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Some new generalized I-convergent difference sequence spaces defined by a sequence of moduli

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Abstract

In this article we introduce the sequence space $c_0^I(F, p, \Delta_v^n)$ and $\ell_\infty^I(F, p, \Delta_v^n)$ for the of sequence of modulii $F = (f_k)$ and given some inclusion relations. These results here proved are analogus to those by M.Aiyub [1](Global Journal of Science Frontier Research Mathematics and Decision Sciences 12(9)(2012),32-36).

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1. Introduction and Preliminaries

Let ω, ℓ_∞, c_0 be the set of all sequences of complex numbers, the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|, \text{ where } k \in \mathbf{N} = \{1, 2, 3.....\}.$$

The idea of difference sequence spaces was introduced by H. Kizmaz [17]. In 1981 , Kizmaz defined the sequence spaces as follow;

$$\ell_\infty(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \ell_\infty\},$$

$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c\},$$

$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0\},$$

where

$$\Delta x = (x_k - x_{k+1}) \text{ and } \Delta^0 x = (x_k),$$

These are Banach space with the norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty.$$

Later Çolak and Et [4] defined the sequence spaces:

$$\ell_\infty(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in \ell_\infty\},$$

$$c(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c\},$$

$$c_0(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c_0\},$$

where $n \in \mathbf{N}, \Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}), \quad \Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$

and this generalized difference notion has the following binomial representation.

$\Delta^n x_k = \sum_{v=0}^{n}(-1)^v \, (nv) \, x_{k+v}$

and showed that these spaces are Banach space with the norm

$$\|x\|_\Delta = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty$$

Esi and Isik [7] defined sequence spaces:

$$\ell_\infty(\Delta_v^n, s, p) = \{x = (x_k) \in \omega : \sup_k k^{-s}|\Delta_v^n x_k|^{p_k} < \infty, s \geq 0\},$$

$$c(\Delta_v^n, s, p) = \{x = (x_k) \in \omega : k^{-s}|\Delta_v^n x_k - L|^{p_k} \to 0 \ (k \to \infty)\,, s \geq 0, \text{ for some L}\},$$

$$c_0(\Delta_v^n, s, p) = \{x = (x_k) \in \omega : k^{-s}|\Delta_v^n x_k|^{p_k} \to 0 \ (k \to \infty) \ , s \geq 0\}.$$

Where $v = (v_k)$ is any fixed sequence of non zero complex numbers $n \in \mathbf{N}$ is fixed number.

$$\Delta_v^0 x_k = (v_k x_k)\,, \Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1}) \text{ and } \Delta_v^n x_k = (\Delta_v^{n-1} x_k - \Delta_v^{n-1} x_{k+1}).$$

And this generalized difference notion has the following binomial representation.

$\Delta_v^n x_k = \sum_{i=0}^{n}(-1)^i \, (ni) \, v_{k+i} x_{k+i},$

when $s = 0,\ m = 1,\ v = (1, 1, 1, ...)$ and $p_k = 1$ for all $k \in \mathbf{N}$, they are just $\ell_\infty(\Delta),\ c(\Delta),\ \mathrm{c}_o(\Delta)$, defined by Kizmaz [17]. When $s = 0$ and $p_k = 1$ for all $k \in \mathbf{N}$, they are the following sequence spaces defined by Et and Esi [9]

$$\ell_\infty(\Delta_v^n) = \{x = (x_k) \in \omega : (\Delta_v^n x_k) \in \ell_\infty\},$$
$$c(\Delta_v^n) = \{x = (x_k) \in \omega : (\Delta_v^n x_k) \in c\},$$
$$c_0(\Delta_v^n) = \{x = (x_k) \in \omega : (\Delta_v^n x_k) \in c_0\}.$$

For more development about difference sequence spaces we refer to Bektas and Çolak [2],M.Et[8] and V.A.khan [14-16]

The idea of modulus was defined by Nakano [22] in 1953. A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(i) $f(t) = 0$ if and only if $t = 0$,

(ii) $f(t + u) \leq f(t) + f(u)$, for all $t, u \geq 0$,

(iii) f is increasing and

(iv) f is continuous from the right at 0.

