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Null controllability on Lie groups

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Abstract

We prove an extension of a classical result for null controllability of linear control systems on Euclidean spaces, to linear control systems on a connected Lie group G, assumed to be simply connected and nilpotent.

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1. Introduction

Controllability property of control systems is one of the most important problems in control theory and there is no general criterion yet to determine it even for a class of invariant systems on Lie groups. However, some partial results are obtained on specific state spaces. For example, Y. Sachkov has studied in ([11],[12]) controllability of invariant control systems on solvable Lie groups, V. Jurdjevic and H. Sussmann, [7], on compact Lie groups, V. Jurdjevic and I. Kupka, [8], on semi-simple Lie groups and their homogeneous spaces, L. San Martin and P. Crouch, [13], on principal fibre bundles with compact structure group, etc.

In particular, linear control systems on R^n are well-known since such class of control systems are important from both practical and theoretical point of views. In fact, there are many applications coming from some mechanical or physical problems where the state space is not the vector space R^n but a Lie group for instance.

A generalization of linear systems on Euclidean spaces to Lie groups has introduced by Ayala and Tirao in [1]. More precisely, let G denote a connected Lie group with Lie algebra g considered as the set of left invariant vector fields. A linear control system on G is defined as a pair

$$\Sigma = (G, \mathcal{D})$$

where the dynamic $\mathcal{D}$ is given by the family of differential equations

$$\dot{x} = X(x) + \sum_{j=1}^{m} u_j Y^j(x), x \in G$$

.

The system Σ has its drift vector field X as an element of the normalizer $n = \{Z \in \mathrm{Vect}^\infty(G) : [Z, g] \subset g\}$ of g in the set $\mathrm{Vect}^\infty(G)$ of smooth vector fields on G, and the control vectors Y^j, $j = 1, 2, \ldots, m$, all belong to the Lie algebra g.

The class of control systems with normalizer generalizes linear control systems on R^n, invariant control systems on Lie groups, etc. The purpose of this paper is to generalize to linear control systems on the class of simply connected and nilpotent Lie groups the classical null controllability result of linear systems on Euclidean spaces.

2. Linear Control Systems on Lie groups

Let G be a connected Lie group of dimension n with Lie algebra g. In [1] the authors introduced on G a generalization of the classical linear control systems on R^n as follows: By definition a linear control system or simply linear system is determined by the pair $\Sigma = (G, \mathcal{D})$ where G is the state space and the dynamic $\mathcal{D}$ is given by the family of differential equations

$$\dot{x} \;= X(x) + \sum_{j=1}^{m} u_j Y^j(x),\; x \in G.$$

The drift vector field X is an element of the normalizer

$$n = \{Z \in \mathrm{Vect}^{\infty}(G) : [Z, g] \subset g\}$$

of g and the control vectors Y^j, $j = 1, 2, \ldots, m$, belong to g. Here g is considered as the set of left invariant vector fields and $[\ ,]$ is the usual Lie bracket. The input functions $u = (u_1, u_2, \ldots, u_m)$ belong to the class $\mathcal{U}$ of unrestricted admissible control functions. The elements of $\mathcal{U}$ are piecewise constant functions of the form $u : [0, T(u)] \to R^m$ where $T(u) > 0$ means the terminal time of u. The set $\mathcal{U}$ of admissible controls is closed under concatenation, that is, if u and v belong to $\mathcal{U}$, then the function $w = u * v$ defined by

$$w(t) = \begin{cases} u(t), & t \in [0, T), \\ v(t - T), & t \in [T, \infty) \end{cases}$$

belongs as well to $\mathcal{U}$.

If $u \in \mathcal{U}$ the associated dynamic of Σ is given by the pair (Y^u, X) where $Y^u = \sum_{j=1}^{m} u_j Y^j \in g$. In [1] the authors prove that the normalizer n of g is isomorphic to the semidirect product $g \otimes_s aut(G)$ where $aut(G)$ is the Lie algebra of $\mathrm{Aut}(G)$, the Lie group of G-automorphisms. Moreover, they have proved for a pair (E, ϕ) in the normalizer $g \otimes_s aut(G)$ the following Theorem which gives an explicit formula for trajectories of Σ.

