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On some refinements of companions of Fejér's inequality via superquadratic functions

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Abstract

In this paper some companions of Fejér's inequality for superquadratic functions are given, we also get refinements of some known results proved in [18].

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1. Introduction

Let $\emptyset \neq I \subseteq R$, $a, b \in I$ with $a < b$, let $f : I \to R$ be a convex function and $p : [a, b] \to R$ be a non-negative integrable and symmetric about $x = \frac{a+b}{2}$. The following two inequalities are of great significance in literature: the first known as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

with the reversed inequality for the concave function f, and the second, known as Fejér's inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx. \tag{1.2}$$

These inequalities attracted the attention of many mathematicians over the decades and they generalize, improve and extend these inequalities in a number of ways, see [6, 7, 8, 9, 11, 19]. Let us now define some mappings and quote the results established by K.L. Tseng, S. R. Hwang and S.S. Dragomir in [18]:

$$G(t) = \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)\right],$$

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx,$$

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right)\right.$$
$$\left. + f\left(t\frac{x+b}{2} + (1-t)\frac{a+b}{2}\right)\right]p(x)dx,$$

$$L_p(t) = \frac{1}{2}\int_a^b \left[f\left(ta + (1-t)x\right) + f\left(tb + (1-t)x\right)\right]p(x)dx,$$

$$L_p(t) = \frac{1}{2(b-a)}\int_a^b \left[f\left(ta + (1-t)x\right) + f\left(tb + (1-t)x\right)\right]dx$$

and

$$S_p(t) = \frac{1}{4}\int_a^b \left[f\left(ta + (1-t)\,\frac{x+a}{2}\right) + f\left(ta + (1-t)\,\frac{x+b}{2}\right)\right.$$

$$\left. + f\left(tb + (1-t)\,\frac{x+a}{2}\right) + f\left(tb + (1-t)\,\frac{x+b}{2}\right)\right] p(x)dx,$$

where $f : [a,b] \to R$ is a convex function and $p : [a,b] \to R$ is non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $t \in [0,1]$.

Now we quote some results from [18]:

Theorem 1. *[18] Let f, p, I be defined as above. Then:*

1. *The following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p\,(x)\,dx \tag{1.3}$$

$$\leq 2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx\right]$$

$$\leq \int_0^1 I\,(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b p\,(x)\,dx\right.$$

$$\left. + \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] p\,(x)\,dx\right].$$

2. *If f is differentiable on $[a,b]$ and p is bounded on $[a,b]$, then for all $t \in [0,1]$ we have the inequality*

$$0 \leq \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] p\,(x)\,dx - I(t) \tag{1.4}$$

$$\leq (1-t)\left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx\right] \|p\|_\infty\,,$$

where $\|p\|_\infty = \sup_{x\in[a,b]} |p\,(x)|\,.$

3. *If f is differentiable on $[a,b]$, then for all $t\in[0,1]$ we have the inequality*

$$0\leq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx-I(t)\leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b p(x)\,dx. \tag{1.5}$$

Theorem 2. *[18] Let f, p, G, I be defined as above. Then:*

1. *The following inequality holds for all $t\in[0,1]$:*

$$I(t)\leq G(t)\int_a^b p(x)dx. \tag{1.6}$$

2. *If f is differentiable on $[a,b]$ and p is bounded on $[a,b]$, then for all $t\in[0,1]$ we have the inequality*

$$0\leq I(t)-f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\leq (b-a)\left[G(t)-H(t)\right]\|p\|_\infty, \tag{1.7}$$

where $\|p\|_\infty=\sup_{x\in[a,b]}|p(x)|$.

Theorem 3. *[18] Let f, p, G, I, S_p be defined as above. Then we have the following results:*

1. *S_p is convex on $[0,1]$.*
2. *The following inequalities hold for all $t\in[0,1]$:*

$$G(t)\int_a^b p(x)\,dx\leq S_p(t) \tag{1.8}$$

$$\leq (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)\,dx$$

$$+t\cdot\frac{f(a)+f(b)}{2}\int_a^b p\left(x\right)dx\leq\frac{f(a)+f(b)}{2}\int_a^b p\left(x\right)dx,$$

$$I(1-t)\leq S_p(t) \tag{1.9}$$

and

$$\frac{I(t)+I(1-t)}{2}\leq S_p(t). \tag{1.10}$$

3. *The following equality holds:*

$$\sup_{x\in[0,1]} S_p\left(t\right)=S_p(1)=\frac{f(a)+f(b)}{2}\int_a^b p\left(x\right)dx. \tag{1.11}$$

They used the following Lemma to prove the above results:

Lemma 4. *[17, p. 3] $f:[a,b]\longrightarrow R$ be convex function and let $a\leq A\leq C\leq D\leq B\leq b$ with $A+B=C+D$. Then*

$$f(A)+f(B)\leq f(C)+f(D).$$

Let us now recall the definition, some of the properties and results related to superquadratic functions to be used in the sequel.