Let X be a sequence spaces.Then Ruckle [25-27] defined the sequence space $X(f)$ for a modulus f as

$$X(f) = \{x = (x_k) \in \omega : (f(|x_k|)) \in X\},$$

Later Kolk [18,19] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$, that is

$$X(F) = \{x = (x_k) \in \omega : (f_k(|x_k|)) \in X\}.$$

Gaur and Mursaleen[13] defined the following sequence spaces:

$$\ell_\infty(F, \Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \ell_\infty(F)\},$$

$$c_0(F, \Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0(F)\}.$$

After then Ç.Bektas and R.Çolak [2]defined the following sequence spaces:

$$\ell_\infty(F, \Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in \ell_\infty(F)\},$$

$$c_0(F, \Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c_0(F)\}.$$

Recently Vakeel A Khan [14] defined the following sequence spaces:

$$X(F, p) = \{x = (x_k) \in \omega : (f(|x_k|)) \in X(p)\},$$

$$\ell_\infty(F, p) = \{x = (x_k) \in \omega : \sup_k f_k(|x_k|^{p_k}) < \infty\},$$

$$c_0(F, p) = \{x = (x_k) \in \omega : f_k(|x_k|^{p_k}) \to 0 \ (k \to \infty)\},$$

$$\ell_\infty(F, p, \Delta^n) = \{x = (x_k) \in \omega : \Delta^n x \in \ell_\infty(F, p)\},$$

$$c_0(F, p, \Delta^n) = \{x = (x_k) \in \omega : \Delta^n x \in c_0(F, p)\}.$$

For a sequence of moduli $F = (f_k)$ and gave the necessary and sufficient conditions for the inclusion relations between $X(\Delta^n)$ and $Y(F, \Delta^n)$, where $X, Y = \ell_\infty$ or c_0. Sequence of moduli have been studied by Ç.A.Bektas and R. Çolak[2] and many other authors.

The notion of statistical convergence was introduced by H.Fast[10]. Later on it was studied by J.A.Fridy [11,12] from the sequence space point view and linked with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. It was studied at initial stage by Kostyrko, Şalat and Wilezynski [20]. Later on it was studied by Şalat [29],Şalat, Tripathy and Ziman [30], Demirci[5]

Let $\mathbf{N}$ be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of $\mathbf{N}$) is said to be an ideal if I is additive i.e $(A, B) \in I \Rightarrow (A \cup B) \in I$ and i.e $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if $\Phi \notin \pounds(I)$ for $A, B \in \pounds(I)$ we have $(A \cap B) \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

An ideal $I \subseteq 2^N$ is called non trivial if $I \neq 2^N$. A non trivial ideal $I \subseteq 2^N$ is called admissible if $\{(x) : x \in N\} \subseteq I$. A non trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter $\pounds(I)$ corresponding to I, i.e $\pounds(I) = \{K \subseteq \mathbf{N} : K^c \in I\}$,where $K^c = N - K$.

Definition 1.1. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in \mathbf{N} : |x_k - L| \geq \epsilon\} \in I$.In this case we write $I - \lim x_k = L$.

Definition 1.2. A sequence $(x_k) \in \omega$ is said to be I-null if L=0. In this case we write $I - \lim x_k = 0$.

Definition 1.3. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$, there exist a number $m = m(\epsilon)$ such that $\{k \in \mathbf{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exist $M > 0$ such that $\{K \in \mathbf{N} : |x_k| \geq M\} \in I$.

We need the following Lemmas.

Lemma 1.5. The condition $\sup_k f_k(t) < \infty, t > 0$ hold if and only if there is a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$ (see [2,13]).

Lemma 1.6. The condition $\inf_k f_k(t) > 0$ hold if and only if there exist is a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$ (see [2,13]).

Lemma 1.7. Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \neq I$ then $M \cap K \neq I$ (see [29]).

Lemma 1.8. If $I \subseteq 2^N$ and $M \subseteq N$. If $M \neq I$ then $M \cap K \neq I$ (see [20]).

