation  $F : (\mathbf{C} \times \mathbf{C}^2, 0) \rightarrow (\mathbf{C}, 0)$ ,  $F(t, y, x) = -x^3 + y^2 - tx^2$ , the singular set of  $F$  in  $\mathbf{C} \times \mathbf{C}^2$  is  $\Sigma(F) = \mathbf{C} \times 0 = V(x, y)$ . As  $\Sigma(F)$  is one dimensional we can have associated to it the polar curve of  $F$  at  $(0,0,0)$  with respect to the parameter  $t$ , defined as  $\Gamma^1(F) = V(J_z(F))/\Sigma(F)$  ( see 3.2 for the definition of the polar varieties). In this example  $\Gamma^1(F) = \{(t, -2/3t, 0)\}$  is not empty, hence  $V(J_z(F)) = \Sigma(F) \cup \Gamma^1(F)$  near  $(0,0,0)$ . Since  $V(J_z(F)) = \cup \Sigma(f_t)$  (the union of all singular sets of  $f_t$  in  $\mathbf{C}^3$ ) we get that  $\cup \Sigma(f_t)$  splits in two sets, the singular set and the polar curve of  $F$ .

We remark here that an easy calculation shows that the conditions 2., 3. and 4. also do not hold for this example.

**Example 2.3.** On the other side we consider the family  $g_t(x,y) = -x^3 + y^2 - tx^4$ , here the Milnor number  $\mu(g_t)$  at  $(0,0)$  is constant and the polar curve of  $F$  is empty, or in other words,  $\cup \Sigma(f_t) = \Sigma(F)$ . An easy calculation shows that the conditions 2., 3. and 4. also hold for this example.

### 3. Lê numbers of hypersurface singularities

An interesting question is to ask if Greuel's result extends to families with singular set having dimension bigger than zero.

For any non-isolated hypersurface singularity there is available the machinery of Lê numbers. As we see in [9], to any given analytic function with a singular locus of arbitrary dimension  $s$ , there are  $\lambda^0, \dots, \lambda^s$  numbers, which perform together, a role roughly analogous to that of the Milnor number in the case  $s = 0$ .

Therefore, our purpose is to characterize families of germs of non-isolated hypersurface singularities which are Lê constant, i.e, which have constant Lê numbers in a neighborhood of the origin. As in the result

for isolated singularity, we are looking for conditions in terms of the non-splitting of the singular set of  $F$  and also in terms of the integral closure of the Jacobian ideal  $J_z(F)$ .

We remark that even in the case of one dimensional singular set, the conditions given by Greuel in the theorem 2.1 are not equivalent at all, as we can see in the example below.

**Example 3.1.** Let  $F : (\mathbf{C} \times \mathbf{C}^2, 0) \rightarrow (\mathbf{C}, 0)$ ,  $F(t, x, y) = y^3 - txy^2$ . Here for all  $t$  we have  $f_t : (\mathbf{C}^2, 0) \rightarrow (\mathbf{C}, 0)$ ,  $f_t(x, y) = y^3 - txy^2$  with singular set  $\Sigma(f_t) = V(-ty^2, -3y^2 - 2txy) = V(y)$ , hence 1-dimensional.

In this example the Lê numbers are not constant at the origin. We remember that since the singular set of each  $f_t$  is one dimensional there are two Lê numbers associated to each  $f_t$ , named  $\lambda^0$  and  $\lambda^1$ , here  $\lambda^1(f_0) = 2$  while  $\lambda^1(f_t) = 1$  for  $t \neq 0$ .

For the second and third conditions, we have  $\partial f / \partial t = xy^2 \notin \overline{J_{(x,y)}(F)}$ . To show this we consider the analytic curve  $\varphi(\lambda) = (\lambda, -2/3\lambda^3, 0, \lambda^2)$ , then  $\frac{\partial F}{\partial t} \circ \varphi = 4/9\lambda^7$  and  $J_{(x,y)}(F) \circ \varphi = \langle -\lambda^6, 0, 0 \rangle$  and this implies that  $\partial f / \partial t \notin \overline{J_{(x,y)}(F)}$ .

