

Proyecciones Journal of Mathematics
Vol. 31, N° 4, pp. 73-90, March 2013.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172013000100006

A new convergence analysis for the two-step Newton method of order three

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Received : September 2012. Accepted : March 2013

Abstract

We present a tighter than before semilocal convergence analysis for the two-step Newton method of order three using recurrent functions. Numerical examples are also provided to show that our convergence criteria are satisfied but earlier studies such as in nine, thirteen, fifteen are not satisfied.

AMS Subject Classification : 65H10; 65G99; 65J15; 47H17; 49M15

Key Words : *Two-step Newton method, Newton's method, Banach space, Kantorovich hypothesis, majorizing sequence, Lipschitz/center-Lipschitz conditions.*

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of equation

$$(1.1) \quad \mathcal{F}(x) = 0,$$

where, \mathcal{F} is Fréchet-differentiable operator defined on a convex subset \mathcal{D} of a Banach space \mathcal{X} with values in a Banach space \mathcal{Y} .

Many problems in computational mathematics can be brought in the form (1.1). The solutions of these equations are rarely found in closed form. Therefore most solution methods for these equations are iterative. Newton's method

$$(1.2) \quad x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1} \mathcal{F}(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D})$$

is undoubtedly the most popular method for generating a sequence $\{x_n\}$ converging quadratically to x^* . Two-step Newton method (TSNM)

$$(1.3) \quad \begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1} \mathcal{F}(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1} \mathcal{F}(y_n) \end{aligned}$$

has also been used to generate a cubically convergent sequence x^* five, nine. Note that (1.3) requires one more evaluation of \mathcal{F} per step than Newton's method (1.2)

In particular Ezquerro, Hernández and Salanova nine used the following conditions (in non-affine invariant form) (\mathbf{C}_K)

$$\mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}) \text{ for some } x_0 \in \mathcal{D};$$

$$\left\| \mathcal{F}'(x_0)^{-1} \mathcal{F}(x_0) \right\| \leq \nu$$

$$\left\| \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(x) - \mathcal{F}'(x_0)] \right\| \leq L_0 \|x - x_0\| \quad \text{for all } x \in \mathcal{D};$$

$$\left\| \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(x) - \mathcal{F}'(y)] \right\| \leq L \|x - y\| \quad \text{for all } x, y \in \mathcal{D};$$

$$h_k = L \eta \leq \frac{1}{2}$$

(1.4)

and

$$U(x_0, \lambda) = \left\{ x \in \nabla[\mathcal{D}] \mid \|x - x_0\| \leq \lambda \right\} \subseteq \mathcal{D},$$

for specified $\lambda \geq 0$.

The same (\mathbf{C}_k) conditions have been used to show the semilocal convergence for the Newton's method (1.2). Note that (1.4) is the, famous for its simplicity and clarity, Kantorovich sufficient convergence hypothesis for the Newton's method (1.2). A current survey on Newton-type methods can be found in [1] and the references therein (see also [13, 15]). We have shown the quadratic convergence of the Newton's method (1.2). Using the set of conditions (\mathbf{C}_{AH})

$$\begin{aligned} & \mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}) \quad \text{for some } x_0 \in \mathcal{D}; \\ & \left\| \mathcal{F}'(x_0)^{-1} \mathcal{F}(x_0) \right\| \leq \text{red} \eta \\ & \left\| \mathcal{F}'(x_0)^{-1} \left[\mathcal{F}'(x) - \mathcal{F}'(x_0) \right] \right\| \leq L_0 \|x - x_0\| \quad \text{for all } x \in \mathcal{D}; \\ & \left\| \mathcal{F}'(x_0)^{-1} \left[\mathcal{F}'(x) - \mathcal{F}'(y) \right] \right\| \leq L \|x - y\| \quad \text{for all } x, y \in \mathcal{D}; \\ & h_{AH} = L \eta \leq \frac{1}{2} \end{aligned} \quad (1.5)$$