2. Main results

Let $F = (f_k)$ be a sequence of moduli, $v = (v_k)$ be any sequence such that $v_k \neq 0$ for all k and $p = (p_k)$ be sequence space of strictly positive real numbers then we define the following sequence spaces.

$$c_0^I(F, p, \Delta_v^n) = \{(x_k) \in \omega : I- \ \lim f_k(|\Delta_v^n x_k|) = 0\},$$

$$\ell_\infty^I(F, p, \Delta_v^n) = \{(x_k) \in \omega : I- \ \sup_k f_k(|\Delta_v^n x_k|) < \infty\}.$$

Theorem 2.1. For a sequence $F = (f_k)$ of moduli and for all $v = (v_k)$ and $p = (p_k)$ the following statements are equivalent:

(a) $\ell_\infty^I(\Delta_v^n) \subseteq \ \ell_\infty^I(F, p, \Delta_v^n)$,

(b) $c_0^I(\Delta_v^n) \subseteq \ c_0^I(F, p, \Delta_v^n)$,

(c) $\sup_k f_k(t) < \infty, (t > 0)$.

Proof. (a) implies (b) is obvious .

(b) implies (c). Let $c_0^I(\Delta_v^n) \subseteq c_0^I(F,p,\Delta_v^n)$. Suppose that (c) is not true. Then by Lemma (1.5)

$$\sup_k f_k(t) = \infty, \text{ for all } t > 0,$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(\frac{1}{i}) > i, \quad \text{for each} \quad i = 1, 2, 3...... \tag{1}$$

Define $x = (x_k)$ as follow

$$x_k = \begin{cases} \frac{1}{i}, \text{ if } k = k_i, i = 1, 2, 3.....; \\ 0, \quad \text{otherwise.} \end{cases}$$

Then $x \in c_0^I(\Delta_v^n)$ but by (1), $x \notin \ell_\infty^I(F,p,\Delta_v^n)$ for $v_k = p_k$ and $k \in \mathbf{N}$ which contradicts (b). Hence (c) must hold.
(c) implies (a). Let (c) be satisfied and $x \in \ell_\infty^I(\Delta_v^n)$. If we suppose that $x \notin \ell_\infty^I(F,p,\Delta_v^n)$.
Then

$$\sup_k f_k(|\Delta_v^n x_k|^{p_k}) = \infty \text{ for } \Delta_v^n x \in \ell_\infty^I.$$

If we take $t = |\Delta_v^n x_k|^{p_k}$. Then $\sup_k f_k(t) = \infty$ which contradicts (c). Hence $\ell_\infty^I(\Delta_v^n) \subseteq \ell_\infty^I(F,p,\Delta_v^n)$.

Theorem 2.2. For a sequence $F = (f_k)$ is a sequence of moduli and for all $v = (v_k)$ and $p = (p_k)$ the following statements are equivalent:

(a) $c_0^I(F,p,\Delta_v^n) \subseteq c_0^I(\Delta_v^n)$,

(b) $c_0^I(F,p,\Delta_v^n) \subseteq \ell_\infty^I(\Delta_v^n)$,

(c) $\inf_k f_k(t) > 0, (t > 0)$.

Proof. (a) implies (b) is obvious.

(b) implies (c). Let $c_0^I(F, p, \Delta_v^n) \subseteq \ell_\infty^I(\Delta_v^n)$. Suppose that (c) is not true. Then by Lemma (1.6)

$$\inf_k f_k(t) = 0, \quad (t > 0) \tag{2}$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \quad \text{for each} \quad i = 1, 2, 3......$$

Define $x = (x_k)$ as follow

$$x_k = \begin{cases} i^2, \text{ if } \ k = k_i \ \ i = 1, 2, 3.....; \\ 0 \quad \text{otherwise.} \end{cases}$$

