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The Nemytskii operator on bounded ϕ-variation in the mean spaces

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Abstract

We introduce the notion of bounded Φ-variation in the sense of L_Φ-norm. We obtain a Riesz type result for functions of bounded Φ-variation in the mean. We also show that if the Nemytskii operator act on the bounded Φ-variation in the mean spaces into itself and satisfy some Lipschitz condition there exist two functions g and h belonging to the bounded Φ-variation in the mean space such that

$$f(t,y) = g(t)y + h(t),\ t \in [0, 2\pi],$$

$y \in \mathbf{R}$.

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1. Introduction

Two centuries ago, around 1880 C. Jordan (See [3]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones; since then a number of authors such as, Yu Medvedv (see [8]), N. Merentes (see [5,6,7], L. Maligranda and W. Orlicz (see[4]), D. Waterman (see[13]), M Schramm (see[12]) and recently R. Castillo (see[1], R. Castillo and Trousselot (see [2]) had been study different spaces with same type of variation. The circle group T is defined as the quotient $\mathbf{R}/2\pi\mathbf{Z}$, where, as indicated by notation, $2\pi\mathbf{Z}$ is the group pf integral multiples of 2π. There is a natural identification between functions on T and 2π-periodic functions on $\mathbf{R}$, which allows an implicit introduction on notions such as continuity, differentiability, etc. for functions on T.

The Lebesgue measure on T also can be defined by means of the preceding identification: a function f is integrable on T if the corresponding 2π-periodic funtion, which we denote again by f, integrable on $[0, 2\pi]$, and we set

$$\int_T f(t)dt = \int_0^{2\pi} f(x)dx.$$

Let f be a real-value function in $L_p(1 < p < \infty)$ on the circle group T. We define the corresponding interval function by f(I) = f(b) - f(a), where I denotes the interval $[a, b]$. Let $0 = t_0 < t_1 < \ldots < t_n = 2\pi$ be a partition of $[0, 2\pi]$ and $I_{kx} = [x + t_{k-1}, x + t_k]$, if

$$V_p^m(f, T) = \sup\{\sum_{k=1}^{n} \int_T \frac{|f(I_{kx})|^p}{|t_k - t_{k-1}|^{p-1}} dx\} < \infty$$

where the supremum is taken over all partition of $[0, 2\pi]$, then f is said to be of p-variation in the mean. We denote the class of all function which are of p-bounded variation in the mean by BV_pM. This concept was introduced by operator act on BV_pM into itself. BV_pM equipped with the norm

$$\|f\|_{BV_pM} = \|f\|_{Lp} + \{V_p^m(f, T)\}^{1/p}$$

is a Banach space (see Theorem 2.8 in [1]). The first author in [1] introduced the above concept. As a matter of fact the latter concept is a generalization of the concept introduced by Mricz and Siddiqi who investigated the convergence in the mean of the partial sums of $S[f]$, the Furier series of f (see[9]).

In 1910 in [11], F. Riesz defined the concept of bounded p-variation $(1 \leq p < \infty)$ and proved that for $1 < p < \infty$ this class coincides with the class of functions f, absolutely continuous with derivative $f' \in L_p[a,b]$. Moreover, the p-variation of a function f on $[a,b]$ is given by $\|f'\|_{Lp[a,b]}$ that is

$$V_p(f;[a,b]) = \|f'\|_{Lp[a,b]} \tag{1.1}$$

For this class we also obtained the following analogous result to (1.1) that is if $f \in BV_pM$ is such that f' is continuous on $[0,2\pi]$ them $f' \in L_p[0,2\pi]$ and

$$V_p^{(m)}(f) = 2\pi\|f'\|Lp \tag{1.2}$$

In this paper we introduced the concept of bounded Φ-variation in the mean, which generalized the above concept.

In this paper we obtain an analogous result as in (1.2) for the class $BV_\Phi M$. More precisely we show that if $f \in BV_\Phi M$ is such that f' is continuous on $[0,2\pi]$, then $f' \in L\Phi[0,2\pi]$ and

$$V_\Phi^m(f) = 2\pi \int_0^{2\pi} \Phi(f'(x))dx.$$

(See Theorem 3.3).

2. Bounded $\Phi-$variation in the mean

In this section, we gather definitions and notations that will be used throughout the paper.

Definition : A function $\Phi : [0,\infty) \to [0,\infty)$ which satisfies the following statements:

1. Φ is continuous.
2. Φ is strictly increasing.
3. $\Phi(t) = 0$ if and only if $t = 0$.
4. $\lim_{t\to\infty} \Phi(t) = +\infty$.

is said to be a $\Phi-$function.

Let us remaind the following, a function $f \in L_\Phi([a,b])$ if:

$$\int_a^b \Phi(f(x))dx < \infty.$$

Now, we are ready for the following:

Definition : Let $f \in L_\Phi([0,2\pi])$ where is a $\Phi-$function and $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0,2\pi]$ if

$$V_\Phi^m(f,\Gamma) = V_\Phi^m(f) = \sup \sum_{k=1}^{n} \int_\Gamma \Phi\left(\frac{|f(x+t_k) - f(x+t_{k-1})|}{|t_k - tk - 1|}\right) |t_k - t_{k-1}| dx,$$

where the supremum is taken over all partitions P of $[0,2\pi]$ the f is said to be a of bounded $\Phi-$variation in the mean. We denote the class of all functions which are of bounded $\Phi-$variation in the mean by $BV_\Phi M$, that is

$$BV_\Phi M = \left\{ f \in L_\Phi([0,2\pi]) : V_\Phi^m(f) < \infty \right\}.$$

Remark : If we choose $\Phi(t) = t^p$ with $1 < p < \infty$ we get back Definition 2.1 in [1].

Next, let us see $V_\Phi^m(\cdot)$ as a functional defined on $BV_\Phi M$ e.g.

$$V_\Phi^m : BV_\Phi M \to [0,+\infty) \; f \mapsto V_\Phi^m(f).$$

In the coming theorem we gather some properties of $V_\Phi^m(\cdot)$.