and

$$U(x_0, \lambda_0) \subseteq \mathcal{D},$$

for some specified $\lambda_0 \geq 0$, where

$$(1.6) \quad L = \frac{1}{8} \left(L + 4L_0 + \sqrt{L^2 + 8L_0L} \right).$$

Note that

$$(1.7) \quad L_0 \leq L$$

holds in general, and L/L_0 can be arbitrarily large (four, five). Moreover, $\text{red} L_0$ the Center-Lipschitz is not an additional condition, since L_0 is a

special case of $\text{red}L$. Furthermore, we have by (1.4)-eq:17

$$(1.8) \quad h_K \leq \frac{1}{2} \implies h_{AH} \leq \frac{1}{2}$$

but not necessarily vice versa unless if $L_0 = \text{red}L$. The error analysis under eq:15 is also tighter than eq:14. Hence, the applicability of Newton's method (1.2) has been extended.

In this study, we provide the sufficient convergence conditions for (TSNM) corresponding to (1.4). The paper is organized as follows: §2 contains the semilocal convergence analysis for (TSNM), whereas the numerical examples are given in §3.

2. Semilocal Convergence Analysis for (TSNM)

We need the following result on majorizing sequence for (TSNM).

Lemma 2.1. *Let L_0, L, η be positive constants. Assume: there exist parameters α and ϕ such that*

$$(2.1) \quad \frac{L\eta}{2} \leq \alpha \leq \frac{L}{2L_2},$$

$$(2.2) \quad \frac{L_1\eta}{2(1 - L_2\eta)} \leq \phi \leq \phi_0$$

and

$$(2.3) \quad \eta \leq \min \left\{ \frac{2}{L_1 + 2L_2(1 + \phi)}, \frac{1}{L_2} \right\}$$

where,

$$(2.4) \quad L_1 = \alpha(2 + \alpha)L, \quad L_2 = (1 + \alpha)L_0,$$

$$(2.5) \quad \phi_0 = \min \left\{ \frac{2L_1}{L_1 + \sqrt{L_1^2 + 8L_1L_2}}, \frac{L - 2\alpha L_2}{L}, \frac{2\alpha(1 - L_2\eta)}{L\eta} \right\}.$$

Then, sequences $\{s_n\}, \{t_n\}$ generated by

$$(2.6) \quad \begin{aligned} t_0 &= 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0 t_n)}, \\ s_{n+1} &= t_{n+1} + \frac{L[2(s_n - t_n) + t_{n+1} - \text{red}s_n](t_{n+1} - \text{red}s_n)}{2(1 - L_0 t_{n+1})} \end{aligned}$$

are non-decreasing, bounded from above by

$$(2.7) \quad t^{**} = \left(\frac{1 + \alpha}{1 - \phi} \right) \eta,$$

and converge to their common least upper bound $t^* \in [0, t^{**}]$. Moreover, the following estimates hold

$$(2.8) \quad 0 \leq t_{n+1} - s_n \leq \alpha(s_n - t_n),$$

and

$$(2.9) \quad 0 \leq s_{n+1} - t_{n+1} \leq \phi(s_n - t_n).$$

Proof. We shall show using induction on k :

$$(2.10) \quad 0 \leq \frac{L(s_k - t_k)}{2(1 - L_0 t_k)} \leq \alpha,$$

and

$$(2.11) \quad 0 \leq \frac{L_1(s_k - t_k)}{2(1 - L_0 t_{k+1})} \leq \phi.$$

Note that estimates (2.8) and (2.9) will then follow from (2.10) and (2.11), respectively. Estimates (2.10) and (2.11) hold by the left hand side hypotheses in (2.1), (2.2), respectively. It follows from (2.6), (2.10) and (2.11) that estimates (2.8) and (2.9) hold for $redn = 0$. Let us assume estimates (2.10) and (2.11) hold for all $k \leq redn$. It then follows that estimates (2.8) and (2.9) hold for $n = redk$. We then have:

$$(2.12) \quad 0 \leq s_k - t_k \leq \phi(s_{k-1} - t_{k-1}) \leq \phi \cdot \phi(s_{k-2} - t_{k-2}) \leq \cdots \leq \phi^k \eta,$$