Theorem 2.1. *The integral curves of the associated differential equation determined by* (E, ϕ) *are given by* $x(t) = X_t(x)\,\exp\zeta(t)$, $x \in G$, *where* $\zeta(t)$ *is a differentiable curve through* 0 *at the Lie algebra level* g *defined by* $\zeta(t) = \Sigma_{n\,\geq 1}(-1)^{n+1} t^n d_n(E, \phi)$. *For each* $n \geq 1$, d_n *is a homogeneous polynomial map of degree* n *from* $g\times\, \partial g$ *into* g.

Additionally, if G is also simply connected they show that

$$n \cong g \otimes_s Der(g)$$

where $Der(g)$ stands for the Lie algebra of all derivations of g, that is, the set of linear transformations $D : g \to g$ satisfying $D([X, Y]) = [D(X), Y] + [X, D(Y)]$, for all $X, Y \in g$. It follows that when we consider a nilpotent and simply connected Lie group G as state space on which linear system Σ evolves it becomes relatively easier to determine explicitly its solutions. See Proposition 2.3 below.

Remark. In [10] it is given a list of nilpotent Lie groups G of low dimension whose underlying set is R^n. The origin is the identity element of G and the multiplication and inverse functions are polynomial maps. So, G is none other than R^n but with a non Abelian algebraic group structure. It follows that for such Lie groups we have a plenty of examples of linear systems which admit computationally tractable solutions.

Also, note that if $X = Y + 0 \in g$ then a linear system Σ is nothing else than an invariant system on G. However, there exists a remarkable difference between the well known class of invariant control systems on G and the linear one. Essentially, the difference depends on the drift vector field. In the first case the drift is invariant while in our case it is induced by a derivation. See Definition 2.2. Actually, in the well known paper by V. Jurdjevic and H. J. Sussmann in [7], the authors proved that the positive orbit of the identity element e of G is a semigroup. Hence, for a connected Lie group controllability on G is equivalent to the local controllability from e. But, in [5], the authors show an example of a linear control system on SL(2) where the positive orbit of e is not a semigroup.

On the other hand, if Σ satisfies the ad-rank condition then it is locally controllable at e. For further details see the paper [1]. Of course, Σ is a natural generalization of the class of linear control systems from Euclidean spaces to any arbitrary Lie group. For a complete review of 40 years of research on invariant control systems we refer to [11]. See also [12].

Definition 2.2. *A vector field X on a Lie group G is said to be an infinitesimal automorphism if the flow $(X_t)_{t\in R}$ induced by X is a* 1*-parameter subgroup of* $\mathrm{Aut}(G)$.

Proposition 2.3. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra g. Any $D \in Der(g)$ induces an infinitesimal automorphism $X = X^D$ as an element in the normalizer n of g. In this particular situation we can compute X explicitly.*

Proof. Consider the 1-parameter group of automorphisms

$$W_t = e^{tD} \in \mathrm{Aut}(g), \ t \in R.$$

By the hypothesis on G, the exponential map $\exp_G : g \to G$ is a global isomorphism from the Abelian group g onto G. For each $t \in R$,

$$X_t = \exp_G(W_t) \in \mathrm{Aut}(G).$$

It turns out that

$$X_t(\exp_G Y) = \exp_G((X_t)_*(Y))$$

where $(X_t)_*$ stands for the differential of X_t given, [14], by

$$(X_t)_* = \sum_{k\geq 0}^{\infty} \frac{t^k}{k!} D^k.$$

By taking derivatives, the flow $(X_t)_{t\in R}$ determines the vector field X in n defined by

$$X(g) = \left(\frac{d}{dt}\right)_{t=0} X_t(g),\ g \in G.$$

If $\exp Y = x \in G$ then through the inverse map of $\exp_G$ we obtain

$$X_t(x) = \exp_G(e^{tD} \log_G\ x)$$

and hence

$$X(x) = \left(\frac{d}{dt}\right)_{t=0} \exp(e^{tD} \log_G\ x).$$

∎

Remark. For the purposes of the paper, the drift vector field X of Σ will be always induced by an element D of $Der(g)$.