However the condition  $\partial f / \partial t \in \sqrt{J_{(x,y)}(F)}$  of the theorem 2.1 holds since  $x^2y^4 = (\partial f / \partial t)^2$  is in  $J_{(x,y)}(F) = \langle -ty^2, 3y^2 - 2txy \rangle$ .

As a consequence of the condition  $\partial f / \partial t \in \sqrt{J_{(x,y)}(F)}$  we also have that  $\Gamma^1(F) = \emptyset$  since  $V(J_{(x,y)}(F)) = \Sigma(F) = V(y)$  and as  $\Gamma^1(F) = V(J_z(F) - V(J(F)))$ , we also obtain that  $\cup \Sigma(f_t) = \Sigma(F)$ .

However, as the singular set of  $F$  is 2-dimensional, there is associated to it the second polar variety  $\Gamma_2(F) = V(3y - 2tx) \subset \mathbf{C}^3$ , here we see that at  $t = 0$ ,  $\Gamma_2(F)|_{t=0} = V(3y)$ , while on the other side  $\Gamma^1(f_0) = \emptyset$ .

As sets we have  $V(J_{(x,y)}(F)) = \Sigma(F) = V(y)$ , but  $\Sigma(f_0) = V(y^2)$ , while  $\Sigma(F)|_{t=0} = V(y)$ . In this example we see that the singular set of  $f_0$ , which consists of the line  $y = 0$  with multiplicity 2, splits in two sets for  $t \neq 0$  the singular set and the polar curve of  $f_t$ .

Now we shall follow the notation of Massey in ([9], pp. 11–15), to define the Lê numbers of a hypersurface with non-isolated singularities.

Fix a linear choice of local coordinates  $z = (z_0, \dots, z_n)$  of  $\mathbf{C}^{n+1}$ , consider the ring  $\mathcal{O}_{n+1}$  of holomorphic germs  $f : (\mathbf{C}^{n+1}, 0) \rightarrow \mathbf{C}$  and denote by  $m_{n+1}$  its maximal ideal. Due to the identification between  $\mathcal{O}_{n+1}$  and the ring of

convergent power series  $\mathbf{C}\{z_0, \dots, z_n\}$  we identify a germ  $f \in \mathcal{O}_{n+1}$  with its power series  $f(z) = \sum a_\alpha z^\alpha$ , where  $z^\alpha = z_0^{\alpha_0} \dots z_n^{\alpha_n}$ ,  $\alpha_i \in \mathbf{Z}^+$ .

The relative polar varieties are defined for any holomorphic function germ  $f : (U, 0) \rightarrow (\mathbf{C}, 0)$ , where  $U$  is an open set of  $\mathbf{C}^{n+1}$ .

**Definition 3.2.** For  $0 \leq k \leq n$ , the  $k$ -th relative polar variety of  $f$  with respect to the coordinates  $z = (z_0, \dots, z_n)$ , which we denote by  $\Gamma_{f,z}^k$ , is the scheme  $V\left(\frac{\partial f}{\partial z_k}, \dots, \frac{\partial f}{\partial z_n}\right) / \Sigma(f)$ .

**Remark 3.3.** On the level of ideals,  $\Gamma_{f,z}^k$  consists of the components of the scheme  $V\left(\frac{\partial f}{\partial z_k}, \dots, \frac{\partial f}{\partial z_n}\right)$  which are not contained in  $\Sigma(f)$ .

As sets we have that  $\Gamma_{f,z}^0$  is empty and  $\Gamma_{f,z}^0 \subseteq \Gamma_{f,z}^1 \subseteq \dots \Gamma_{f,z}^{n+1} = U$ . If  $\dim \Sigma(f) < k$ , then  $\Gamma_{f,z}^k = V\left(\frac{\partial f}{\partial z_k}, \dots, \frac{\partial f}{\partial z_n}\right)$ .

The polar varieties are not necessarily reduced and the dimension of the critical locus of  $f$ , denoted by  $\Sigma(f)$ , is allowed to be arbitrary.