$$(2.13) \quad 0 \leq t_{k+1} - s_k \leq \alpha(s_k - t_k) \leq \alpha \phi^k \eta,$$

and

$$\begin{aligned} t_{k+1} &\leq s_k + \alpha \phi^k \eta \leq t_k + \alpha \phi^k \eta + \phi^k \eta \\ &\leq s_{k-1} + \alpha \phi^{k-1} \eta + \alpha \phi^k \eta + \phi^k \eta \end{aligned}$$

$$\begin{aligned}
&\leq t_{k-1} + \phi^{k-1}\eta + \alpha\phi^{k-1}\eta + \alpha\phi^k\eta + \phi^k\eta \\
&= t_{k-1} + (\phi^{k-1} + \phi^k)\eta + \alpha(\phi^{k-1} + \phi^k)\eta \leq \dots \\
&\leq s_0 + \alpha(\eta + \phi\eta + \dots + \phi^k\eta) + \alpha(\phi\eta + \dots + \phi^k\eta) \\
&= (1+\alpha)(1 + \phi + \dots + \phi^k\eta) \leq t^{**}.
\end{aligned}
\tag{2.14}$$

In view of (2.12) and (2.14), estimate (2.10) certainly holds, if

$$(2.15) \quad 0 \leq \frac{L\phi^k\eta}{2[1 - L_2(1 + \phi + \dots + \phi^{k-1})\eta]} \leq \alpha,$$

or

$$(2.16) \quad L\phi^k\eta + 2\alpha L_2(1 + \phi + \dots + \phi^{k-1})\eta - 2\alpha \leq 0.$$

Estimate (2.16) motivates us to introduce recurrent functions f_k on $[0, 1)$ by

$$(2.17) \quad f_k(t) = L\eta t^k + 2\alpha L_2(1 + t + \dots + t^{k-1})\eta - 2\alpha.$$

We need a relationship between two consecutive functions f_k :

$$\begin{aligned}
f_{k+1}(t) &= Lt^{k+1}\eta + 2\alpha L_2(1 + t + \dots + t^k)\eta - 2\alpha - Lt^k\eta - \\
&2\alpha L_2(1 + t + \dots + t^{k-1})\eta + 2\alpha + f_k(t) \\
&= f_k(t) + Lt^{k+1}\eta - Lt^k\eta + 2\alpha L_2 t^k\eta \\
&= f_k(t) + g(t)t^k\eta,
\end{aligned}
\tag{2.18}$$

where

$$(2.19) \quad g(t) = Lt - L + 2\alpha L_2.$$

Note that $g(\phi) \leq 0$ by (2.2). Using (2.17) we see that (2.16) holds

$$(2.20) \quad \begin{array}{rcl} redif & f_k(\phi) & \leq 0 \\ redor & redf_1(\phi) & \leq 0, \end{array}$$

$$(2.21) \text{ since, } g(\phi) \leq 0 \text{ and } f_{k+1}(\phi) = f_k(\phi) + g(\phi)\phi^k\eta \leq f_k(\phi),$$

where ϕ is chosen as in the right hand side inequality of (2.1). But (2.20) also holds by (2.2). Moreover, define function f_∞ on $[0, 1)$ by

$$(2.22) \quad f_\infty(t) = \lim_{k \rightarrow \infty} f(t).$$

Then, we have by (2.19) that

$$(2.23) \quad f_\infty(\phi) \leq 0.$$

Hence, (2.8) and (2.10) hold for all k . Similarly, (2.11) holds, if

$$(2.24) \quad L_1\phi^k\eta \leq 2\phi \left[1 - L_2(1 + \phi + \dots + \phi^k)\eta \right]$$

or

$$(2.25) \quad L_1\phi^k\eta + 2\phi L_2(1 + \phi + \dots + \phi^k)\eta - 2\phi \leq 0.$$