Theorem : Let Φ be a $\Phi-$function

1. $V_\Phi^m(-f) = V_\Phi^m(f)$ for all $f \in BV_\Phi M$.
2. $V_\Phi^m(\cdot)$ is a convex function if and only if Φ is convex.
3. If f is a constant function, then $V_\Phi^m(f) = 0$.
4. f is a $2\pi-$periodic function if and only if $V_\Phi^m(f) = 0$.
5. If Φ is convex and $0 \le \lambda \le 1$, then $V_\Phi^m(\lambda f) \le \lambda V_\Phi^m(f)$.

Proof :

1. is just a straightforward application of the definition.
2. Assume Φ convex, let $f, g \in BV_\Phi M$ and $\lambda, \mu \in [0,1]$ such that $\lambda + \mu = 1$. Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0,2\pi]$. Then since Φ is an increasing and convex function, we have

$$\sum_{k=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|(\lambda f + \mu g)(x+t_k) - (\lambda f + t_k)|}{|t_k - t_{k-1}|}\right) dx$$

$$\leq \sum_{k=1}^{n}\int_0^{2\pi}\Phi\Big(\lambda\frac{|f(x+t_k)-f(x+t_{k-1})|}{|t_k-t_{k-1}|}$$
$$+\mu\frac{|g(x+t_k)-g(x-t_{k-1})|}{|t_k-t_{k-1}|}|t_k-t_{k-1}|dx$$
$$\leq \lambda\sum_{k=1}^{n}\int_0^{2\pi}\frac{|f(x+t_k)-f(x+t_{k-1})|}{|t_k-t_{k-1}|}|t_k-t_{k-1}|dx$$
$$+\mu\sum_{k=1}^{n}\int_0^{2\pi}\frac{|g(x+t_k)-g(x+t_{k-1})|}{|t_k-t_{k-1}|}|t_k-t_{k-1}|dx$$
$$\leq \lambda V_\Phi^m(f)+\mu V_\Phi^m(g).$$

Finally

$$V_\Phi^m(\lambda f+\mu g)\leq \lambda V_\Phi^m(f)+\mu V_\Phi^m(g).$$

Which means that:
If $f, g \in BV_\Phi M$ then $\lambda f+\mu g \in BV_\Phi M$ with $\lambda+\mu=1$.

Conversely, assume $V_\Phi^m(\cdot)$ is a convex function, then let us take r, s in $[0,\infty)$ and define f(x)=rx ; x∈ $[0, 2\pi]$,
$g(x)=sx\,;\, x\in[0,2\pi]$.

Let λ, $\mu\in[0,1]$ with $\lambda+\mu=1$ and $P: 0=t_0<t_1<\cdots<t_n=2\pi$ be a partition of $[0,2\pi]$, then $\sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{|f(x+t_k)-f(x+t_{k-1})|}{|t_k-t_{k-1}|}\right)|t_k-t_{k-1}|dx$
$=\sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{r|t_k-t_{k-1}}{|t_k-t_{k-1}|}\right)|t_k-t_{k-1}|dx$
$=4\pi^2\Phi(r)<\infty,$ note that this holds for any partition of $[0,2\pi]$. Thus,

$$V_\Phi^m(f)=4\pi^2\Phi(v)<\infty,$$

hence $f\in BV_\Phi M$.

In a similar way we have

$$V_\Phi^m(g)=4\pi^2\Phi(s)<\infty \text{ and } g\in BV_\Phi M,$$

and also

$$V_\Phi^m(\lambda f+\mu g)=4\pi^2\Phi(\lambda r+\mu s)<\infty.$$

By hypothesis

$$V_\Phi^m(\lambda f+\mu g)\leq \lambda V_\Phi^m(f)+\mu V_\Phi^m(g).$$

Hence $4\pi^2\Phi(\lambda r+\mu s)\leq 4\pi^2[\lambda\Phi(r)+\mu\Phi(s)]$

$\Phi(\lambda r+\mu s)\leq \lambda\Phi(r)+\mu\Phi(s)$.

So then Φ is a convex function.

3. If f is a constant function on $[0,2\pi]$, then $V_\Phi^m(f)=0$ since $\Phi(0)=0$.

4. Let f be a $2\pi-$periodic function and $P:0=t_0<t_1<\cdots<t_n=2\pi$ be a partition of $[0,2\pi]$, then an easy computation gives us the result.

   Now, assume $V_\Phi^m(f)=0$ for the same partition as above, then after some easy calculations we have

$$2\pi\int_0^{2\pi}\Phi\left(\frac{|f(x+2\pi)-f(x)|}{2\pi}\right)dx=0,$$

   thus

$$\Phi\left(\frac{|f(x+2\pi)-f(x)|}{2\pi}\right)=0,$$

   by Definition1(c) we obtain

$$|f(x+2\pi)-f(x)|=0.$$

   Therefore $f(x+2\pi)=f(x)$.

5. By (ii) and (iii) we get $\mathrm{V}_\Phi^m(\lambda f)=V_\Phi^m(\lambda f+(1-\lambda)\cdot 0)$
   $\leq\lambda V_\Phi^m(f)+(1-\lambda)V_\Phi^m(0)$
   $V_\Phi^m(\lambda f)\leq\lambda V_\Phi^m(f)$.

□

Theorem : Let Φ be a convex function and $f\in BV_\Phi M$. Then

1. If $0<k<k_1$, then $V_\Phi^m(kf)\leq V_\Phi^m(k_1f)$.
2. $\lim_{\beta\to 0}V_\Phi^m(\beta f)=0$.
3. $\{\varepsilon>0:V_\Phi^m(f/\varepsilon)\leq 1\}\neq\emptyset$.

Proof :

1. Let $0<k<k_1$ and $P:0=t_0<t_1<\cdots<t_n=2\pi$ be a partition of $[0,2\pi]$, then

$$|kf(x+t_j)-kf(x+t_{j-1})|\leq|k_1f(x+t_j)-kf(x+t_{j-1})|,$$

since Φ is an increasing function, we have
$\sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{|kf(x+t_j)-kf(x+t_{j-1})|}{|t_j-t_{j-1}|}\right)|t_j-t_{j-1}|dx$
$\leq \sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{|k_1f(x+t_j)-k_1f(x+t_{j-1})|}{|t_j-t_{j-1}|}\right)|t_j-t_{j-1}|dx$ for any partition P of $[0,2\pi]$. Hence

$$V_\Phi^m(kf)\leq V_\Phi^m(k_1f).$$

2. Let $f\in BV_\Phi M$ note that for $\lambda>0$ then $\lambda f\in BV_\Phi M$, if $0<\beta\leq\lambda(\frac{\beta}{\lambda}\leq 1)$.