Definition 5. *[3, Defintion 2.1] Let $I=[0,a]$ or $[0,\infty)$ be an interval in R. A function $f:I\longrightarrow R$ is superquadratic if for each x in I there exists a real number $C(x)$ such that*

$$f(y)-f(x)\geq C(x)(y-x)+f\left(|y-x|\right) \tag{1.12}$$

for all $y\in I$. If $-f$ is superquadratic then f is called subquadratic.

For examples of superquadratic functions see [2, p. 1049].

Theorem 6. *[3, Theorem 2.3] The inequality*

$$f\left(\int gd\mu\right)\leq\int\left(f(g(s))-f\left(\left|g(s)-\int gd\mu\right|\right)\right)d\mu(s) \tag{1.13}$$

holds for all probability measure μ and all non-negative μ-integrable function g, if and only if f is superquadratic.

The following discrete version of the above theorem will be helpful in the sequel of the paper:

Lemma 7. *[2, Lemma A, p.1049] Suppose that f is superquadratic. Let $x_r \geq 0$, $1 \leq r \leq n$, and let $\bar{x} = \sum_{r=1}^{n} \lambda_r x_r$ where $\lambda_r \geq 0$ and $\sum_{r=1}^{n} \lambda_r = 1$. Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^{n} \lambda_r f\left(|x_r - \bar{x}|\right). \tag{1.14}$$

The following Lemma shows that positive superquadratic functions are also convex:

Lemma 8. *[3, Lemma 2.2] Let f be superquadratic function with $C(x)$ as in Definition 1. Then*

1. *$f(0) \leq 0$.*
2. *If $f(0) = f^{'}(0) = 0$ then $C(x) = f'(x)$ whenever f is differentiable at $x > 0$.*
3. *If $f \geq 0$, then f convex and $f(0) = f'(0) = 0$.*

In [4] a converse of Jensen's inequality for superquadratic functions was proved:

Theorem 9. *[4, Theorem 1] Let (Ω, A, μ) be a measurable space with $0 < \mu(\Omega) < \infty$ and let $f : [0, \infty) \to R$ be a superquadratic function. If $g : \Omega \to [m, M] \subseteq [0, \infty)$ is such that g, $f \circ g \in L_1(\mu)$, then we have for $\bar{g} = \frac{1}{\mu(\Omega)} \int g d\mu$,*

$$\frac{1}{\mu(\Omega)} \int f(g) d\mu \leq \frac{M - \bar{g}}{M - m} f(m) + \frac{\bar{g} - m}{M - m} f(M) \tag{1.15}$$

$$- \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int \left((M - g)\, f\left(g - m\right) + (g - m) f\left(M - g\right) \right) d\mu.$$

The discrete version of this theorem is:

Theorem 10. *[4, Theorem 2] Let $f : [0, \infty) \to R$ be a superquadratic function. Let $(x_1, ..., x_n)$ be an n-tuple in $[m, M]^n$ $(0 \leq m \leq M < \infty)$, and $(p_1, ..., p_n)$ be a non-negative n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$. Denote $\bar{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$, then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M) \tag{1.16}$$

$$-\frac{1}{P_n(M-m)}\sum_{i=1}^{n} p_i \left[(M-x_i) f(x_i-m) + (x_i-m) f(M-x_i)\right]$$

For recent results on Fejér and Hermite-Hadamard type inequalities for superquadratic functions, we refer interested readers to [4], [5] and [2]. In this paper we deal with mappings $G(t)$, $I(t)$, $S_p(t)$ and $L(t)$ when f is superquadratic function. In case when superquadratic function f is also non-negative and hence convex we get refinements of some parts of Theorem 1, Theorem 2 and of Theorem 3.

