On square sum graphs

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Received : July 2012. Accepted : March 2013

Abstract

A (p,q)-graph G is said to be square sum, if there exists a bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that the induced function $f^* : E(G) \to N$ given by $f^*(uv) = (f(u))^2 + (f(v))^2$ for every $uv \in E(G)$ is injective. In this paper we initiate a study on square sum graphs and prove that trees, unicyclic graphs, mC_n, $m \geq 1$, cycle with a chord, the graph obtained by joining two copies of cycle C_n by a path P_k and the graph defined by path union of k copies of C_n, when the path $P_n \cong P_2$ are square sum.

Key Words : *Square sum graphs.*

1. Introduction

If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling. A dynamic survey on graph labeling is regularly updated by Gallian [4]. Pairs of positive integers whose sum of squares are distinct, constitute an important part of number theory. For example, in number theory a positive integer n has a representation as a sum of two squares if $n = a^2 + b^2$ for some $a, b \in Z$. Exploration of the following problems were of great interest which inspired the curiosity of number theorists.

Problem 1.1. *What numbers can be written as sum of squares of two numbers? In general does there exists a number k so that all numbers can be written as sum of k squares?*

In additive number theory, Pierre de Fermat's theorem on sums of two squares states that an odd prime p is expressible as $p = x^2 + y^2$ with x and y integers, if and only if $p \cong (1 \ mod \ 4)$. For example, the primes $5, 13, 17, 29, 37$ and 41 are all congruent to $1 \ mod \ 4$, and they can be expressed as sums of two squares in the following ways: $5 = 1^2 + 2^2$; $13 = 2^2 + 3^2$; $17 = 1^2 + 4^2$; $29 = 2^2 + 5^2$; $37 = 1^2 + 6^2$; $41 = 4^2 + 5^2$ etc. On the other hand, the primes $3, 7, 11, 19, 23$ and 31 are all congruent to $3 \ modulo \ 4$, and none of them can be expressed as the sum of two squares. Albert Girard was the first to make the observation (in 1632) and Fermat was first to claim a proof of it. Fermat announced this theorem in a letter to Marin Mersenne dated December 25, 1640; for this reason this theorem is sometimes called Fermat's Christmas Theorem. Since the Brahmagupta-Fibonacci identity implies that the product of two integers that can be written as the sum of two squares is itself expressible as the sum of two squares, by applying Fermat's theorem to the prime factorization of any positive integer n, we see that if all of n's odd prime factors congruent to $3 \ mod \ 4$ occur to an even exponent, it is expressible as a sum of two squares. The converse also holds.

These wide-angular history of sum of squares of numbers motivated the authors to study the particular graphs named square sum graphs.

Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [3]. Square sum graphs are vertex labeled graphs with the labels from the set $\{0, 1, 2, \ldots, p - 1\}$ such that the induced edge labels as the sum of the squares of the labels of the end vertices are all distinct. Not every graph is square sum. For

example, any complete graph K_n, where $n \geq 6$ is not square sum [2]. We are interested to study different classes of graphs, which are square sum. In this paper, we shall show that trees, unicyclic graphs, mC_n, $m \geq 1$, cycle with a chord, the graph obtained by joining two copies of cycles C_n by a path P_k and the graph defined by path union of k copies of C_n, when the path $P_n \cong P_2$ are square sum.

2. Square sum graphs

Acharya and Germina defined a square sum labeling of a (p, q)-graph G as follows [1, 2].

Definition 2.1. *A (p, q)-graph G is said to be square sum, if there exists a bijection $f : V(G) \to \{0, 1, 2, \ldots, p - 1\}$ such that the induced function $f^* : E(G) \to N$ is given by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective.*

Some square sum graphs are depicted in Figure 1.
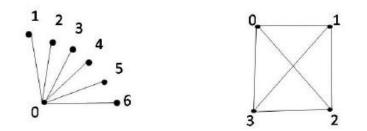
Theorem 2.2. *P_n is square sum for every n.*

Figure 1: Example of square sum graphs

Proof. Let $V(P_n) = \{u_1, u_2, u_3, \ldots, u_n\}$. Define a function $f : V(P_n) \to \{0, 1, 2, \ldots, n-1\}$ by $f(u_i) = i - 1$; $1 \leq i \leq n$. Clearly, the edge labels of P_n are distinct as it form a strictly increasing sequence of odd numbers. $\square$