   By Theorem1(v) we have

$$V_\Phi^m(\beta f)=V_\Phi^m\left(\frac{\beta\lambda}{\lambda}f\right)\leq\frac{\beta}{\lambda}V_\Phi^m(\lambda f)<+\infty.$$

   From the later inequality we obtain

$$0\leq\lim_{\beta\to 0}V_\Phi^m(\beta f)\leq\lim_{\beta\to 0}\frac{\beta}{\lambda}V_\Phi^m(\lambda f)=0$$

   and the result follows.

3. In view of part (ii) we could see that there exist an $\varepsilon>0$ such that $V_\Phi^m(f/\varepsilon)\leq 1$, that is

$$\{\varepsilon>0:V_\Phi^m(f/\varepsilon)\leq 1\}\neq\emptyset.$$

Remark : This latter result allow us to take for granted that infimum of $\{\varepsilon>0:V_\Phi^m\left(\frac{f}{\varepsilon}\right)\leq 1\}$ exists, since this non empty set is bounded below by 0.
Definition : Let Φ be a convex function. Then

$$BV_\Phi^m M=\{f:[0,2\pi]\to\mathbf{R}:f\in BV_\Phi M \text{ and } f(0)=0\}$$

is the linear space of bounded $\Phi-$variation in the mean functions which are nulls at zero.

Let us denote
$|\cdot|_\Phi^m :, :\ BV_\Phi^0 M\to\mathbf{R}^+$
$f\mapsto|f|_\Phi=\inf\{\varepsilon>0:V_\Phi^m(f/\varepsilon)\leq 1\}.$

According to Remark 2 this infimum exists. We will now show that $|\cdot|_\Phi^m$ is a norm on $BV_\Phi^0 M$. In order to do that we will need a previous lemma.
Lemma : Let Φ be a convex function and $f\in BV_\Phi^0 M$. Then:

1. $|f|_\Phi^m \neq 0$ implies $V_\Phi^m(\frac{f}{|f|_\Phi^m}) \leq 1$.
2. $|f|_\Phi^m < k$ if and only if $V_\Phi^m(\frac{f}{k}) \leq 1$ $k > 0$.
3. $0 \leq |f|_\Phi^m \leq 1$ then $V_\Phi^m(f) \leq |f|_\Phi^m$.
4. $\{\varepsilon > 0 : V_\Phi^m(\frac{f}{\varepsilon}) \leq 1\} = (|f|_\Phi^m, +\infty)$.

Proof :

1. Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$ and $k > |f|_\Phi^m$. Then

$$\sum_{j=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|f(x+t_j) - f(x+t_{j-1})|}{k|t_j - t_{j-1}|}\right) |t_j - t_{j-1}| \, dx \leq V_\Phi^m(\frac{f}{k}) \leq 1$$

and $\sum_{j=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|f(x+t_j)-f(x+t_{j-1})|}{|f|_\Phi^m |t_j - t_{j-1}|}\right) |t_j - t_{j-1}| \, dx$
$= \lim_{k \to |f|_\Phi m} \sum_{j=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|f(x+t_j)-f(x+t_{j-1})|}{k|t_j-t_{j-1}|}\right) |t_j - t_{j-1}| \, dx \leq 1$ where $V_\Phi^m(\frac{f}{|f|_\Phi^m}) \leq 1$.

2. Let $|f|_\Phi^m < k$.

   1. If $|f|_\Phi^m = 0$, then there exists k' such that $0 < k' < k$ and $V_\Phi^m(\frac{f}{k'}) \leq 1$, since $\frac{1}{k} < \frac{1}{k'}$, by Theorem2 (i) we have:

   $$V_\Phi^m(f/k) \leq V_\Phi^m(f/k') \leq 1.$$

   2. If $0 \leq |f|_\Phi^m < k$, then $\frac{1}{k} < \frac{1}{|f|_\Phi^m}$, again using Theorem2(i) we obtain

   $$V_\Phi^m(\frac{f}{k}) \leq V_\Phi^m(\frac{f}{|f|_\Phi^m}) \leq 1.$$

   Conversely, if $V_\Phi^m(\frac{f}{k}) \leq 1$ then $\{\varepsilon > 0 : V_\Phi^m(\frac{f}{\varepsilon}) \leq 1\}$ implies $k > |f|_\Phi^m$.

3. If $|f|_\Phi^m = 0$, then by part (ii)(*) for $k > 0$, we have $V_\Phi^m\left(\frac{f}{k}\right) \leq 1$, that is $k \in \{\varepsilon > 0 : V_\Phi^m\left(\frac{f}{\varepsilon}\right) \leq 1\}$.
Let $0 < k < 1$, we invoke Theorem 1(v) to obtain

$$V_\Phi^m(f) = V_\Phi^m\left(k\frac{f}{k}\right) \leq kV_\Phi^m\left(\frac{f}{k}\right) \leq k.$$

Hence $V_\Phi^m(f)$ is the lower bound of the set $\{\varepsilon > 0 : V_\Phi^m(k\frac{f}{\varepsilon}) \leq 1\}$ and therefore $V_\Phi^m(f) \leq |f|_\Phi^m$.
If $k > 1$ such that $k \in \{\varepsilon > 0 : V_\Phi^m(\frac{f}{\varepsilon}) \leq 1\}$ then there exists k' such that $0 < k' < 1 < k$ and this $V_\Phi^m(f)$ is a lower bound of the set $\{\varepsilon > 0 : V_\Phi^m(\frac{f}{\varepsilon}) \leq 1\}$; then $V_\Phi^m(f) \leq |f|_\Phi^m$.
If $0 < |f|_\Phi^m \leq 1$ by Theorem 1(v)

$$V_\Phi^m(f) = V_\Phi^m\left(|f|_\Phi^m \frac{f}{|f|_\Phi^m}\right) \leq |f|_\Phi^m V_\Phi^m\left(\frac{f}{|f|_\Phi^m}\right)$$

also, by part (i) we have

$$\frac{1}{|f|_\Phi^m} V_\Phi^m(f) \leq V_\Phi^m\left(\frac{f}{|f|_\Phi^m}\right) \leq 1,$$

from this last inequality we obtain

$$V_\Phi^m(f) \leq |f|_\Phi^m.$$

4. k$\in \{\varepsilon > 0 : V_\Phi^m(f/\varepsilon) \leq 1\} \Leftrightarrow V_\Phi^m(\frac{f}{k}) \leq 1$
$\Leftrightarrow |f|_\Phi^m < k$ by (ii)
$\Leftrightarrow k \in (|f|_\Phi^m, +\infty)$.