By (2) $x \in c_0^I(F, p, \Delta_v^n)$ but $x \notin \ell_\infty^I(\Delta_v^n)$ for $v_k = (p_k)$ and $k \in \mathbf{N}$ which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) be satisfied and $x \in c_0^I(F, p, \Delta_v^n)$ that is

$$I - \lim_k f_k(|\Delta_v^n x_k|) = 0.$$

Suppose that $x \notin c_0^I(\Delta_v^n)$. Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $|\Delta_v^n x_k| < \epsilon_0$ for $k \geq k_0$. Therefore $f_k(\epsilon_0) \geq f_k(|\Delta_v^n x_k|)$ for $k \geq k_0$ and hence $\lim_k f_k(\epsilon_0) = 0$, which contradicts our assumption that $x \notin c_0^I(\Delta_v^n)$.

Thus $c_0^I(F, p, \Delta_v^n) \subseteq c_0^I(\Delta_v^n)$.

Theorem 2.3. The inclusion $\ell_\infty^I(F, p, \Delta_v^n) \subseteq c_0^I(\Delta_v^n)$ holds if and only if

$$\lim_k f_k(t) = \infty, \quad \text{for} \quad t > 0. \tag{3}$$

Proof. Let $\ell^I_\infty(F, p, \Delta^n_v) \subseteq \ c^I_0(\Delta^n_v)$ such that $\lim_k f_k(t) = \infty$ for, $t > 0$ doesn't hold. Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integer such that

$$f_{k_i}(t_0) \leq M < \infty. \tag{4}$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, \text{ if } \ k = k_i \ \ i = 1, 2, 3.....; \\ 0, \quad \text{otherwise.} \end{cases}$$

Thus $x \in \ell^I_\infty(F, p, \Delta^n_v)$ by (4). But $x \notin c^I_0(\Delta^n_v)$, for $v_k = p_k$ and $k \in \mathbf{N}$ so that (3) must hold.If
$\ell^I_\infty(F, p, \Delta^n_v) \subseteq \ c^I_0(\Delta^n_v)$. Conversely, let (3) hold. If $x \in \ell^I_\infty(F, p, \Delta^n_v)$,then $\lim_k f_k(|\Delta^n_v x_k|^{p_k}) \leq \lim_k M < \infty$, for $k = 1, 2, 3$.......Suppose that $x \notin c^I_0(\Delta^n_v)$. Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $|\Delta^n_v x_k| < \epsilon_0$ for $k \geq k_0$. Therefore $f_k(\epsilon_0) \leq f_k(|\Delta^n_v x_k|^{p_k}) \leq M$ for $k \geq k_0$, which contradicts (3).Hence $x \in c^I_0(\Delta^n_v)$.

Theorem 2.4. The inclusion $\ell^I_\infty(\Delta^n_v) \subseteq \ c^I_0(F, p, \Delta^n_v)$ holds if and only if

$$\lim_k f_k(t) = 0, \quad \text{for } t > 0. \tag{5}$$

Proof. Suppose that $\ell^I_\infty(\Delta^n_v) \subseteq \ c^I_0(F, p, \Delta^n_v)$ but (5) doesn't hold,

Then

$$\lim_k f_k(t_0) = l \neq 0, \quad \text{for some } t_0 > 0 \qquad (6).$$

Define the sequence $x = (x_k)$ by

$\mathrm{x}_k = t_0 \sum_{v=0}^{k-n} (-1)^n \, (\, n \, + k - v - 1k - v)$
for $k = 1, 2, 3, ...$ Then $x \notin c_0^I(F, p, \Delta_v^n)$ by (6)for $v_k = p_k$ and $k \in \mathbf{N}$. Hence (5) must hold.
Conversely, let $x \in \ell_\infty^I(\Delta_v^n)$ and suppose that (5) holds.Then $|\Delta_v^n x_k| \leq M < \infty$ for $k = 1, 2, 3....$ There for $f_k(|\Delta_v^n x_k|) \leq \ f_k(M)$ for $k = 1, 2, 3.....$ and $\lim_k f_k(|\Delta_v^n x_k|) \leq \lim_k f_k(M) = 0$ by (5). Hence $x \in c_0^I(F, p, \Delta_v^n)$

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