A particular class of such dynamics comes from inner automorphisms on G. More precisely, consider an element $W \in g$. Since W is complete its flow

$$W_t(z) = \exp_G(tW)z,\ z \in G$$

defines by conjugation a 1-parameter group of inner automorphisms on G as follows:

$$X_t(x) = W_t(e)\, x\, W_{-t}(e),\ x \in G.$$

Therefore $X_t \in \mathrm{Aut}(G)$ for any $t \in R$. In this case, the associated derivation $D : g \to g$ is defined by $D(Y) = \mathrm{ad}(X) = [X, Y]$ for any Y in g. However, this subclass of drift vector fields is far from determining $Der(g)$ since there could exist a big difference of cardinality of derivations and inner derivations.

3. Null controllability

In this section we prove a null controllability result for linear systems on nilpotent and simply connected Lie groups.

Let G denote a connected Lie group with the identity element e, and let Σ be a linear system on G. We denote by $x(g, u, t)$ the integral curve associated to the control u with initial condition $g \in G$. First we remember some definitions.

Definition 3.1. *A linear system Σ on G is said to be*

1. *Controllable if for any $g, h \in G$ there exists an admissible control $u \in \mathcal{U}$ and a time $t \geq 0$ such that $x(g, u, t) = h$.*
2. *Locally controllable at $e \in G$ if the reachable set*

$$\mathcal{A}(e, t) = \{x(e, u, t) : u \in \mathcal{U}\}$$

   of Σ in time t is a neighborhood of e.

3. *Σ is said to be null controllable if for each $g \in G$ there exist a control $u \in \mathcal{U}$ and $t > 0$ such that $x(g, u, t) = e \in G$.*

We denote by

$$\Im(t) = \{g \in G : \exists\ u \in \mathcal{U},\ x(g, u, t) = e\ \}$$

the set of controllable points to e at time t and by

$$\Im = \bigcup_{t>0} \Im(t)$$

the null controllable set of Σ. We use the shape of the Σ-solutions to give some topological properties of the set $\Im$ as follows:

Proposition 3.2. *Let Σ be a linear system on G. Then, (i) $\Im$ is path-connected and (ii) $\Im$ is an open set if and only if $e \in \operatorname{int}(\Im)$.*

Proof. To prove the path- connectedness of $\Im$ we have to show that given p and q in $\Im$ there exists a curve $\gamma : [0, 1] \to \Im$ such that $\Im(0) = p$ and $\Im(1) = q$. That $p \in \Im$ implies that $p \in \Im(t)$ for some $t > 0$ so that there exists a control $u \in \mathcal{U}$ whose terminal time $T(u)$ is t satisfying $x(0) = p$

and $x(p,u,t)=e$. Therefore, to any $p'\in G$ on the integral curve $x(t)$ there corresponds a control u' defined by

$$u'(s)=\begin{cases} u(t'+s) & \text{if } \ 0\le s\le t-t' \\ 0 & \text{if } \ t-t'\le s\le t \end{cases}$$

where $T(u')=t-t'$ and $x(t')=p'$. Since the drift vector field X is an infinitesimal automorphism of G it satisfies $X_t(e)=e$ for each $t\in R$ so that $p'\in\Im$. Let us denote by $U(p)$ the set

$$\{x(p,u,s):0\le s\le t\}\subset\Im(t).$$

It follows that the curve γ given by $U(p)\cup U(q)\subset\Im$ joins p to q.