If the intersection of  $\Gamma_{f,z}^k$  and  $V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})$  is purely 0-dimensional at a point  $P = (p_0, \dots, p_n) \in (U, 0)$ , i.e., either  $p$  is an isolated point of the intersection or  $p$  is not in the intersection, it is possible to define the  $k+1$  polar multiplicities.

**Definition 3.4.** For any point  $P = (p_0, p_1, \dots, p_n)$  in  $\mathbf{C}^{n+1}$  and  $k = 0, \dots, s$ , the  $k$ -th **polar multiplicity** of  $f$  at  $P$  with respect to the coordinates  $(z_0, \dots, z_n)$  denoted by  $m_{f,z}^k(P)$ , is defined as the intersection number:

$$\left(\Gamma_{f,z}^k \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})\right)$$

The L $\hat{e}$  numbers are defined in terms of the L $\hat{e}$  varieties and unlike the polar varieties, the L $\hat{e}$  varieties are supported on the critical set of  $f$  itself.

**Definition 3.5.** The  $k$ -th **L $\hat{e}$  variety** of  $f$ :

$$\Lambda_{f,z}^k = \left(\Gamma^k \cap V\left(\frac{\partial f}{\partial z_k}\right)\right) \triangleleft \Sigma(f),$$

where  $\triangleleft$  denotes the components of  $\left(\Gamma^k \cap V\left(\frac{\partial f}{\partial z_k}\right)\right)$  which are in  $\Sigma(f)$ .

If the intersection of  $\Lambda_{f,z}^k$  and  $V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})$  is purely 0-dimensional at a point  $p = (p_0, \dots, p_n)$ , i.e., either  $p$  is an isolated point of the intersection or  $p$  is not in the intersection, it is possible to define the Lê numbers as follows:

**Definition 3.6.** The  $k$ -th **Lê number**  $\lambda_{f,z}^k(p)$  at  $p$ , is defined as the intersection number:

$$\left( \Lambda_{f,z}^k \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1}) \right)$$

Note that if  $\lambda_{f,z}^k(p)$  is defined at  $P$ , it is defined at all points near  $P$ . If  $\lambda_{f,z}^k(P)$  is defined, then  $\Gamma_{f,z}^k$  is purely  $k$ -dimensional at  $P$  and  $\Gamma_{f,z}^k$  has no embedded components at  $P$ .

A deformation  $F: (\mathbf{C}^{n+1} \times \mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$  of  $f$  is said to be Lê-constant if all Lê numbers at 0 of each  $f_t(z) = F(z, t)$  are constant for small values of  $t$ .

#### 4. Lê constant deformations

We seek now for complementary conditions which guarantee a characterization of Lê constant families in terms of the integral closure of ideals and also in terms of the non-splitting of the singular set and consequently will allow us to generalize Greuel's result.

Consider  $F: (\mathbf{C}^{n+1} \times \mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$  be a one parameter deformation of a holomorphic germ  $f: (\mathbf{C}^{n+1}, 0) \rightarrow (\mathbf{C}, 0)$  such that for all  $t$  in a neighborhood of the origin,  $\dim \Sigma(f_t) = s$  for some  $s \geq 0$ .

For such  $F$  we give next six conditions which will allow us to characterize the Lê constant deformations.

1.  $F$  is a Lê-constant deformation of  $f$ ;
2.  $\frac{\partial F}{\partial t} \in \overline{J_z(F)}^+$ ;
3.  $\frac{\partial F}{\partial t} \in \overline{J_z(F)}$ ;
4.  $\frac{\partial F}{\partial t} \in \sqrt{J_z(F)}$ ;
5.  $F$  satisfies the non-splitting condition, i.e.  $\Sigma(f_t) = \Sigma(F) \cap V(t)$  for all  $t$  in a neighborhood of the origin.

6. For all  $i = 1, \dots, s$ , the Lê numbers  $\lambda_{F(t,x)}^{i+1}(P)$  and the sum  $m_{F(t,x)}^1(P) + \lambda_{F(t,x)}^1(P)$  of  $F$  are independent of the parameter  $t$  in the family of points  $P = (t, 0, \dots, 0)$  for all  $t$  in a neighborhood of the origin.

