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An approximation formula for n!

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Abstract

We prove the following very accurate approximation formula for the factorial function:

$$n! \approx n^n e^{-n}\sqrt{2\pi\left(n+\frac{1}{6}+\frac{1}{72(n+\frac{31}{90})}-\frac{5929}{2332800(n+\frac{3055123}{11205810})^3}\right)}.$$

This gives better results than the following approximation formula

$$n! \approx \sqrt{2\pi}n^n e^{-n}\sqrt{n+\frac{1}{6}+\frac{1}{72n}-\frac{31}{6480n^2}-\frac{139}{155520n^3}+\frac{9871}{6531840n^4}},$$

which is established by the author [5] and C. Mortici [16] independently, and gives similar results with

$$n! \approx \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[8]{16n^4+\frac{32}{3}n^3+\frac{32}{9}n^2+\frac{176}{405}n-\frac{128}{1215}},$$

which is established by C. Mortici in his very new paper [8].

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1. Introduction

For a positive real number z the gamma function Γ and its logarithmic derivative ψ, so-called psi function or digamma function, are defined by

$$\Gamma(z)=\int_0^\infty u^{z-1}e^{-u}du,\ \ \psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}\ (z>0).$$

These two functions are considered to be the most important special functions next to the Riemann zeta function. As it is well known, the gamma function and factorials are related with the identity $\Gamma(n+1)=n!$ for $n\in\mathbf{N}$. The problem of approximating the gamma function, in particular, the factorial function has been attracted the attention of many mathematicians recently, and a lot of paper concerning this problem have been published; see for example [1-20]. The most well known approximation formula for $n!$ is the classical Stirling formula

$$n!\approx n^ne^{-n}\sqrt{2\pi n}.$$

Bauer [7] defined the sequence (δ_n) by the relation

$$n!=n^ne^{-n}\sqrt{2\pi(n+\delta_n)}$$

and numerical computations led him to infer that

$$\lim_{n\to\infty}\delta_n=0.166666...=1/6. \tag{1.1}$$

Consequently, he conjectured the approximation formula

$$n!\approx\sqrt{2\pi}n^ne^{-n}\sqrt{n+1/6}. \tag{1.2}$$

In [2] the author shoved the superiority of (1.2) over the Stirling formula. In [6] the author obtained the best possible scaler pairs (a,b) of real numbers such as the approximation $n!\approx n^ne^{-n-a}\sqrt{2\pi(n+b)}$ gives the best accurate values for $n!$. We note that (1.2) corresponds to $(a,b)=(0,1/6)$. The author [3] proved (1.1) and established the following inequalities, for $n\in\mathbf{N}$

$$\mathrm{n}^ne^{-n}\sqrt{2\pi\left(n+\frac{1}{6}\right)}<n!\le n^ne^{-n}\sqrt{2\pi\left(n+\frac{e^2}{2\pi}-1\right)}.$$

Very recently and independently, the author [5] and C. Mortici [16] obtained the following approximation formula for $n!$ to improve (1.2).

$$\mathrm{n}!\approx\sqrt{2\pi}n^ne^{-n}\sqrt{n+\frac{1}{6}+\frac{1}{72n}-\frac{31}{6480n^2}-\frac{139}{155520n^3}+\frac{9871}{6531840n^4}}.$$

As a first aim of this work we determine the best scalers α, β, a, b in such a way that the following formula offers the best approximation for $n!$.

$$n! \approx n^n e^{-n} \sqrt{2\pi \left(n + \frac{1}{6} + \frac{\alpha}{n+a} + \frac{\beta}{(n+b)^3} \right)}. \tag{1.3}$$

In particular, we show that the best approximation of this form is obtained for

$$\alpha = \frac{1}{72},\ \beta = \frac{5929}{2332800},\ a = \frac{31}{90},\ b = \frac{3055123}{11205810}.$$

Our second aim is to find the best scalers α, β such that the following inequalities hold true:

$$\alpha \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x+\frac{31}{90})}} \le \frac{\Gamma(x+1)}{x^x e^{-x}} < \beta \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x+\frac{31}{90})}}.$$

In order to prove our main results we need the following elementary but very useful lemmas. The first lemma is proved in [19]. The algebraic and numerical computations have been carried out with the computer software Mathematica 8.

Lemma 1.1. *If $(\omega_n)_{n\ge1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = c \in \mathbf{R},$$

with $k > 1$, then there exists the limit

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{c}{k-1}.$$

It is clear from this lemma that the speed of convergence of the sequence (ω_n) is as higher as the value of k is greater.