To show the second assertion assume that $\Im$ is an open subset of G. Since $e\in\Im$ it follows that e is in the interior $\mathrm{int}(\Im)$. Suppose conversely that $e\in\mathrm{int}(\Im)$. Then, there exists an open set O_e of e contained in $\Im$. Let $g\in\Im$ be arbitrary and given $u\in\mathcal{U}$. Since the corresponding solution of Σ depends continuously on the initial state it follows by standard continuity arguments that for the curve connecting the initial state g to e there exists an open set O_g of g in $\Im$. This completes the proof. ∎

Definition 3.3. *Let $\Sigma=(G,\mathcal{D})$ be a linear system*

1. *Σ is said to be transitive if for any $g,h\in G$ there exist an admissible control $u\in\mathcal{U}$ and a time $t\in R$ such that $x(g,u,t)=h$.*
2. *We say Σ satisfies the ad-rank condition if $\dim(V)=\dim(G)$ where V denotes the vector space*

$$\mathrm{Span}\left\{\mathrm{ad}^i(X)(Y^j):i\ge 0,\ 1\le j\le m\right\}.$$

Let Σ be a transitive linear system on a connected Lie group G such that the drift X is an infinitesimal automorphism. According to Theorem 3.5 in [1], if Σ satisfies the ad-rank condition then it is locally controllable at e. Since $\mathrm{ad}(X)(Y)=D(Y)$ one can determine explicitly ad-rank sequence just by matrix multiplication with consecutive iterations.

Given a matrix $A\in gl(n,R)$ the spectrum of A is given by $\mathrm{Spec}(A)=\{\lambda:\lambda \text{ is an eigenvalue of } A\}$. We present the null controllability result as follows:

Theorem 3.4. *Assume that G is connected and simply connected nilpotent Lie group. Let Σ be a linear system on G such that the drift X is provided by a* **non-inner** *derivation D. If Σ satisfies ad-rank condition and* $\mathrm{Spec}(D) \subset C^-$ *then $\Im = G$, that is, Σ is null controllable.*

Proof. Consider the reverse system Σ^-, which is obtained by replacing X by $-X$. Since, we are assuming that the original system satisfies the ad-rank condition the same is true also for Σ^-. In particular, Theorem 3.5 in [1] implies that Σ^- is locally controllable at e. So, there exists a neighborhood U of e such that e can be reached by any state in U through an integral curve of Σ. On the other hand, since G is assumed to be simply connected and any eigenvalues of D has negative real part we obtain at the Lie algebra level

$$e^{tD} \cdot \log_G\ x \longrightarrow 0 \in g \quad \text{as } t \longrightarrow \infty$$

and hence

$$X_t(x) \longrightarrow e \in G$$

on the corresponding group. In particular, for each $x \in G$ there exists a T_x such that $X_{T_x}(x) \in U$. The proof is now complete. ∎

Remark. The assumption that X is associated to a **non-inner** derivation cannot be dropped, otherwise Theorem above fails to be true. This is because all the adjoint operators $\mathrm{ad}(Z), Z \in g$, of nilpotent Lie algebras have zero spectrum! We recall that a controllability (and hence, null controllability) result for linear systems on nilpotent Lie groups is recently obtained by P. Jouan in (Theorem 4, [9]) where the drift X is inner. Hence, although our theorem states a weaker result (null controllability does not imply controllability!) it gives an affirmative answer to the case when the infinitesimal automorphism X is not inner.

On the other hand, according to the Theorem above it would be very useful to have outer derivations of a given Lie algebra g to determine systems on the corresponding Lie group G which are null controllable. Of course, effective computation of derivations is not an easy task. That is why computer programs for calculation of Lie algebra characteristics such as automorphisms, ideals, derivations, etc are in use for a long time. However, such computer programs have reasonably limited applications since some of them do not compute with Lie algebras having pure real or complex non-rational structure constants while others have not been conceived to consider parameters. See the paper by Ayala-Kizil-Tribuzy in [6] where the structure tensor is considered to obtain conditions that a linear transformation $g \to g$ must satisfy in order to be a derivation of g.