**Theorem 4.1.** 1. implies 2. implies 3. implies 4. implies 5. and 1. implies 6..

**Proof.** First we show that 1. implies 2.. For this we apply the Theorem 6.5 of Massey [9] where it is shown that the constancy of the Lê numbers imply that the Thom condition  $A_F$  is satisfied. Then we apply the result of [3], Theorem 2.1., where it is shown that the Thom condition  $A_F$  holds if, and only if, the strict integral dependence condition  $\frac{\partial F}{\partial t} \in \overline{J_z(F)}^+$  holds.

We have that 2. implies 3. and 3. implies 4. trivially.

To show that (4.) implies (5.) we see that (3.) implies the equality  $\overline{J_z(F)} = \overline{J(F)}$ , and as  $J_z(F) \subset J(F)$  the ideal  $J_z(F)$  is a reduction of  $J(F)$ , therefore we obtain  $V(J_z(F)) = V(J(F))$ . As we know that  $V(J_z(F)) = \cup \Sigma(f_t)$ ,  $V(J(F)) = \Sigma(F)$  and the sets  $\Sigma(f_t)$  are disjoint for each  $t$  we obtain the non-splitting condition  $\Sigma(f_t) = \Sigma(F) \cap V(t)$ .

To finish the proof we need to show that 1. implies 6., but we have that 1. implies 5., hence for all  $i = 1, \dots, s$  we apply the proposition 1.21 of Massey [9], which relates the Lê numbers of hyperplane sections of hypersurfaces with any singular set.

In this proposition Massey shows that  $\lambda_{f_t(z)}^i(0) = \lambda_{F(t,x)}^{i+1}(t, 0)$ , and from 1. we see that  $\lambda_{f_t(z)}^i(0)$  is constant for all  $t$ , hence  $\lambda_{F(t,x)}^{i+1}(t, 0)$  is constant also.

To conclude that  $m_{F(t,x)}^1(t, 0) + \lambda_{F(t,x)}^1(t, 0)$  is also independent of  $t$  again we apply the proposition 1.21 of Massey to get  $\lambda_{f_t(x)}^0(0) = m_{F(t,x)}^1(t, 0) + \lambda_{F(t,x)}^1(t, 0)$  and since  $\lambda_{f_t(x)}^0(0)$  is constant, we obtain the constancy of the sum.  $\square$

**Corollary 4.2.** The following statements are equivalent: 1.  $\iff$  5. + 6.  $\iff$  4. + 6.  $\iff$  3. + 6.  $\iff$  2. + 6..

**Proof.** We just need to show that 5. + 6. implies 1., but this we obtain again from the proposition 1.21 of Massey. If we suppose that 5. holds, then we get

$$\lambda_{f_t(x)}^0(0) = m_{F(t,x)}^1(t, 0) + \lambda_{F(t,x)}^1(t, 0) \text{ and } \lambda_{f_t(z)}^i(0) = \lambda_{F(t,x)}^{i+1}(t, 0)$$



for all  $i = 1, \dots, s$  and all  $t$ , therefore we obtain the constancy of the Lê numbers of  $f_t$  from 6..  $\square$

**Corollary 4.3.** *In the case of isolated singularity, the following statements are equivalent: 1.  $\iff$  2  $\iff$  3  $\iff$  4.  $\iff$  5..*

**Proof.** To prove this Corollary we just need to show that in this case the condition 5 implies the condition 6. For this we follow the proof done by Greuel. First we remember that  $\mu_{f_0}(0) = \sum \mu_{f_t}(Q)$  for all  $Q \in \Sigma(f_t)$ , now if we suppose that the non-splitting condition holds, then  $\Gamma^1(F) = \emptyset$ , hence each  $f_t$  has only the origin as an isolated singularity and  $\mu_{f_0}(0) = \mu_{f_t}(0)$  for all  $t$  (hence condition 1. is satisfied). Now as we are supposing that the nonsplitting condition holds we obtain from the proposition 1.21 of Massey that  $\mu_{f_t}(0) = \lambda_{f_t}^0(0) = \lambda_F^1(t, 0) + m_F^1(t, 0)$  and as  $\Gamma^1(F) = \emptyset$ ,  $m_F^1(t, 0) = 0$  and  $\mu_{f_t}(0) = \lambda_F^1(t, 0)$  from which we conclude that  $\lambda_F^1(t, 0)$  is constant, hence the condition 6 is also satisfied.  $\square$