2. Main Results

In this section we prove our main results by using the same techniques as used in [17] and [2]. Moreover, we assume that all the considered integrals in this section exist.

In order to prove our main results we go through some calculations. From Lemma 2 and Theorem 6 for $n = 2$, we get that

$$f(z) \leq \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \frac{M-z}{M-m} f(z-m) - \frac{z-m}{M-m} f(M-z) \tag{2.1}$$

and

$$f(M+m-z) \leq \frac{z-m}{M-m} f(m) + \frac{M-z}{M-m} f(M) - \frac{z-m}{M-m} f(M-z) - \frac{M-z}{M-m} f(z-m) \tag{2.2}$$

hold for superquadratic function f, $0 \leq m \leq z \leq M$, $m < M$.

Therefore from (2.1) and (2.2), we have

$$f(z) + f(M+m-z) \leq f(m) + f(M) - 2\frac{z-m}{M-m} f(M-z) - 2\frac{M-z}{M-m} f(z-m). \tag{2.3}$$

Now for $0 \leq t \leq \frac{1}{2}$ and $0 \leq a \leq x \leq \frac{a+b}{2}$, we obtain from (2.3) the following inequalities:

By setting $z=\frac{a+b}{2}, M=\frac{3(a+b)}{4}-\frac{x}{2}, m=\frac{x}{2}+\frac{a+b}{4}$ in (2.3), we have that

$$2f\left(\frac{a+b}{2}\right)\leq f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3\left(a+b\right)}{4}-\frac{x}{2}\right)-2f\left(\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right) \tag{2.4}$$

holds.

Also, by replacing $z=\frac{x}{2}+\frac{a+b}{4}, M=tx+(1-t)\frac{a+b}{2}, m=t\frac{a+b}{2}+(1-t)x$ in (2.3), we get that

$$\begin{aligned}2f\left(\frac{x}{2}+\frac{a+b}{4}\right)&\leq f\left(t\frac{a+b}{2}+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right)\\&-2f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right)\end{aligned} \tag{2.5}$$

holds.

Further, for $z=\frac{3(a+b)}{4}-\frac{x}{2}, M=t\frac{a+b}{2}+(1-t)(a+b-x)$, $m=t(a+b-x)+(1-t)\frac{a+b}{2}$ in (2.3), we observe that

$$\begin{aligned}2f\left(\frac{3\left(a+b\right)}{4}-\frac{x}{2}\right)&\leq f\left(t\left(a+b-x\right)+(1-t)\frac{a+b}{2}\right)\\+f\left(t\frac{a+b}{2}+(1-t)\left(a+b-x\right)\right)&-2f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right)\end{aligned} \tag{2.6}$$

holds.

Again, for $z=t\frac{a+b}{2}+(1-t)x, M=\frac{a+b}{2}, m=x$ in (2.3), we observe that

$$\begin{aligned}&f\left(t\frac{a+b}{2}+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right)\\&\leq f(x)+f\left(\frac{a+b}{2}\right)-2tf\left((1-t)\left(\frac{a+b}{2}-x\right)\right)-2(1-t)f\left(t\left(\frac{a+b}{2}-x\right)\right)\end{aligned} \tag{2.7}$$

holds.

Finally, by setting $z=t(a+b-x)+(1-t)\frac{a+b}{2}, M=a+b-x, m=\frac{a+b}{2}$ in (2.3), we get that

$$f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2}\right)+f\left(t\frac{a+b}{2}+\left(1-t\right)\left(a+b-x\right)\right)$$

$$\leq f\left(\frac{a+b}{2}\right)+f\left(a+b-x\right)-2tf\left(\left(1-t\right)\left(\frac{a+b}{2}-x\right)\right) \tag{2.8}$$

$$-2\left(1-t\right)f\left(t\left(\frac{a+b}{2}-x\right)\right)$$

holds.

Now we are ready to state and prove our main results based on the calculations done above.