As in (2.17) we define functions p_k on $[0, 1)$ by

$$(2.26) \quad p_k(t) = L_1t^k\eta + 2tL_2(1 + t + \dots + t^k)\eta - 2\phi.$$

We need a relationship between two consecutive functions $redh_k$:

$$\begin{aligned} p_{k+1}(t) &= [t]L_1t^{k+1}\eta + 2tL_2(1 + t + \dots + t^{k+1})\eta - 2\phi - L_1t^k\eta \\ &\quad - 2tL_2(1 + t + \dots + t^k)\eta + 2\phi + p_k(t) \\ &= p_k(t) + L_1t^{k+1}\eta - L_1t^k\eta + 2L_2t^{k+2}\eta \\ &= p_k(t) + g_1(t)t^k\eta \end{aligned} \quad (2.27)$$

where

$$(2.28) \quad g_1(t) = 2L_2t^2 + L_1t - L_1.$$

Note that $g_1(\phi) \leq 0$ by (2.2) and that

$$(2.29) \quad redr = red \frac{2L_1}{L_1 + \sqrt{L_1^2 + 8L_1L_2}}$$

redis the positive root of g_1 . In view of (2.26), estimate (2.25) holds

$$(2.30) \quad \text{if } p_k(\phi) \leq 0 \quad \text{or} \quad p_1(\phi) \leq 0$$

since, $g_1(\phi) \leq 0$ and $p_{k+1}(\phi) = p_k(\phi) + g_1(\phi)\phi^k\eta \leq p_k$, where ϕ is chosen as in the right hand side of (2.2). Note now that (2.30) holds by (2.3). Furthermore, define functions p_∞ on $[0, 1]$ by

$$(2.31) \quad p_\infty(t) = \lim_{k \rightarrow \infty} p_k(t).$$

We then have

$$(2.32) \quad p_\infty(\phi) \leq 0.$$

That completes the induction for (2.9) and (2.11). Finally, in view of (2.8), (2.9) and (2.14), sequences $\{t_n\}$, $\{s_n\}$ converge to t^* . That completes the proof of the Lemma. \square

We need an Ostrowski-type relationship between iterates $\{x_n\}$ and $\{y_n\}$ fourteen.

Lemma 2.2. *Let us assume iterates $\{x_n\}$ and $\{y_n\}$ in (TSNM) are well defined for all $n \geq 0$. Then, the following identities hold:*

$$(2.33) \quad \mathcal{F}(x_{n+1}) = \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(x_n)](x_{n+1} - y_n)d\theta,$$

and

$$(2.34) \quad \mathcal{F}(y_n) = \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n)d\theta.$$

Proof. Identity (2.34) follows from the Taylor's theorem and the first iteration in (TSNM), whereas (2.35) follows from Taylor's theorem and the second iteration in (TSNM). That completes the proof of the Lemma. \square

We can show the following semilocal convergence result for (TSNM).

Lemma 2.3. *Let $\mathcal{F} : \mathcal{D} \subset \mathcal{X} \rightarrow \mathcal{Y}$ be Fréchet-differentiable operator. Assume: there exist $x_0 \in \mathcal{D}$, $L_0 > 0$, $L > 0$ and $\eta > 0$ such that for all $x, y \in \mathcal{D}$:*

$$(2.35) \quad \mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}),$$

$$(2.36) \quad \left\| \mathcal{F}'(x_0)^{-1} \mathcal{F}(x_0) \right\| \leq \eta,$$

$$(2.37) \quad \left\| \mathcal{F}'(x_0)^{-1} (\mathcal{F}'(x) - \mathcal{F}'(x_0)) \right\| \leq L_0 \|x - x_0\|,$$

$$(2.38) \quad \left\| \mathcal{F}'(x_0)^{-1} (\mathcal{F}'(x) - \mathcal{F}'(y)) \right\| \leq L \|x - y\|,$$

$$(2.39) \quad U(x_0, t^*) \subseteq \mathcal{D}.$$

Hypotheses of Lemma 2.1 hold, where t^* is given in Lemma 2.1. Then, sequences $\{x_n\}$ and $\{y_n\}$ generated by (TSNM) are well defined, remain in $U(x_0, t^*)$ for all $n \geq 0$ and converge to a solution $x^* \in U(x_0, t^*)$ of equation $\mathcal{F}(x) = 0$.