We are in a good position now to show the following.
Theorem : Let Φ be a convex function, then $|\cdot|_\Phi^m$ is a norm on $BV_\Phi^m M$.
Proof : We are going just to check the triangle inequality property. Indeed, let $f, g \in BV_\Phi^0 M$. If $f = 0$ or $g = 0$, then $|f+g|_\Phi^m = |f|_\Phi^m + |g|_\Phi^m$ holds trivially.

Now, let us consider the case when $f \neq 0$ and $g \neq 0$. Thus

$$\begin{aligned} & V_\Phi^m\left(\frac{f+g}{|f|_\Phi^m + |g|_\Phi^m}\right) \\ &= V_\Phi^m\left(\frac{|f|_\Phi^m}{|f|_\Phi^m + |g|_\Phi^m} \cdot \frac{f}{|f|_\Phi^m} + \frac{|g|_\Phi^m}{|f|_\Phi^m + |g|_\Phi^m} \cdot \frac{g}{|g|_\Phi^m}\right) \\ &\leq \frac{|f|_\Phi^m}{|f|_\Phi^m + |g|_\Phi^m} V_\Phi^m\left(\frac{f}{|f|_\Phi^m}\right) + \frac{|g|_\Phi^m}{|f|_\Phi^m + |g|_\Phi^m} V_\Phi^m\left(\frac{g}{|g|_\Phi^m}\right) \\ &\leq \frac{|f|_\Phi^m}{|f|_\Phi^m + |g|_\Phi^m} + \frac{|g|_\Phi^m}{|f|_\Phi^m + |g|_\Phi^m} = 1. \end{aligned}$$

Hence, by Lemma1(ii) we have

$$|f+g|_\Phi^m \leq |f|_\Phi^m + |g|_\Phi^m.$$

Our next goal is to systematically define a norm on $BV_\Phi M$ spaces. The proof of the following Lemma is just a straightforward application of the definition.

Lemma : Let Φ be a Φ function, then $f \in BV_\Phi M$ if and only if $f - f(0) \in BV_\Phi^0 M$.

Now we are ready to announce the following:

Definition : Let Φ be a convex $\Phi-$function and

$$\|\cdot\|_\Phi^m : BV_\Phi M \to \mathbf{R}^+$$
$$f \mapsto \|f\|_\Phi^m = |f(0)| + |f - f(0)|_\Phi^m$$
$$= |f(0)| + \inf\{\varepsilon > 0 : V_\Phi^m\left(\frac{f - f(0)}{\varepsilon}\right) \leq 1\}.$$

Since $f - f(0) \in BV_\Phi^0 M$, Lemma2 and Definition3 implies

$$\|f\|_\Phi^m = |f(0)| + \inf\{\varepsilon : V_\Phi^m\left(\frac{f}{\varepsilon}\right) \leq 1\}.$$

Now, is just routine to check that $\|\cdot\|_\Phi^m$ define a norm on $BV_\Phi M$ spaces.

Conclusion : If Φ is a convex function, the

1. $(\mathbf{R}, BV_\Phi^0, +, |\cdot|_\Phi^m)$ is a normed vector spaces.
2. $(\mathbf{R}, BV_\Phi, +, |\cdot|_\Phi^m)$ is a normed vector spaces.

Theorem : $Lip[0,2\pi] \subset BV_\Phi M$, where $Lip[0,2\pi]$ denotes the class of all function which are Lipschitz on $[0,2\pi]$.

Proof : Let $f \in Lip[0,2\pi]$, then there exists a positive constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x-y|$$

for all $x, y \in [0,2\pi]$ Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0,2\pi]$, thus

$$|f(x+t_k) - f(x+t_{k-1})| \leq M|t_k - t_{k-1}|,$$

then

$$\sum_{k=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|f(x+t_k) - f(x+t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx \leq 4\pi\Phi(M).$$

This last inequality means that $f \in BV_\Phi M$.

$BV_\Phi M$ is a Banach spaces.

In order to prove that $BV_\Phi M$ is a Banach space we will need two lemmas.

Lemma : Let Φ be a convex function defines on $[0,\infty]$ with $\Phi(0)=0$. Then the function $\Psi : (0,\infty) \to \mathbf{R}$
$x \mapsto \Psi(x) = \frac{\Phi(x)}{x}$ is increasing on $(0,\infty)$.

We omitted the proof of Lemma3 because is just a routine calculations.

In the proof of the coming Lemma we do not use (∞_1) condition (see Definition 5) as was used in [4], [7], [10] Let Φ be a $\Phi-$function which is convex. If $f \in BV_\Phi^0 M$, then

$$\|f\|_{L_1[0,2\pi]} \le M|f|_\Phi^m$$

with

$$M = \max\{\frac{1}{2\pi\Phi(\frac{1}{2\pi})}, 2\pi\Phi^1\left(\frac{1}{2\pi}\right)\}.$$

Proof : If $|f|_\Phi^m = 0$, there is nothing to prove.
Next, let us consider the case $|f|_\Phi^m \neq 0$ and thus we define the following set

$$E = \left\{t \in [0,2\pi] : \left|\frac{f(x+t)}{|f|_\Phi^m}\right| \frac{t}{2\pi}\right\}.$$

If $t \in E$, then

$$\frac{1}{2\pi} \le \frac{\left|\frac{f(x+t)}{|f|_\Phi^m}\right|}{t},$$

by Lemma 3 we have

$$\frac{\Phi\left(\frac{1}{2\pi}\right)}{\frac{1}{2\pi}} \le \frac{\Phi\left(\frac{\frac{|f(x+t)|}{|f|_\Phi^m}}{t}\right)}{\frac{\left|\frac{f(x+t)}{|f|_\Phi^m}\right|}{t}},$$