Theorem 1. *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$, $0\leq a<b$. Let I be defined as above, then we have the following inequalities:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p\left(x\right)dx \leq 2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx\right. \tag{2.9}$$

$$\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx\right]-\int_a^b f\left(\frac{1}{4}\left(b-x\right)\right)p(x)dx,$$

$$2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx\right] \tag{2.10}$$

$$\leq \int_0^1 I(t)dt-\int_a^b\int_0^1 f\left(\left|\frac{1}{2}-t\right|\left(\frac{b-x}{2}\right)\right)p\left(x\right)dtdx$$

and

$$\int_0^1 I(t)dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b p\left(x\right)dx+\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)\right.\right. \tag{2.11}$$

$$\left.\left.+f\left(\frac{x+b}{2}\right)\right]p\left(x\right)dx\right]-2\int_a^b\int_0^1\left(1-t\right)f\left(t\frac{b-x}{2}\right)p\left(x\right)dtdx.$$

Proof. Using simple techniques of integration and by the assumptions on p, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = 4\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}\right)p(2x-a)dtdx.$$

Therefore from (2.4), we get that

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \tag{2.12}$$

$$\leq 2\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]p(2x-a)dtdx$$

$$-4\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} f\left(\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right)p(2x-a)dtdx.$$

But

$$2\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]p(2x-a)dtdx$$

(2.13)

$$=2\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]p(4x-2a-b)dx$$

$$=2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx\right].$$

From (2.12), (2.13) and by the change of variable $x\to\frac{x+a}{2}$, we get (2.9). From (2.5), (2.6) and (2.13), we have

$$2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,p(4x-2a-b)\,dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\,p(4x-a-2b)\,dx\right]$$

(2.14)

$$\leq \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(t\frac{a+b}{2} + (1-t)\,x \right) + f\left(tx + (1-t)\,\frac{a+b}{2} \right) \right.$$

$+\mathrm{f}\Big(t\,(a+b-x) + (1-t)\,\frac{a+b}{2}\Big)$

$+f\left(t\frac{a+b}{2} + (1-t)\,(a+b-x)\right)\Big]\,p\,(2x-a)\,dtdx$

-$4\int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right) p\,(2x-a)\,dtdx.$

But

$$\int_0^1 I(t)dt = \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(t\frac{a+b}{2} + (1-t)\,x \right) + f\left(tx + (1-t)\,\frac{a+b}{2} \right) \right. \tag{2.15}$$

$$+f\left(t\,(a+b-x) + (1-t)\,\frac{a+b}{2} \right)$$

$+f\left(t\frac{a+b}{2} + (1-t)\,(a+b-x)\right)\Big]\,p\,(2x-a)\,dtdx.$

From (2.14) and (2.15), we get that

$$2\left[\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f\,(x)\,p\,(4x-2a-b)\,dx + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f\,(x)\,p\,(4x-a-2b)\,dx \right] \tag{2.16}$$

$$\leq \int_0^1 I(t)dt - 4\int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right) p\,(2x-a)\,dtdx.$$

By the change of variables $t \to 1-t$ and $x \to \frac{x+a}{2}$ in (2.16), we get (2.10).

From (2.7), (2.8) and (2.15), we have

$$\int_0^1 I(t)dt \leq \int_0^{\frac{1}{2}} \int_a^{\frac{a+b}{2}} \left[f\,(x) + 2f\left(\frac{a+b}{2}\right) - 4tf\left((1-t)\left(\frac{a+b}{2}-x\right)\right) \right. \tag{2.17}$$

$$- 4(1-t) f\left(t\left(\frac{a+b}{2} - x\right)\right) + f(a+b-x)\Big] p(2x-a)\, dtdx.$$

But

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx + \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] p(x)\,dx\right]$$

(2.18)

$$= \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f(x) + 2f\left(\frac{a+b}{2}\right) + f(a+b-x)\right] p(2x-a)\, dtdx.$$

From (2.17), (2.18) and by the change of variables $x \to \frac{x+a}{2}$ and $t \to 1-t$, we get (2.11).

This completes the proof of the theorem as well. □

Remark 2. *If the superquadratic function f is non-negative and hence convex, then from (2.9) we get refinement of the first inequality of (1.3) in Theorem 1; from (2.10) we get refinement of the middle inequality of (1.3) in Theorem 1 and from (2.11) we get refinement of the last inequality of (1.3) in Theorem 1.*