Moreover, the following estimates hold

$$(2.40) \quad \|y_n - x_n\| \leq s_n - t_n,$$

$$(2.41) \quad \|x_{n+1} - y_n\| \leq t_{n+1} - s_n,$$

$$(2.42) \quad \|x_{n+1} - x_n\| \leq t_{n+1} - t_n,$$

$$(2.43) \quad \|y_{n+1} - y_n\| \leq s_{n+1} - s_n,$$

$$(2.44) \quad \|x_n - x^*\| \leq t^* - t_n,$$

$$(2.45) \quad \|y_n - x^*\| \leq t^* - s_n.$$

Furthermore, if there exists $R \geq t^*$ such that

$$(2.46) \quad U(x_0, R) \subseteq \mathcal{D}$$

and

$$(2.47) \quad L_0(t^* + R) < 2,$$

then, x^* is the only solution of $\mathcal{F}(x) = 0$ in $U(x_0, R)$.

Proof. We shall show using induction on k that (TSNM) is well defined, the iterates remain in $U(x_0, t^*)$ for all $n \geq 0$ and estimates (2.41) and (2.42) hold for all $n \geq 0$. Iterate y_0 is well defined by the first equation in (TSNM) for $n = 0$ and (2.36). We also have by (2.6) and (2.37)

$$\|y_0 - x_0\| = \|\mathcal{F}'(x_0)^{-1} \mathcal{F}(x_0)\| \leq \eta = reds_0 = s_0 - t_0 \leq t^*.$$

That is (2.41) holds for $n = 0$ and $y_0 \in U(x_0, t^*)$. Using (TSNM) for $n = 0$, we see that x_1 is well defined. Moreover, in view of (2.35) for $n = 0$, (TSNM), (2.6) and (2.37)-(2.39), we get

$$\begin{aligned} \|x_1 - y_0\| &= \left\| \int_0^1 \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(x_0 + \theta(y_0 - x_0)) - \mathcal{F}'(x_0)] d\theta (y_0 - x_0) \right\| \\ &\leq L_0 \int_0^1 \theta \|y_0 - x_0\|^2 d\theta = \frac{L_0}{2} \|y_0 - x_0\|^2 \\ &\leq \frac{L_0}{2} (s_0 - t_0)^2 = t_1 - s_0 \end{aligned}$$

which shows (2.42) for $n = 0$. We also have

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 - t_0 \leq t^*,$$

which implies (2.43) holds for $n = 0$ and $x_1 \in U(x_0, t^*)$.

Let $w \in U(x_0, t^*)$. Then, we have by Lemma 2.1 and (2.38) that

$$(2.48) \quad \left\| \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(w) - \mathcal{F}'(x_0)] \right\| \leq L_0 \|w - x_0\| \leq L_0 t^* < 1.$$

It follows from (2.49) and the Banach lemma on invertible operators five,thirteen,fifteen that $\mathcal{F}'(w)^{-1}$ exists and

$$(2.49) \quad \left\| \mathcal{F}'(w)^{-1} \mathcal{F}'(x_0) \right\| \leq \frac{1}{1 - L_0 \|w - x_0\|}.$$

In particular, for $x_1 \in U(x_0, t^*)$, we have

$$\begin{aligned} \|\mathcal{F}'(x_1)^{-1} \mathcal{F}'(x_0)\| &\leq \frac{1}{1 - L_0 \|x_1 - x_0\|} \leq \frac{1}{1 - L_0(t_1 - t_0)} = \frac{1}{1 - L_0 t_1}. \end{aligned} \quad (2.50)$$