Lemma 1.2. *Let f be a function defined on an interval I and $\lim_{x\to\infty} f(x) = 0$. If $f(x+1)-f(x) > 0$ for all $x \in I$, then $f(x) < 0$. If $f(x+1)-f(x) < 0$, then $f(x) > 0$.*

Proof. Let $f(x+1)-f(x) > 0$ for all $x \in I$. By mathematical induction we have $f(x) < f(x+n)$ for all $n \in \mathbf{N}$. Letting $n \to \infty$, we have $f(x) < \lim_{n\to\infty} f(x+n) = 0$. The proof of second part of the lemma follows from the same argument. $\square$

2. Main results

The following two theorems are our main results.

Theorem 2.1. *For all $n \in \mathbf{N}$, the best approximation of the form (1.3) is obtained for*

$$\alpha = \frac{1}{72},\ \beta = \frac{5929}{2332800},\ a = \frac{31}{90},\ b = \frac{3055123}{11205810},$$

namely

$$n! \approx \quad n^n e^{-n} \sqrt{2\pi \left(n + \tfrac{1}{6} + \tfrac{1}{72(n+\frac{31}{90})} - \tfrac{5929}{2332800(n+\frac{3055123}{11205810})^3}\right)}$$

Proof. We define for $n \in \mathbf{N}$

$$T_n = \log n! - n \log n + n - \frac{1}{2}\log(2\pi)$$
$$- \frac{1}{2}\log\left(n + \frac{1}{6} + \frac{\alpha}{n+a} + \frac{\beta}{(n+b)^3}\right). \tag{2.1}$$

Successive differences of the sequence (T_n) are given by

$$T_n - T_{n+1} = n \log\left(1 + \frac{1}{n}\right) - 1$$
$$- \frac{1}{2}\log\left(\frac{n + \frac{1}{6} + \frac{\alpha}{n+a} + \frac{\beta}{(n+b)^3}}{n + \frac{7}{6} + \frac{\alpha}{n+1+a} + \frac{\beta}{(n+1+b)^3}}\right). \tag{2.2}$$

If we expand the right hand side of (2.2) as a power series of n^{-1} we get

$$\begin{aligned} T_n - T_{n+1} = {} & \tfrac{1}{72}(1 - 72\alpha)\tfrac{1}{n^3} + \tfrac{1}{540}\Big(- 17 + 945\alpha + 810a\alpha\Big)\tfrac{1}{n^4} \\ & + \tfrac{1}{12960}\Big(641 - 33120\alpha - 43200a\alpha - 25920a^2\alpha + 12960\alpha^2 \\ & -25920\beta\Big)\tfrac{1}{n^5} + \tfrac{1}{27216}\Big(- 1831 + 94815\alpha + 160650a\alpha \\ & +147420a^2\alpha + 68040a^3\alpha - 79380\alpha^2 - 68040a\alpha^2 + 147420\beta \\ & +204120b\beta\Big)\tfrac{1}{n^6} + \tfrac{1}{653184}\Big(55609 - 2981664\alpha - 6132672a\alpha \\ & -7402752a^2\alpha 5225472a^3\alpha - 1959552a^4\alpha + 4164048\alpha^2 \\ & +5552064a\alpha^2 + 2939328a^2\alpha^2 - 653184\alpha^3 - 7402752\beta \end{aligned}$$

$$-15676416b\beta - 11757312b^2\beta + 1959552\alpha\beta\Big)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \tag{2.3}$$

Faster convergences are are obtained by imposing the first six coefficients vanish. This results in

$$\alpha = \frac{1}{72},\ \beta = \frac{5929}{2332800},\ a = \frac{31}{90},\ b = \frac{3055123}{11205810}, \tag{2.4}$$

that is, the best approximation of the form (1.3) is

$$n! \approx n^n e^{-n}\sqrt{2\pi\left(n + \frac{1}{6} + \frac{1}{72(n+\frac{31}{90})} - \frac{5929}{2332800(n+\frac{3055123}{11205810})^3}\right)}.$$

□

If we put the values of α, β, a and b obtained in (2.4) in (2.3) we find

$$T_n - T_{n+1} = \frac{39977573013907}{10979183698560000}\frac{1}{n^7} + O\left(n^{-8}\right). \tag{2.5}$$

By Lemma 1 this proves that

$$\lim_{n\to\infty} n^6 T_n = \frac{39977573013907}{65875102191360000},$$

namely, the sequence (T_n) converges to zero like n^{-6}. Our second theorem provides new and elegant bounds for the gamma function.

Theorem 2.2. *Let x be a non-negative real number. Then we have*

$$\alpha \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x+\frac{31}{90})}} \le \frac{\Gamma(x+1)}{x^x e^{-x}} < \beta \cdot \sqrt{x + \frac{1}{6} + \frac{1}{72(x+\frac{31}{90})}} \tag{2.6}$$

where $\alpha = \frac{1452e}{1709} = 2.30951...$ and $\beta = \sqrt{2\pi} = 2.50663...$ are the best possible constants.