**Remark 4.4.** The condition 6. above is given in terms of the Lê numbers of the deformation germ  $F$  and it should be interesting if we could give another condition which does not depend on Lê numbers. In the sequel we shall show how to translate this condition in terms of the blowing up of  $\Sigma(F)$  along the parameter space  $T = \mathbf{C} \times \{0\}$ , using some results of Lipman [8].

First we give some notation: Call  $b : Bl_T(\Sigma(F)) \rightarrow T$  the blow up of  $\Sigma(F)$  along  $T$  and denote by  $D$  its exceptional divisor, then for all  $i = 1, \dots, s$ , the Lê numbers  $\lambda_{F(t,x)}^i(P)$  of  $F$  are independent of the parameter  $t$  in the family of points  $P = (t, 0, \dots, 0)$  for all  $t$  in a neighborhood of the origin if, and only if, the restriction of  $b$  to the intersection  $D \cdot b^\dagger \Lambda_F^i$  is equidimensional over  $T$ , where  $b^\dagger \Lambda_F^i$  is the proper transform of the cycle  $\Lambda_F^i$  in  $\Sigma(F)$  by  $b$ .

Thus we can rephrase the condition 6 in terms of the equidimensionality of  $D \cdot b^\dagger \Lambda_F^i$  over  $T$  and the constancy of the polar multiplicities  $m_{F(t,x)}^1(P)$  along  $T$ .

**Remark 4.5.** The constancy of the Lê numbers  $\lambda^i(f_t)$  is also characterized by Gaffney and Gassler in [2, Proposition 4.6], in terms of the equidimensionality over the parameter space  $T$  of the components of suitable divisors on the blow-up of the hypersurface  $X = F^{-1}(0)$  along the product of the ideal of  $T$  with the ideal of  $\Sigma(F)$ .

In fact the results of Gaffney and Gassler are done for the Segre numbers of any ideal, and when we consider the Jacobian ideal of any germ  $f$ , the Lê numbers and the Segre numbers coincide.

**Remark 4.6.** While in the case of isolated singularities the constancy of the Milnor numbers are sufficient to guarantee the topological triviality of the family in the non-isolated singularity case, this is not true, as observed by Massey in [9] and shown by Bobadilla in [1]. However Massey shows in the Theorem 9.4 of [9] a generalization of part of the result of Lê-Ramanujam, essentially he proves that the constancy of the Lê numbers in a family implies the constancy of the Milnor fibrations in the family.

The importance of the constancy of the Lê numbers to express the topological triviality or equisingularity of germs of hypersurfaces is shown in the results of Massey [9], Gaffney-Gassler [2], Gaffney-Massey [3] and Bobadilla [1]. The results given in [1] are given for function germs with one dimensional critical set and the main interest of Bobadilla is to obtain geometrical conditions to ensure when the constancy of the Lê numbers implies the topological triviality of families of such functions.

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**Roberto Callejas-Bedregal**

Departamento de Matemática,  
CCEN, UFPb - João Pessoa - Paraíba,  
Brazil  
e-mail : Roberto@mat.ufpb.br

**V. H. Jorge Pérez**

Departamento de Matemática,  
ICMC-USP, São Carlos, SP,  
Brazil  
e-mail : vhjperes@icmc.usp.br

**M. J. Saia**

Departamento de Matemática,  
ICMC-USP, São Carlos, SP,  
Brazil  
mjsaia@icmc.usp.br

and

**J. M. Tomazella**

Departamento de Matemática,  
UFSCar, São Carlos, SP,  
Brazil  
e-mail : tomazella@dm.ufscar.br