Using (TSNM), (2.6), (2.34) (for $n = 0$) and (2.51), we get

$$\begin{aligned} \|y_1 - x_1\| &= \left\| \left[\mathcal{F}'(x_1)^{-1} \mathcal{F}'(x_0) \right] \left[\mathcal{F}'(x_0)^{-1} \mathcal{F}(x_1) \right] \right\| \\ &\leq \|\mathcal{F}'(x_1)^{-1} \mathcal{F}'(x_0)\| \|\mathcal{F}'(x_0)^{-1} \mathcal{F}(x_1)\| \\ &\leq \frac{1}{1 - L_0 t_1} \left\| \int_0^1 \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(y_0 + \theta(x_1 - y_0)) - \mathcal{F}'(x_0)] d\theta (x_1 - y_0) \right\| \\ &\leq \frac{L_0}{1 - L_0 t_1} \int_0^1 [\|y_0 - x_0\| + \theta \|x_1 - y_0\|] d\theta \|x_1 - y_0\| \\ &\leq \frac{redL}{1 - L_0 t_1} \left[(s_0 - t_0) + \frac{1}{2}(t_1 - s_0) \right] (t_1 - s_0) = s_1 - t_1, \end{aligned}$$

which implies (2.41) for $n = 1$. We then have that

$$\|y_1 - y_0\| \leq \|y_1 - x_1\| + \|x_1 - y_0\| \leq s_1 - t_1 + t_1 - s_0 = s_1 - s_0,$$

$$\|y_1 - x_0\| \leq \|y_1 - y_0\| + \|y_0 - x_0\| \leq s_1 - s_0 + s_0 - t_0 = s_1 \leq t^*,$$

which imply (2.44) for $n = 0$ and $y_1 \in U(x_0, t^*)$. Let us now assume (2.41)-(2.44), $y_n, x_k \in U(x_0, t^*)$ for all $n \leq k$. Using (TSNM), (2.6), (2.34), (2.35), (2.39) and the induction hypotheses, we have in turn that

$$\begin{aligned} \|x_{k+1} - x_0\| &\leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \\ &\leq t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 = t_{k+1} \leq t^*, \end{aligned} \quad (2.51)$$

$$\begin{aligned}
\|y_{k+1} - x_{k+1}\| &= \left\| \left[\mathcal{F}'(x_{k+1})^{-1} \mathcal{F}'(x_0) \right] \left[\mathcal{F}'(x_0)^{-1} \mathcal{F}(x_{k+1}) \right] \right\| \\
&\leq \left\| \mathcal{F}'(x_{k+1})^{-1} \mathcal{F}'(x_0) \right\| \left\| \mathcal{F}'(x_0)^{-1} \mathcal{F}(x_{k+1}) \right\| \\
&\leq \frac{1}{1 - L_0 \|x_{k+1} - x_0\|} \int_0^1 \left\| \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(y_k + \theta(x_{k+1} - y_k)) \right. \\
&\quad \left. - \mathcal{F}'(x_k)] d\theta(x_{k+1} - y_k) \right\| \\
&\leq \frac{L}{1 - L_0 t_{k+1}} \int_0^1 \left[\|y_k - x_k\| + \theta \|x_{k+1} - y_k\| \right] d\theta \|x_{k+1} - y_k\| \\
&\leq \frac{L}{1 - L_0 t_{k+1}} \left[s_k - t_k + \frac{1}{2}(t_{k+1} - s_k) \right] (t_{k+1} - s_k) \\
&= s_{k+1} - t_{k+1}, \\
(2.52)
\end{aligned}$$

$$\begin{aligned}
\|x_{k+2} - y_{k+1}\| &= \left\| \left[\mathcal{F}'(x_{k+1})^{-1} \mathcal{F}'(x_0) \right] \left[\mathcal{F}'(x_0)^{-1} \mathcal{F}(y_{k+1}) \right] \right\| \\
&\leq \frac{1}{1 - L_0 t_{k+1}} \int_0^1 \left\| \mathcal{F}'(x_0)^{-1} [\mathcal{F}'(x_{k+1} + \theta(y_{k+1} - x_{k+1})) \right. \\
&\quad \left. - \mathcal{F}'(x_{k+1})] d\theta(y_{k+1} - x_{k+1}) \right\| \\
&\leq \frac{L}{1 - L_0 t_{k+1}} \int_0^1 \theta \|y_{k+1} - x_{k+1}\|^2 d\theta \\
&\leq \frac{L}{2(1 - L_0 t_{k+1})} (s_{k+1} - t_{k+1})^2 = t_{k+2} - s_{k+1}, \\
(2.53)
\end{aligned}$$