Corollary 3. *Let f be superquadratic integrable function on $[0, b]$. If $p(x) = \frac{1}{b-a}$, $x \in [a, b]$ and $0 \le a < b$, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx - \frac{1}{b-a}\int_a^b f\left(\frac{1}{4}(b-x)\right) dx, \tag{2.19}$$

$$\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \le \int_0^1 H(t)dt - \frac{1}{b-a}\int_a^b \int_0^1 f\left(\left|\frac{1}{2} - t\right|\left(\frac{b-x}{2}\right)\right) dtdx \tag{2.20}$$

and

$$\int_0^1 H(t)dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx\right] - \frac{2}{b-a}\int_a^b \int_0^1 (1-t) f\left(t\frac{b-x}{2}\right) dtdx, \tag{2.21}$$

where

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx, \ t \in [0, 1].$$

Proof. If $p(x) = \frac{1}{b-a}$, $x \in [a,b]$, then $I(t) = H(t)$, $t \in [0,1]$, and therefore the proof of the corollary follows directly from the above theorem. □

Remark 4. *If the superquadratic function f is non-negative and therefore convex, then the inequalities in Corollary 1 refine the inequalities in (1.3) of Theorem B from [18, p.2].*

To proceed to our next result, we go again through the similar calculations as given before Theorem 7.

For $0 \leq a \leq x \leq \frac{a+b}{2}$, $t \in [0,1]$, we have that

$$a \leq ta + (1-t)\frac{a+b}{2} \leq tx + (1-t)\frac{a+b}{2} \leq t(a+b-x) + (1-t)\frac{a+b}{2}$$
$$\leq tb + (1-t)\frac{a+b}{2} \leq b.$$

Therefore, by replacing $z = tx + (1-t)\frac{a+b}{2}$, $M = tb + (1-t)\frac{a+b}{2}$, $m = ta + (1-t)\frac{a+b}{2}$ in (2.3),
we get that

$$f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) + f\left(tx + (1-t)\frac{a+b}{2}\right)$$
$$\leq f\left(tb + (1-t)\frac{a+b}{2}\right) + f\left(ta + (1-t)\frac{a+b}{2}\right)$$
$$-\frac{2(x-a)}{b-a} f\left(t(b-x)\right) - \frac{2(b-x)}{b-a} f\left(t(x-a)\right) \tag{2.22}$$

holds.

Now we are ready to state and prove our next result based on the above calculations.

Theorem 5. *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I and G be defined as above, then the following inequality holds for all $t \in [0,1]$:*

$$I(t) \leq G(t)\int_a^b p(x)\,dx - \int_a^b \frac{1}{2}\left[\frac{x-a}{b-a} f\left(t\left(\frac{2b-x-a}{2}\right)\right)\right.$$
$$\left. + \frac{2b-x-a}{b-a} f\left(t\left(\frac{x-a}{2}\right)\right)\right] p(x)dx. \tag{2.23}$$

Proof. Using simple techniques of integration and by the assumptions on p, we have that the following identity holds for all $t \in [0,1]$:

$$G(t)\int_a^b p(x)dx = \int_a^{\frac{a+b}{2}} \left[f\left(ta+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left. + f\left(tb+(1-t)\frac{a+b}{2}\right)\right] p(2x-a)dx. \tag{2.24}$$

Arguing similarly as in obtaining (2.15), by using (2.22) and (2.24), we get that

$$I(t) \le G(t)\int_a^b p(x)dx \tag{2.25}$$

$$-\int_a^{\frac{a+b}{2}} \left[\frac{2(x-a)}{b-a} f(t(b-x)) + \frac{2(b-x)}{b-a} f(t(x-a))\right] p(2x-a)dx,$$

for all $\in [0,1]$.

By the change of variable $x \to \frac{x+a}{2}$ in (2.25), we get (2.23).

This completes the proof of the theorem. □

Remark 6. *If the superquadratic function f is non-negative and hence convex, then the inequality (2.23) represents a refinement of the inequality (1.6) in Theorem 2.*

Corollary 7. *Let f be superquadratic integrable function on $[0,b]$, let $p(x) = \frac{1}{b-a}$, $0 \le a < b$ and G, H be defined as above. Then for all $t \in [0,1]$, we have the following inequality*

$$H(t) \le G(t) - \int_a^b \frac{1}{2(b-a)} \left[\frac{x-a}{b-a} f\left(t\left(\frac{2b-x-a}{2}\right)\right)\right.$$
$$\left. + \frac{2b-x-a}{b-a} f\left(t\left(\frac{x-a}{2}\right)\right)\right] dx. \tag{2.26}$$

Proof. This is a direct consequence of the above theorem, since for $p(x) = \frac{1}{b-a}$, $x \in [a,b]$, $I(t) = H(t)$, for all $t \in [0,1]$. □

Remark 8. *If the superquadratic function f is non-negative and hence convex, then the inequality (2.26) represents refinement of the inequality (1.6) in [18, Theorem C, p. 2].*

Now again we give some calculations for our next result.