Since $f(0)=0$, from this we have

$$2\pi\left|\frac{f(x+t)}{|f|_\Phi^m}\right|\Phi\left(\frac{1}{2\pi}\right) \le \Phi\left(\frac{|f(x+t)-f(0)|}{\frac{|f|_\Phi^m}{|t-0|}}\right)|t-0|,$$

then

$$\frac{2\pi\Phi(\frac{1}{2\pi})}{|f|_{\Phi}^{m}}\int_0^{2\pi}|f(x+t)|dx \le \int_0^{2}\pi\Phi\left(\frac{|f(x+t)-f(0)|}{|f|_{\Phi}^{m}|t-0|}\right)|t-0|dx.$$

Now, for the partition $0<t<2\pi$ of $[0,2\pi]$ and from the fact that the Lebesque measure is invariant translation we have

$$\frac{2\pi\Phi(\frac{1}{2\pi})}{|f|_{\Phi}^{m}}\int_0^{2\pi}|f(x)|dx \le V_{\Phi}^{m}\left(\frac{f}{|f|_{\Phi}^{m}}\right)\le 1.$$

Thus

$$\int_0^{2\pi}|f(x)|dx \le \frac{|f|_{\Phi}^{m}}{2\pi\Phi(\frac{1}{2\pi})}.$$

If $t\notin E$, then

$$\left|\frac{f(x+t)}{|f|_{\Phi}^{m}}\right| < \frac{t}{2\pi} < 1,$$

since $\frac{t}{2\pi}<1$, Φ is convex and $\Phi(0)=0$, then

$$\begin{aligned}\Phi\left(\frac{|f(x+t)|}{\frac{|f|_{\Phi}^{m}}{2\pi}}\right) &= \Phi\left(\frac{\frac{|f(x+t)|}{|f|_{\Phi}^{m}}}{t}\cdot\frac{t}{2\pi}\right)\\ &\le \frac{t}{2\pi}\Phi\left(\frac{|f(x+t)|}{|f|_{\Phi}^{m}t}\right).\end{aligned}$$

Hence, for the partition $0<t<2\pi$ of $[0,2\pi]$ and so then

$$\begin{aligned}\int_0^{2\pi}\Phi\left(\frac{|f(x+t)|}{2\pi|f|_{\Phi}^{m}}\right)dx &\le \frac{1}{2\pi}\int_0^{2\pi}\Phi\left(\frac{|f(x+t)-f(0)|}{|t-0||f|_{\Phi}^{m}}\right)|t-0|dx\\ &\le \frac{1}{2\pi}V_{\Phi}^{m}(\frac{f}{|f|_{\Phi}^{m}})\\ &\le \frac{1}{2\pi}.\end{aligned}$$

Finally, by Jensen's inequality

$$\Phi\left(\frac{1}{|f|_{\Phi}^{m}}\int_0^{2\pi}|f(x+t)|\frac{1}{2\pi}dx\right) \le \frac{1}{2\pi}\int_0^{2\pi}\Phi\left(\frac{|f(x+t)|}{|f|_{\Phi}^{m}}\right)dx \le \frac{1}{2\pi}.$$

Thus
$$\frac{1}{2\pi|f|_{\Phi}^{m}}\int_{0}^{2\pi}|f(x)|dx \leq \Phi^{-1}\left(\frac{1}{2\pi}\right).$$
Therefore
$$\int_{0}^{2\pi}|f(x)|dx \leq 2\pi\Phi^{-1}\left(\frac{1}{2\pi}\right)|f|_{\Phi}^{m}$$
and the result of the Lemma holds.

Theorem : Let Φ be a $\Phi-$function which is convex, then $(\mathbf{R}, BV_{\Phi}^{0}M, +, |\cdot|_{\Phi}^{m})$ is a complete.

Proof : Let $\{f_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in $BV_{\Phi}^{0}M$. Given $\varepsilon > 0$, let us choose $\varepsilon' = \varepsilon M$, $(M > 0)$ the there exists a positive integer N such that:
$$|f_p - f_q|_{\Phi}^{m} < \frac{\varepsilon'}{M} = \varepsilon$$
for all $\quad p, q \geq N.$

By Lemma 4
$$\|f_p - f_q\|_{L_1[0,2\pi]} < \varepsilon$$
for all $p, q \geq N$

This implies that $\{f_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $(L_1[0.2\pi], \|\cdot\|_{L_1[0,2\pi]})$ which is a Banach spaces.

Therefore $\{f_n\}_{n\in\mathbf{N}}$ converges in norm $\|\cdot\|_{L_1[0,2\pi]}$ to some $f \in L_1[0, 2\pi]$.

Next, we like to define:
$$f : [0, 2\pi] \to \mathbf{R}$$
$$x \mapsto f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if } x \neq 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Our next task is to show that:

1. $f \in BV_{\Phi}^{0}M$.
2. The entire sequence $\{f_n\}_{n\in\mathbf{N}}$ converges to f in $BV_{\Phi}^{0}M$

By Lemma1 (ii) we have
$$V_{\Phi}^{m}\left(\frac{f_p - f_q}{\varepsilon}\right) \leq 1.$$

Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$. Then

$$\sum_{k=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|(f_p - f)(x + t_k) - (f_p - f)(x + t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx$$

$$= \sum_{k=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|(f_p - \lim f_q)(x + t_k) - (f_p - \lim f_q)(x + t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx$$

$$= \lim_{q\to\infty} \sum_{k=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|(f_p - f_q)(x + t_k) - (f_p - f_q)(x + t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx$$

$$\leq \lim_{q\to\infty} V_\Phi^m\left(\frac{f_p - f_q}{\varepsilon}\right)$$

$$\leq 1$$

for any partition $[0, 2\pi]$.

Hence

$$V_\Phi^m\left(\frac{f_p - f_q}{\varepsilon}\right) \leq 1 \quad for \quad p > N$$

and so $f_p - f \in BV_\Phi^0 M$ is a vector space $f = f_p - (f_p - f) \in BV_\Phi^0 M$. Since $V_\Phi^m\left(\frac{f_p - f}{\varepsilon}\right) \leq 1$ one more time Lemma 1 (ii) implies that

$$|f_p - f|_\Phi^m < \varepsilon \quad if \quad p > N.$$

And the proof is now complete.
Theorem : Let Φ be a $\Phi-$function which is convex.