$$\begin{aligned}
\|y_{k+2} - y_{k+1}\| &\leq \|y_{k+2} - x_{k+2}\| + \|x_{k+2} - y_{k+1}\| \\
&\leq s_{k+2} - t_{k+2} + t_{k+2} - s_{k+1} = s_{k+2} - s_{k+1}, \\
(2.54)
\end{aligned}$$

$$\begin{aligned}
 & \|x_{k+2} - x_{k+1}\| \leq \|x_{k+2} - y_{k+1}\| + \|y_{k+1} - x_{k+1}\| \\
 & \leq t_{k+2} - s_{k+1} + s_{k+1} - t_{k+1} = redt_{k+2} - t_{k+1} \\
 (2.55)
 \end{aligned}$$

which show (2.41)-(2.44) hold for all $n \geq 0$. Estimates (2.45) and (2.46) follow from (2.43) and (2.44), respectively by using standard majorization technique five,thirteen,fifteen. Moreover, from Lemma 2.1 and (2.41)-(2.44) we deduce that (TSNM) is Cauchy in a Banach space \mathcal{X} and as such it converges to some $x^* \in U(x_0, t^*)$ (since $U(x_0, t^*)$ is a closed set).

Moreover, we have by (2.53)

$$\begin{aligned}
 & \left\| \mathcal{F}'(x_0)^{-1} \mathcal{F}'(x_{k+1}) \right\| \leq L \left[\|y_k - x_k\| + \frac{1}{2} \|x_{k+1} - y_k\| \right] \|x_{k+1} - y_k\| \\
 & \rightarrow 0, \quad \text{as } k \rightarrow \infty. \\
 (2.56)
 \end{aligned}$$

That is $\mathcal{F}(x^*) = 0$. Finally to show uniqueness, let $y^* \in U(x_0, R)$ be a solution of equation $\mathcal{F}(x) = 0$. Let us define linear operator M by

$$(2.57) \quad M = \int_0^1 \mathcal{F}'(y^* + \theta(x^* - y^*)) \theta.$$

Then, using (2.38), (2.47) and (2.48), we get in turn that

$$\begin{aligned}
 & \left\| \mathcal{F}'(x_0) [M - \mathcal{F}'(x_0)] \right\| \leq L_0 \int_0^1 \|y^* + \theta(x^* - y^*) - x_0\| \theta \\
 & \leq L_0 \int_0^1 [(1 - \theta) \|y^* - x_0\| + \theta \|x^* - x_0\|] \theta \\
 & \leq \frac{L_0}{2} (R + t^*) < 1. \\
 (2.58)
 \end{aligned}$$

It follows from (2.59) and the Banach Lemma on invertible operators that M^{-1} exists. Then, in view of the identity

$$(2.59) \quad 0 = \mathcal{F}(x^*) - \mathcal{F}(y^*) = M(x^* - y^*),$$

we conclude that $x^* = y^*$. That completes the proof of the Theorem. \square

Remarks 2.4.

Limit point t^* can be replaced by t^{**} , given in closed form by (2.7), in hypotheses (2.40) and (2.48).

The verification of conditions (2.1)-(2.3) require simple algebra (see also Example 3.1).