For $0 \le t \le \frac{1}{2}$ and $0 \le a \le x \le \frac{a+b}{2}$, we obtain from (2.3) the following inequalities:

By setting $m = ta + (1-t)x$, $M = ta + (1-t)(a+b-x)$ and $z = ta + (1-t)\frac{a+b}{2}$ in (2.3), we obseve that

$$2f\left(ta + (1-t)\frac{a+b}{2}\right) \le f(ta + (1-t)x) + f(ta + (1-t)(a+b-x)) - 2f\left((1-t)\left(\frac{a+b}{2} - x\right)\right) \tag{2.27}$$

holds.

Also, by replacing $m = tb + (1-t)x$, $M = tb + (1-t)(a+b-x)$ and $z = tb + (1-t)\frac{a+b}{2}$ in (2.3), we get that

$$2f\left(tb + (1-t)\frac{a+b}{2}\right) \le f(tb + (1-t)x) + f(tb + (1-t)(a+b-x)) - 2f\left((1-t)\left(\frac{a+b}{2} - x\right)\right) \tag{2.28}$$

holds.

Theorem 9. *Let f be superquadratic integrable function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \le a < b$. Let S_p and G be defined as above, then the following inequality holds for all $t \in [0,1]$:*

$$G(t)\int_a^b p(x)dx \le S_p(t) - \int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right)p(x)dx. \tag{2.29}$$

Proof. By the simple techniques of integration and by the assumptions on p, we have the following identity for all $t \in [0,1]$:

$$S_p(t) = \frac{1}{2}\int_a^{\frac{a+b}{2}} [f(ta + (1-t)x) + f(ta + (1-t)(a+b-x)) + f(tb + (1-t)x) + f(tb + (1-t)(a+b-x))]\, p(2x-a)dx. \tag{2.30}$$

From (2.27), (2.28) and (2.30), we have that

$$\int_a^{\frac{a+b}{2}} \left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]p(2x-a)dx \tag{2.31}$$

$$\leq \; S_p(t)-2\int_a^{\frac{a+b}{2}} f\left((1-t)\left(\frac{a+b}{2}-x\right)\right)p(2x-a)dx,$$

holds for all $t\in[0,1]$.

From (2.24) and by the change of variable $x\to\frac{a+x}{2}$, we get from (2.31) that

$$G(t)\int_a^b p(x)dx \leq S_p(t)-\int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right)p(x)dx,$$

for all $t\in[0,1]$. Which is (2.29) and this completes the proof of the theorem as well. □

Remark 10. *The result of the above theorem refines the first inequality of Theorem 3, when superquadratic function f is non-negative and hence convex.*

Corollary 11. *Let f be superquadratic integrable function on $[0,b]$ and let $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, $0\leq a<b$. Let G be defined as above, then the following inequality holds for all $t\in[0,1]$:*

$$G(t)\leq L(t)-\frac{1}{b-a}\int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right)dx. \tag{2.32}$$

Proof. Since for $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, $S_p(t)=L_p(t)=L(t)$, for all $t\in[a,b]$. Therefore the proof of the crollary follows directly from the above theorem. □

3. Inequalities for differentiable superquadratic functions

In this section we give results when f is a differentiable superquadratic function. Those results give refinements of (1.4) and (1.5) in Theorem 1 and refine (1.7) of Theorem 2 when superquadratic function f is non-negative and hence convex. Here we quote very important result which will be helpful in the sequel of the paper.

Theorem 1. *[14, Theorem 10, p. 5] Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$, $0\le a<b$. Let I be defined as above and let fp be integrable on $[a,b]$, then for $0\le s\le t\le 1$, $t>0$, we have the following inequality:*

$$I(s)\le I(t)-\int_a^b \frac{t+s}{2t} f\left(\left(\frac{t-s}{2}\right)(b-x)\right)p(x)\,dx \tag{3.1}$$
$$-\int_a^b \frac{t-s}{2t} f\left(\left(\frac{t+s}{2}\right)(b-x)\right)p(x)\,dx.$$

Now we state and prove the first result of this section.