Proof. Let g be as following

$$g(x) = \log\Gamma(x+1) - x\log x + x - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(x + \frac{1}{6} + \frac{1}{72(x+\frac{31}{90})}\right).$$

Differentiation gives

$$\begin{aligned} g'(x) &= \psi(x+1) - \log x \\ &\quad - \frac{6(9833 + 129735x + 376650x^2 + 364500x^3)}{0541 + 438216x + 2310120x^2 + 5248800x^3 + 4374000x^4} \end{aligned}$$

and

$$g''(x) = \psi'(x+1) - \frac{1}{x} - \frac{a(x)}{b(x)},$$

where

$$\begin{aligned} a(x) &= -18(31+90x)^2(-2579729 + 7079760x + 46704600x^2 \\ &\quad +90396000x^3 + 65610000x^4) \end{aligned}$$

and

$$b(x) = (90541 + 438216x + 2310120x^2 + 5248800x^3 + 4374000x^4)^2.$$

Using the functional relation $\psi'(x+1) - \psi'(x) = -1/x^2$, we obtain

$$g''(x+1) - g''(x) = \frac{P(x)}{Q(x)},$$

where

$$\begin{aligned} -P(x) &= 243646234638147649 + 4312614519882734208x \\ &\quad +30540379834894285224x^2 + 113179004500484499840x^3 \\ &\quad +240030776577049466400x^4 + 299615216845238784000x^5 \\ &\quad +217070653175475840000x^6 + 84383307377280000000x^7 \\ &\quad +13611947136480000000x^8 \end{aligned}$$

and

$$\begin{aligned} &Q(x) = x(1+x)^2(31+90x)^2(121+90x)^2(77+552x+1080x^2)^2 \\ &\cdot(1709 + 2712x + 1080x^2)^2. \end{aligned}$$

Since both P and Q are positive in $(0,\infty)$, we have $g''(x+1) - g''(x) > 0$ for $x \geq 0$. Utilizing the relation $\lim_{x\to\infty} \psi'(x) = 0$, we see that $\lim_{x\to\infty} g''(x) = 0$. Consequently, by the help of Lemma 2 we prove that g' is strictly decreasing in $(0,\infty)$. It is well known that $\lim_{x\to\infty}(\psi(x) - \log x) = 0$; see, for example, [1], so that $\lim_{x\to\infty} g'(x) = 0$. This shows that g is strictly increasing in $(0,\infty)$. Also, by Stirling formula we get $\lim_{x\to\infty} g(x) = 0$. Consequently, we have

$$g(0) = \frac{1}{2}\log(372/77) - \frac{1}{2}\log(2\pi) < g(x) < \lim_{x\to\infty} g(x) = 0,$$

which is equivalent to (2.6). $\square$

3. Numerical computations

In this section we want to compare our estimation to some recent estimations due to the author and C. Mortici. First, we set

$$a_n = \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n + \frac{1}{6} + \frac{1}{72(n+\frac{31}{90})} - \frac{5929}{2332800(n+\frac{3055123}{11205810})^3}\right)}. \tag{3.1}$$

and

$$b_n = \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n + \frac{1}{6} + \frac{1}{72(n+\frac{31}{90})}}. \tag{3.2}$$

As shown in Table 1, the formula $n! \approx a_n$ gives better results than the following estimate

$$n! \approx \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n + \frac{1}{6} + \frac{1}{72n} - \frac{31}{6480n^2} - \frac{139}{155520n^3} + \frac{9871}{6531840n^4}} = \alpha_n, \tag{3.3}$$

which is established by the author [5] and C. Mortici [16] independently, and can be compared with the formula [5, 16]

$$n! \approx \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[8]{16n^4 + \frac{32}{3}n^3 + \frac{32}{9}n^2 + \frac{176}{405}n - \frac{128}{1215}} = \beta_n \tag{3.4}$$

which is established by C. Mortici in his very new paper [10].

n	$\lvert a_n - n!\rvert$	$\lvert\alpha_n - n!\rvert$	$\lvert\beta_n - n!\rvert$
1	0.0001023	0.0002085	0.0001633
2	0.0000078	0.00122768	0.0000189
10	0.00188	0.011978	0.000283
50	1.1399×10^{51}	3.0925×10^{52}	2.3968×10^{50}
100	6.0971×10^{142}	2.9144×10^{144}	1.2194×10^{142}

Table 3.1: A comparison between some terms of a_n, α_n, and β_n.

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