Then $(\mathbf{R}, BV_\Phi M, +, \|\cdot\|_\Phi^m)$ is complete.
Proof : Let $\{f_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in $BV_\Phi M$ for all $\varepsilon > 0$ there exists a positive integer N such that

$$\|f_p - f_q\|_\Phi^m < \varepsilon \quad \text{for all} \quad p, q > N.$$

That is

$$|(f_p - f_q)(0)| + |(f_p - f_q) - (f_p - f_q)(0)|_\Phi^m < \varepsilon \quad \text{for all} \quad p, q > N.$$

Let $g_p = f_p - f_q(0)$, $p \in \mathbf{N}$, by Lemma 2 $g_p \in BV_\Phi^0 M$, then

$$|g_p - g_q|_\Phi^m < \varepsilon \quad \text{for all} \quad p, q > N,$$

thus $\{g_p\}_{p\in\mathbf{N}}$ is a Cauchy sequence in $(BV_\Phi^0 M, |\cdot|_\Phi^m)$ which is complete, therefore the entire sequence $\{g_p\}_{p\in\mathbf{N}}$ converges to g in $BV_\Phi^0 M$.

On the other hand

$$|f_p(0) - f_q(0)| < \varepsilon \ \ for\ all \ \ p, q > N,$$

this tell us that $f_p(0)_{p\in\mathbf{N}}$ is a Cauchy sequence in $\mathbf{R}$ and so converges to $f_0 \in \mathbf{R}$.

Let $f = g + f_0$, note that $f \in BV_\Phi M$ and

$$f(0) = (g + f_0)(0) = g(0) + f_0 = f_0.$$

Then

$$g = f - f(0),$$

moreover

$$\begin{aligned}\|f_n - f\|_\Phi^m &= |(f_n - f)(0)| + |(f_n - f) - (f_n - f)(0)|_\Phi^m \\ &= |f_n(0) - f(0)| + |g_n - g|_\Phi^m.\end{aligned}$$

Since $\{f_n(0)\}_{n\in\mathbf{N}}$ converges to $f_0 = f(0)$ and $\{g_p\}_{p\in\mathbf{N}}$ converges to g in $BV_\Phi^0 M$.

This completes the proof of the Theorem 5

1. $(\mathbf{R}, BV_\Phi^0, +, |\cdot|_\Phi^m)$ is a Banach spaces.
2. $(\mathbf{R}, BV_\Phi, +, \|\cdot\|_\Phi^m)$ is a Banach spaces.

Theorem : Let $f \in BV_\Phi M$ such that f' is continuous on $[0, 2\pi]$, then $f' \in L_\Phi([0, 2\pi])$ and

$$V_\Phi^m(f) = 2\pi \int_0^{2\pi} \Phi(f'(x))dx.$$

Definition : Let $P : 0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$. By the Mean Value Theorem there exists $\xi_k(x) \in (x + t_{k-1}, x + t_k)$ for any $x \in [0, 2\pi]$

Such that

$$|f(x + t_k) - f(x + t_{k-1})| = |f'(\xi_k(x))||t_k - t_{k-1}|, \quad (*)$$

by (*) we have

$$\pi \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Phi(f'(\xi_k(x)))(t_k - t_{k-1})$$
$$\leq \sum_{k=1}^{n} \int_0^{2\pi} \Phi\left(\frac{|f(x+t_k) - f(x+t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx.$$

From (**) we obtain

$$2\pi \int_0^{2\pi} \Phi(f'(\xi_k(x)))(t_k - t_{k-1}) \leq V_\Phi^m(f). \quad (***)$$

(***) shows that $f' \in L_\Phi([0, 2\pi])$.

On the other hand $\int_0^{2\pi} \Phi\left(\frac{|f(x+t_k)-f(x+t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx$
$= \int_0^{2\pi} \Phi\left(\frac{\left|\int_{x+t_{k-1}}^{x+t_k} f'(t)dt\right|}{\int_{t_{k-1}}^{t_k} dt}\right) |t_k - t_{k-1}| dx$
$\leq \int_0^{2\pi} \Phi\left(\frac{\int_{x+t_{k-1}}^{x+t_k} |f'(t)|dt}{\int_{t_{k-1}}^{t_k} dt}\right) |t_k - t_{k-1}| dx$

Invoking the Jensen inequality we have
$\int_0^{2\pi} \Phi\left(\frac{\int_{x+t_{k-1}}^{x+t_k} |f'(t)|dt}{\int_{t_{k-1}}^{t_k} dt}\right) |t_k - t_{k-1}| dx$
$\leq 2\pi \frac{\int_{x+t_{k-1}}^{x+t_k} \Phi(f'(t))dt}{\int_{t_{k-1}}^{t_k} dt} (t_k - t_{k-1}) dx$
$= 2\pi \int_{x+t_{k-1}}^{x+t_k} \Phi(f'(t))dt.$

Then

$$\int_0^{2\pi} \Phi\left(\frac{|f(x+t_k) - f(x+t_{k-1})|}{|t_k - t_{k-1}|}\right) |t_k - t_{k-1}| dx \leq 2\pi \int_{x+t_{k-1}}^{x+t_k} \Phi(f'(t))dt.$$

Thus, the latter inequality means that

$$V_\Phi^m(f) \leq \int_0^{2\pi} \Phi(f'(x))dx. \quad (*v)$$

Combining (***) and (xv) we easily have

$$V_\Phi^m(f) = \int_0^{2\pi} \Phi(f'(x))dx.$$

As we claimed.

In what follows, we will need the next:

Let Φ be a convex $\Phi-$function. If $\lim_{x\to} \frac{\Phi(x)}{x} = +\infty$, then it is said that Φ satisfy the (∞_1) condition.

Remark :

1. Observe that the limit exists since Φ is convex.

2. If the convex $\Phi-$function does not satisfy the (∞_1) condition, the there exist $r > 0$ such that $\lim_{x\to} \frac{\Phi(x)}{x} < +\infty$, that is, there exists $M > 0$ such that $\Phi(x) \leq x$ for $x \geq M$.