Theorem 2. *Let f be superquadratic function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$, $0\le a<b$. Let f be differentiable on $[a,b]$ such that $f(0)=f^{'}(0)=0$ and p is bounded on $[a,b]$, then the following inequalities hold for all $t\in[0,1]$:*

$$\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx-I(t) \tag{3.2}$$
$$\le (1-t)\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)dx\right]\|p\|_\infty$$
$$-\int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right)p(x)dx,$$

where $\|p\|_\infty=\sup\limits_{x\in[a,b]}|p(x)|$ and

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx-I(t)\le\left[\frac{(f'(b)-f'(a))(b-a)}{4}\right. \tag{3.3}$$
$$\left.-f\left(\left|\frac{a-b}{2}\right|\right)\right]\int_a^b p(x)dx-\int_a^b f\left(t\left(\frac{b-x}{2}\right)\right)p(x)dx.$$

Proof. By integration by parts, we have that

$$\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right]dx=\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)dx$$

(3.4)

$$= \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx.$$

Using the substitution rules for integration, under the assumptions on p, we have

$$\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] p(x)dx = \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)\right. \tag{3.5}$$

$$\left. + f\left(\frac{a+2b-x}{2}\right)\right] p(x)dx$$

$$= \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x)\right] p(2x-a)dx$$

and

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right)\right. \tag{3.6}$$

$$\left. + f\left(t\frac{a+2b-x}{2} + (1-t)\frac{a+b}{2}\right)\right] p(x)dx$$

$$= \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right)\right.$$

$$\left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] p(2x-a)dx,$$

for all $t \in [0,1]$.

Now by the assumptions on f, we have that

$$\left[f(x) - f\left(tx + (1-t)\frac{a+b}{2}\right)\right] p(2x-a) + \left[f(a+b-x)\right. \tag{3.7}$$

$$\left. - f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] p(2x-a)$$

$$\leq (1-t)\left(\frac{a+b}{2} - x\right)\left[f'(a+b-x) - f'(x)\right] p(2x-a)$$

$$-2f\left((1-t)\left(\frac{a+b}{2} - x\right)\right) p(2x-a)$$

$$\leq (1-t)\left(\frac{a+b}{2}-x\right)[f'(a+b-x)-f'(x)]\,\|p\|_\infty$$
$$-2f\left((1-t)\left(\frac{a+b}{2}-x\right)\right)p\,(2x-a)\,,$$

for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$.

From (3.4), (3.5), (3.6) and (3.7) and by the change of variable $x \to \frac{a+x}{2}$, we get (3.2).

By the assumptions on f and from Lemma 3, we get that

$$\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2} \leq \frac{a-b}{4}f'\,(a)-\frac{1}{2}f\left(\left|\frac{a-b}{2}\right|\right)$$

and

$$\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2} \leq \frac{b-a}{4}f'\,(b)-\frac{1}{2}f\left(\left|\frac{a-b}{2}\right|\right).$$

Adding these inequalities we get that

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right) \leq \frac{\left(f'\,(b)-f'\,(a)\right)(b-a)}{4}-f\left(\left|\frac{a-b}{2}\right|\right).$$

Thus

$$\frac{f(a)+f(b)}{2}\int_a^b p\,(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p\,(x)\,dx \tag{3.8}$$
$$\leq \left[\frac{\left(f'\,(a)-f'\,(b)\right)(b-a)}{4}-f\left(\left|\frac{a-b}{2}\right|\right)\right]\int_a^b p\,(x)\,dx.$$

From (3.1), for $s=0$, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b p\,(x)\,dx \leq I(t)-\int_a^b f\left(t\left(\frac{b-x}{2}\right)\right)p(x)dx, \tag{3.9}$$

for all $t \in [0,1]$.