Example. Let $G = R^3$ the connected and simply connected Heisenberg Lie group of dimension 3. In this case, the non-Abelian group operation $*$ is defined by

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) =: (x_1 + y_1 + x_3 y_2, x_2 + y_2, x_3 + y_3).$$

The Lie algebra g of G is generated by the vector fields

$$Y^1 = \frac{\partial}{\partial x_1} \qquad Y^2 = x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \qquad Y^3 = \frac{\partial}{\partial x_3}$$

and the only non-vanishing Lie bracket is $[Y^3, Y^2] = Y^1$. The exponential map is defined by

$$\exp(a_1 Y^1 + a_2 Y^2 + a_3 Y^3) = (a_1 + \frac{1}{2} a_2 a_3,\ a_2,\ a_3)$$

while the logarithm map is given by

$$\log(x_1, x_2, x_3) = (x_1 - \frac{1}{2} x_2 x_3) Y^1 + x_2 Y^2 + x_3 Y^3.$$

We consider the linear control system $\Sigma = (G, \mathcal{D})$ where the available dynamic comes from

$$\mathcal{D} = \left\{ X(x) + u Y^2(x) : u \in R \right\}.$$

Here X is the infinitesimal automorphism which gives rise to the derivation

$$D = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

In coordinates, the system Σ under consideration has the following form:

$$\begin{aligned} \dot{x_1} &= -3x_1 + x_3 + \frac{1}{2} x_2^2 + u x_3 \\ \dot{x_2} &= -x_2 + u \\ \dot{x_3} &= x_2 - 2x_3. \end{aligned}$$

Thus, $\mathcal{H} = \mathrm{Span}\{Y^2\}$ and since $\mathrm{ad}(X)(\mathcal{H}) = \mathrm{Span}\{Y^2, Y^3, Y^1\} = g$, Σ is locally controllable. It follows at once from Theorem 3.4 that $\mathrm{Spec}(D) \subset C^-$ implies that Σ is null controllable. ∎

3.1. Almost null controllability

Definition 3.5. *Let Σ be a control system on a Lie group G and U a neighborhood of $e \in G$. We say that Σ is U-null controllable if for each $g \in G$ there exists a control u such that the trajectory $x(g,u,\cdot)$ meets* $\mathrm{int}(U)$ *for some $t > 0$. If Σ is U-null controllable for every neighborhood U of e we say that Σ is almost null controllable.*

It is clear that controllability implies null controllability, which also implies almost null controllability. However, (any) converse is false. See the example below.

Example. Let g denote the Heisenberg Lie algebra generated by the vector fields $Y^1 = \frac{\partial}{\partial x_1}$, $Y^2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$ and $Y^3 = \frac{\partial}{\partial x_3}$ with non-trivial Lie bracket $[Y^1, Y^2] = Y^3$. Consider the system Σ given by

$$\dot{x} = X(x) + uY^1(x), \; x \in G$$

where X is defined by the derivation $D = (d_{ij}) \in Der(g)$ with $d_{11} = d_{22} = -1$, $d_{33} = -2$, $d_{21} = 1$ and 0 otherwise. It is quite obviously that $\mathrm{Spec}(D) \subset C^-$. Hence, Σ is almost null controllable. However it is not null controllable since we do not have ad-rank condition (even it is transitive!):

$$\begin{aligned} \mathrm{ad}(X)(Y^1) &= -Y^1 + Y^2 \\ \mathrm{ad}(X)(-Y^1 + Y^2) &= Y^1 - 2Y^2. \end{aligned}$$

Indeed, it is enough to find a state that cannot be steered to the identity in finite time. For this we need to calculate the trajectories. A straightforward calculation shows that

$$e^{tD} = \begin{pmatrix} e^{-t} & 0 & 0 \\ te^{-t} & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

Now, take $x = (0,0,1)$. It follows that

$$X_t(x) = \exp(e^{tD}\log(x)) = \exp(0,0,e^{-2t}) = (0,0,e^{-2t})$$

In particular, for $t = 1/2$ we find that the Σ-trajectory starting at x meets a neighborhood U of the origin but ends at $(0,0,0.367)$.

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