If $L_0 = L$, then scalar sequences $\{s_n\}$, $\{t_n\}$ given by (2.6) reduce essentially to the ones used in nine. In particular, we have in this case

$$(2.60) \quad \begin{aligned} redt_0 &= 0, \quad reds_0 = \eta, \quad t_{n+1} = s_n + \frac{L(s_n - t_n)^2}{2(1 - Lt_n)}, \\ s_{n+1} &= t_{n+1} + \frac{L[2(s_n - t_n) + t_{n+1} - s_n](t_{n+1} - s_n)}{2(1 - Lt_{n+1})} \end{aligned}$$

If $L_0 < L$ iteration (2.6) is tighter than eq:261. Moreover, in view of the proof of the Theorem 2.3, we note that sequence

$$(2.61) \quad \begin{aligned} t_0 &= 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{L^*(s_n - t_n)^2}{2(1 - L_0 t_n)}, \\ s_{n+1} &= t_{n+1} + \frac{L^*[2(s_n - t_n) + t_{n+1} - s_n](t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}, \end{aligned}$$

is also majorizing for (TSNM), where

$$L^* = \begin{cases} L_0, & \text{if } n = 0 \\ L, & \text{if } n > 0. \end{cases}$$

In case $L_0 < L$, (2.26) is even a tighter majorizing sequence than (2.61). Furthermore, L, L_1 can be replaced by $L_0, L_1^* = \alpha(\alpha + 2)L_0$ red at the left hand sides of (2.1) and (2.2), respectively.

If $\alpha = 0$, redefine $L_1 = L$, then it is simple algebra to show that conditions of Lemma 2.1 reduce to (1.5). Moreover, if $L_0 = L$, these conditions reduce to (1.4). That is we have Newton's method (1.2), and iteration (2.6) reduces to

$$(2.62) \quad t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0 t_{n+1})}.$$

In the case of Newton's method for $L_0 = L$, we have the well-known Kantorovich majorizing sequence four,five,thirteen,fifteen

$$(2.63) \quad \nu_0 = 0, \quad \nu_1 = \eta, \quad \nu_{n+2} = \nu_{n+1} + \frac{L(\nu_{n+1} - \nu_n)^2}{2(1 - L\nu_{n+1})}.$$

Note that if $L_0 < L$, $\{t_n\}$ is a tighter majorizing sequence than $\{\nu_n\}$ for the Newton's method five,thirteen,fifteen.

3. Numerical Examples

Let $\mathcal{X} = \mathcal{Y} = R^2$ be equipped with the max-norm, $x_0 = (1, 1)^T$, $\mathcal{D} = U(x_0, 1 - p)$, $p \in [0, 1/2)$ and define \mathcal{F} on \mathcal{D} by

$$(3.1) \quad \mathcal{F}(x) = \left(\xi_1^3 - p, \xi_2^3 - p \right)^T, \quad x = (\xi_1, \xi_2)^T.$$

Using (2.35)-(2.37), we get

$$(3.2) \quad \eta = \frac{1-p}{2}, \quad L_0 = 3 - p \quad \text{and} \quad L = 2(2 - p) > L_0.$$

The Newton-Kantorovich hypothesis (1.4) is violated, since

$$\frac{4}{3}(1-p)(2-p) > 1 \quad \text{for all } p \in [0, 1/2).$$

Hence, there is no guarantee that (TSNM) converges to $x^* = (\sqrt[3]{p}, \sqrt[3]{p})$. That is the results in rednine,thirteen,fifteen cannot apply to solve equation (3.1).

Using (2.1)-(2.5) and (TSNM) for $p = 0.49$, we get

$$\eta = 0.17, \quad L_0 = 2.51, \quad L = 3.02, \quad L_1 = 1.774552, \quad L_2 = 3.1626.$$

So, (2.1)-(2.3) become

$$0.2567 < 0.26 < 0.477455258,$$

$$0.326234049 < 0.33 < 0.407656274,$$

$$\eta \leq 0.196327344.$$

Moreover, we have

$$t^{**} = 0.319701493 < 1 - p = 1 - 0.49 = 0.51,$$

$$t^{**} \leq R < \frac{2}{L_0} - t^{**} = 0.47711256 < 1 - p = 0.51.$$

Hence, the conclusions of Theorem 2.2 apply and (TSNM) converges to

$$x^* = \left(\sqrt[3]{0.49}, \sqrt[3]{0.49} \right)^T = (0.788373516, 0.788373516)^T.$$

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