From (3.8) and (3.9), we get (3.3). This completes the proof of the theorem. □

Remark 3. *The inequalities (3.2) and (3.3) refine the inequalities (1.4) and (1.5) in Theorem 1, when the superquadratic function f is non-negative and therefore convex.*

Corollary 4. *Let f be superquadratic function on $[0,b]$ and differentiable on $[a,b]$ such that $f(0)=f^{'}(0)=0$. If $p(x)=\frac{1}{b-a}$, then we have the following inequalities:*

$$\int_a^b \frac{1}{2(b-a)}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx-H(t) \tag{3.10}$$

$$\leq \frac{1-t}{b-a}\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)dx\right]$$

$$-\frac{1}{b-a}\int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right)dx$$

and

$$\frac{f(a)+f(b)}{2}-H(t)\leq \frac{\left(f'(b)-f'(a)\right)(b-a)}{4}-f\left(\left|\frac{a-b}{2}\right|\right)dx \tag{3.11}$$

$$-\frac{1}{b-a}\int_a^b f\left(t\left(\frac{b-x}{2}\right)\right)dx,$$

for all $t\in[0,1]$.

Now we give our last result and summarize the results related to it in the remark followed by Theorem 12.

Theorem 5. *Let f be superquadratic function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$, $0\leq a<b$. Let f be differentiable on $[a,b]$ such that $f(0)=f^{'}(0)=0$ and p is bounded on $[a,b]$, then for all $t\in[0,1]$ we have the inequality:*

$$I(t)-f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\leq(b-a)\left[G(t)-H(t)\right]\|p\|_\infty \tag{3.12}$$

$$-\int_a^b f\left(\frac{t}{2}(b-x)\right)p(x)dx,$$

where $\|p\|_\infty=\sup\limits_{x\in[a,b]}|p(x)|$.

Proof. By integration by parts, we observe that

$$
\begin{aligned}
(3.13)\qquad & t\int_a^{\frac{a+b}{2}} \left[\left(x-\frac{a+b}{2}\right) f^{'}\left(tx+(1-t)\frac{a+b}{2}\right)\right. \\
&+\left.\left(\frac{a+b}{2}-x\right) f^{'}\left(t\,(a+b-x)+(1-t)\frac{a+b}{2}\right)\right] dx \\
&= t\int_a^{b} \left(x-\frac{a+b}{2}\right) f^{'}\left(tx+(1-t)\frac{a+b}{2}\right) dx \\
&= (b-a)\,G(t)-H(t),
\end{aligned}
$$

hold for sll $t\in[0,1]$.

Under the assumptions on f, we have that

$$
\begin{aligned}
(3.14)\qquad & \left[f\left(tx+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right] p(2x-a) \\
&+\left[f\left(t\,(a+b-x)+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right] p(2x-a) \\
&\le t\left(x-\frac{a+b}{2}\right) f^{'}\left(tx+(1-t)\frac{a+b}{2}\right) p(2x-a) \\
&+t\left(\frac{a+b}{2}-x\right) f^{'}\left(t\,(a+b-x)+(1-t)\frac{a+b}{2}\right) p(2x-a) \\
&-2f\left(t\left|\frac{a+b}{2}-x\right|\right) p(2x-a) \\
&= t\left(\frac{a+b}{2}-x\right)\left[f^{'}\left(t\,(a+b-x)+(1-t)\frac{a+b}{2}\right)\right. \\
&\left.-f^{'}\left(tx+(1-t)\frac{a+b}{2}\right)\right] p(2x-a)-2f\left(t\left|\frac{a+b}{2}-x\right|\right) p(2x-a) \\
&\le t\left(\frac{a+b}{2}-x\right)\left[f^{'}\left(t\,(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.
\end{aligned}
$$

$$-f^{'}\left(tx+(1-t)\frac{a+b}{2}\right)\Big]\,\|p\|_{\infty}-2f\left(t\left|\frac{a+b}{2}-x\right|\right)p(2x-a)$$

hold for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.
Integrating (3.14) over x on $\left[a,\frac{a+b}{2}\right]$, using (3.13), by the change of variable $x\rightarrow\frac{x+a}{2}$, under the assumptions on p, we get (3.12). This completes the proof of the theorem. □

Remark 6. *The result of Theorem 12 refines (1.7) of Theorem 2, when superquadratic function f is non-negative and therefore convex.*

Corollary 7. *Let f be superquadratic function on $[0,b]$. Let f be differentiable on $[a,b]$ such that $f(0)=f^{'}(0)=0$. If $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, then*

$$H(t)-f\left(\frac{a+b}{2}\right)\leq[G(t)-H(t)]-\frac{1}{b-a}\int_{a}^{b}f\left(t\left(\frac{b-x}{2}\right)\right)dx,$$

(3.15)

for all $t\in[0,1]$.

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