3. Since $\frac{\Phi(x)}{x}$ is increasing (Lemma 1) we have

$$\lim_{x\to} \frac{\Phi(x)}{x} = \sup_{x\in(0,\infty)} \left\{ \frac{\Phi(x)}{x} \right\}.$$

3. Nemytskii Operator

Let $\Omega \subset \mathbf{R}$ be a bounded open set. A function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is said it satisfy the Caratheodory conditions if:

1. For every $t \in \mathbf{R}$, the function $f(\cdot, t) : \Omega \to \mathbf{R}$ is Lebesgue measurable.

2. For a.e. $x \in \Omega$, the function $f(x, \cdot) : \Omega \to \mathbf{R}$ is continuous.

Set

$$M = \{\varphi : \Omega \to \mathbf{R} : \varphi \quad \text{is Lebesgue measurable}\},$$

for each $\varphi \in M$ define the operator

$$(Nf\varphi)(t) = f(t, \varphi(t)).$$

The operator Nf is said Nemytskii operator generated by the function f.

The purpose of this section is to present one condition on $BV_\Phi M$ into itself.

Also if Nf satisfy the hypothesis condition from Lemma5 below, we will show that there exist two functions g and h which belong to the bounded $\Phi-$variation in the mean space such that

$$f(t, y) = g(t)y + h(t) \ , \ t \in [0, 2\pi] \ , \ y \in \mathbf{R}.$$

Lemma : Let Φ be a $\Phi-$function.
$N_f : BV_\Phi M \to BV_\Phi M$ if there exist a constant $L > 0$ such that $|f(s,\varphi(s)) - f(t,\varphi(t))| \leq L|\varphi(s) - \varphi(t)|$ for every $\varphi \in M$.
Proof : Let $\varphi \in BV_\Phi M$, then

$$\sup\left\{\sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{|(N_f\varphi)(x+t_k) - (N_f\varphi)(x+t_{k-1})|}{|t_k - t_{k-1}|}\right)|t_k - t_{k-1}|dx\right\}$$

$$= \sup\left\{\sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{|f(x+t_k,\varphi(x+t_k)) - f(x+t_{k-1},\varphi(x+t_{k-1}))}{|t_k - t_{k-1}|}\right)|t_k - t_{k-1}|dx\right\}$$

$$\leq \sup\left\{\sum_{k=1}^{n}\int_0^{2\pi}\Phi\left(\frac{|\varphi(x+t_k) - \varphi(x+t_{k-1})|}{|t_k - t_{k-1}|}\right)|t_k - t_{k-1}|dx\right\} < \infty.$$

Thus $N_f \in BV_\Phi M$.
Theorem : Let Φ be a convex $\Phi-$function which satisfy (∞_1) condition. Let $f : [0,2\pi] \times \mathbf{R} \to \mathbf{R}$and the Nemytskii operator N_f generated by f and defined by $\mathrm{N}_f : BV_\Phi \to BV_\Phi$
$u \mapsto N_f u$, with $(N_f u)(t) = (ft, u(t))$, $t \in [0,2\pi]$.
If there exists a constant $k > 0$ such that

$$\|N_f u_1 - N_f u_2\|_\Phi^m \leq k\|u_1 - u_2\|_\Phi^m,$$

for u_1, $u_2 \in BV_\Phi^m$. Then there exists $g, h \in BV_\Phi^m$ such that

$$f(t,y) = g(t)y + h(t) \quad , \quad t \in [0,2\pi] \quad , \quad y \in \mathbf{R}.$$

Proof :
Let $y \in \mathbf{R}$, define $\mathrm{u}_0 : [0,2\pi] \to \mathbf{R}$
$t \mapsto u_0(t) = y$ a constant function, and

$\mathrm{N}_f : BV_\Phi M \to BV_\Phi M$
$u_0 \mapsto N_f u_0$ with $N_f u_0(t) = f(t, u_0(t))$. Note that $f(t,y) \in BV_\Phi M$, $\forall y \in \mathbf{R}$ by hypothesis.

Next, let $t, t' \in [0,2\pi]$, $t < t_1$; $y_1, y_2, y_1', y_2' \in \mathbf{R}$.

Now, we define u_1 and u_2 by

$$u_i : [0,2\pi] \to \mathbf{R}$$

$$s \mapsto u_i(s) = \begin{cases} y_i & \text{if } 0 \le s < t \\ \dfrac{y_i' - y_i}{t' - t}(s - t) & \text{if } t \le s \le t' \\ y_i' & \text{if } t' < s \le 2\pi \end{cases}$$

i=1,2.

Note that each u_i belong to $Lip[0, 2\pi]$, thus $u_1 - u_2 \in Lip[0, 2\pi]$. Then

$$(u_1 - u_2)(s) = \begin{cases} y_1 - y_2 & \text{if } 0 \le s < t \\ \dfrac{y_1' - y_1 - y_2' + y_2}{t' - t}(s - t) + y_1 - y_2 & \text{if } t \le s \le t' \\ y_1' - y_2' & \text{if } t' < s \le 2\pi \end{cases}$$

Observe that

$$(u_1 - u_2)'(s) = \begin{cases} 0 & \text{if } 0 \le s < t \\ \dfrac{y_1' - y_1 - y_2' + y_2}{t' - t} & \text{if } t \le s \le t' \\ 0 & \text{if } t' < s \le 2\pi \end{cases}$$

And also that $(u_1 - u_2)'$ is a continuous function on $[0, 2\pi]$. Now, we can apply Theorem 7 obtaining:

$$\begin{aligned} & 2\pi \int_0^{2\pi} \Phi\left(\frac{(u_1 - u_2)'(s)}{\varepsilon}\right) ds \\ &= 2\pi \int_t^{t'} \Phi\left(\frac{|y_1' - y_2' + y_2 - y_1|}{\varepsilon|t' - t|}\right) \\ &= 2\pi\Phi\left(\frac{|y_1' - y_2' + y_2 - y_1|}{\varepsilon|t' - t|}\right)|t' - t|. \end{aligned}$$

Hence

$$V_\Phi^m\left(\frac{u_1 - u_2}{\varepsilon}\right) = 2\pi\Phi\left(\frac{|y_1' - y_2' + y_2 - y_1|}{\varepsilon|t' - t|}\right)|t' - t|,$$

and

$$V_{\Phi}^{m}\left(\frac{u_1-u_2}{\varepsilon}\right)\leq 1 \Leftrightarrow 2\pi\Phi\left(\frac{|y_1'-y_2'+y_2-y_1|}{\varepsilon|t'-t|}\right)|t'-t|\leq 1$$

$$\Leftrightarrow \frac{|y_1'-y_2'+y_2-y_1|}{\varepsilon|t'-t|}\leq \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)$$

$$\Leftrightarrow \frac{|y_1'-y_2'+y_2-y_1|}{|t'-t|\Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)}.$$

Thus by Definition 4

$$\|u_1-u_2\|_{\Phi}^{m}=|(u_1-u_2)(0)|+\inf\left\{\varepsilon>0: V_{\Phi}^{m}\left(\frac{u_1-u_2}{\varepsilon}\right)\leq 1\right\}$$

$$|y_1-y_2|+\frac{|y_1'-y_2'+y_2-y_1|}{|t'-t|\Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)}.$$

By hypothesis $N_f u_1$, $N_f u_2$ belong to $BV_{\Phi}M$ and thus $N_f u_1 - N_f u_2 \in BV_{\Phi}M$ with

$$N_f u_i : [0,\pi] \Leftrightarrow \mathbf{R}$$

$$s \mapsto (N_f u_i)(s) = f(s, u_i(s))$$

where

$$f\left(s,u,(s)\right)=\begin{cases} f\left(f(s,y_i),\ 0\leq s\leq< t\right.\\ f\left(s,\frac{y_i'-y_i}{t'-t}(s-t)+y_i\right),\ t\leq s\leq t'\\ f(s,y_i'),\ t'<s\leq 2\pi.\end{cases}$$

Next, let us consider the partition $\pi : 0 < t < t' < 2\pi$, then

$$\int_0^{2\pi}\Phi\left(\frac{|(N_fu_1-N_fu_2)(t')-(N_fu_1-N_fu_2)(t)|}{\varepsilon|t'-t|}\right)|t'-t|dx$$

$$\leq V_{\Phi}^{m}\left(\frac{N_fu_1-N_fu_2}{\varepsilon}\right)$$

Hence

$$\frac{|(N_fu_1-N_fu_2)(t')-(N_fu_1-N_fu_2)(t)|}{|t'-t|\cdot\Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)}\leq\varepsilon,$$

$$\frac{|(N_f u_1 - N_f u_2)(t') - (N_f u_1 - N_f u_2)(t)|}{|t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)}$$
$$\leq \left\{ \varepsilon > 0 : V_\Phi^m \left(\frac{N_f u_1 - N_f - u_2}{\varepsilon} \right) \leq 1 \right\}.$$

Finally using the hypothesis we have

$$\frac{|(N_f u_1 - N_f u_2)(t') - (N_f u_1 - N_f u_2)(t)|}{|t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)}$$
$$\leq |(N_f u_1 - N_f u_2)(0)| + \inf \left\{ \varepsilon > 0 : V_\Phi^m \left(\frac{N_f u_1 - N_f - u_2}{\varepsilon} \right) \leq 1 \right\}$$
$$\leq \|N_f u_1 - N_f u_2\|_\Phi^m$$
$$\leq k\|u_1 - u_2\|_\Phi^m$$
$$= k \left[|y_1 - y_2| + \frac{|y_1' - y_2' + y_2 - y_1|}{|t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)} \right].$$

Therefore

$$\frac{|f(t', y_1') - f(t', y_2') - f(t, y_1) + f(t, y_2)|}{|t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)}$$
$$\leq k \left[|y_1 - y_2| + \frac{|y_1' - y_2' + y_2 - y_1|}{|t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right)} \right].$$

Thus

$$|f(t', y_1') - f(t', y_2') - f(t, y_1) + f(t, y_2)|$$
$$\leq k \left[|y_1 - y_2||t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right) + |y_1' - y_2' + y_2 - y_1| \right].$$

Since Φ satisfy the (∞_1) condition we have

$$\lim_{t' \to t} |t'-t| \cdot \Phi^{-1}\left(\frac{1}{2\pi|t'-t|}\right) = 0,$$

more over $f(\cdot, y)$ is continuous, then

$$|f(t, y_1') - f(t', y_2') - f(t, y_1) + f(t, y_2)| \leq k|y_1' - y_2' + y_2 - y_1| \quad (*)$$

Next, we make the following substitution:
y'$_1 = w + z$
$y_2' = w$
$y_1 = z$
$y_2 = 0$. $(**)$ Putting (**) into (*) we get

$$|f(t, w + z) - f(t, w) + f(t, 0) - f(t, z)| \leq k|w + z - w - z| = 0,$$

thus

$$f(t, w + z) - f(t, w) + f(t, 0) - f(t, z) = 0,$$

from this latter equation we have

$$f(t, w + z) - f(t, 0) = f(t, w) - f(t, 0) + f(t, z) - f(t, 0)$$

writing

$$P_t(\cdot) = f(t, \cdot) - f(t, 0), \quad , then P_t(w + z) = P_t(w) + P_t(z),$$

which means that P_t is additive and also $P_t(\cdot) = f(t, \cdot) - f(t, 0)$ is a continuous function, thus $P_t(\cdot)$ satisfy the functional Cauchy equation and its unique solution is given by

$$P_t(y) = g(t)y,$$

with $g : [0, 2\pi] \to \mathbf{R}$, $y \in \mathbf{R}$.
Let h:[0,2π] $\to \mathbf{R}$
$t \mapsto h(t) = f(t, 0)$ then $h \in BV_\Phi M$ and $P_t(y) = f(t, y) - f(t, 0)$.
Can be reduce to

$$g(t)y = f(t, y) - h(t)$$

and thus

$$f(t, y) = g(t)y + h(t).$$

Finally, since

$$f(t, 1) - f(t, 0) = (P_t(1) + f(t, 0)) - f(t, 0) = g(t),$$

for $t \in [0.2\pi]$, we conclude that $g \in BV_\Phi M$. $\square$

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