Theorem 2.3. *Trees are square sum.*

Proof. Let $v_{0,0}$ be a vertex with the maximum degree in a tree T. Choose $v_{0,0}$ as the root vertex. Let k denote the height of the tree T. Let n_1 be the number of vertices adjacent to $v_{0,0}$ and let these vertices be denoted by $v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}$. These vertices are at a distance 1 from the root vertex and are taken as the first level vertices. Let n_2 be the number of vertices adjacent to first level vertices. Denote the vertices adjacent to $v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}$ by $v_{2,1}, v_{2,2}, \ldots, v_{2,n_2}$ in such a manner that the vertices adjacent to $v_{1,i}$ are labeled first in an increasing order when compared to that of $v_{1,j}$, where $1 \leq i < j \leq n_1$. These vertices are at a distance 2 from the root vertex and are taken as the second level vertices. Proceeding like this, let n_k be the number of vertices adjacent to the vertices of $(k-1)^{th}$ level. The vertices in k^{th} level are denoted by $v_{k,1}, v_{k,2}, \ldots, v_{k,n_k}$. Thus we have $|V(T)| = \sum_{i=1}^{k} n_i + 1$. Define $f : V(T) \to \{0, 1, \ldots, \sum_{i=1}^{k} n_i\}$ as follows.

$$
\begin{aligned}
f(v_{0,0}) &= \; 0 \\
f(v_{1,i}) &= \; f(v_{0,0}) + i; \;\; 1 \leq i \leq n_1 \\
f(v_{2,i}) &= \; f(v_{0,0}) + (n_1 + i); \;\; 1 \leq i \leq n_2 \\
f(v_{3,i}) &= \; f(v_{0,0}) + (n_1 + n_2 + i); \;\; 1 \leq i \leq n_3
\end{aligned}
$$

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$f(v_{k,i}) = \; f(v_{0,0}) + (n_1 + n_2 + \ldots + n_{k-1} + i); \;\; 1 \leq i \leq n_k$.

Clearly f is injective. Let $e_1, e_2 \in E(T)$ be any two arbitrary edges of T. If e_1 and e_2 are two incident edges, then $f^*(e_1) \neq f^*(e_2)$. Let $e_1 = u_1u_2$ and $e_2 = u_3u_4$ be the edges such that e_1 and e_2 have no common vertex. Without loss of generality, assume that $f(u_1)$ is least among $\{f(u_1), f(u_2), f(u_3), f(u_4)\}$. Since T is a tree, at least one of the vertices u_3, u_4 is not adjacent to u_1. If u_4 is not adjacent to u_1 then $f(u_4) > f(u_2)$ and if u_3 is not adjacent to u_1 then $f(u_3) > f(u_2)$. Hence $f^*(e_1) \neq f^*(e_2)$, showing that T is square sum. $\square$

Theorem 2.4. *Forests are square sum.*

Proof. Let $F = \bigcup_{i=1}^{k} T_i$ be a forest, where each T_i is a tree. Let $|V(T_i)| = n_i$; $1 \le i \le k$, so that $T_1 \cup T_2 \cup \ldots \cup T_k$ is of order $p = \sum_{i=1}^{k} n_i$.

Define $f : V(\bigcup_{i=1}^{k} T_i) \to \{0, 1, \ldots, p-1\}$ as follows. Let v be a vertex with maximum degree in tree T_1. Choose v as the root vertex. Visit all the vertices of T_1 using BFS algorithm and label the vertices with the consecutive numbers $0, 1, 2, 3, \ldots, n_1 - 1$ in ascending order, in the order in which they are visited in a one to one manner. By Theorem 2.3, T_1 is square sum. Then label the vertices of T_2 as follows. Let u be a vertex with maximum degree in tree T_2. Choose u as the root vertex. Visit the vertices of T_2 using BFS algorithm and label the vertices with the consecutive numbers $n_1, n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 - 1$ in ascending order, in the order in which they are visited in a one to one manner. Clearly $T_1 \cup T_2$ is square sum. Now label the vertices of T_3 in a similar way applying BFS algorithm, by choosing maximum degree vertex as the root vertex, with label $n_1 + n_2$. Proceeding in this way, we label the vertices of T_k as follows. Let w be a vertex with maximum degree in T_k. Choose w as the root vertex. Visit the vertices of T_k using BFS algorithm and label the vertices with consecutive numbers $(n_1 + n_2 + \ldots + n_{k-1}), (n_1 + n_2 + \ldots + n_{k-1} + 1), \ldots, (n_1 + n_2 + \ldots + n_{k-1} + n_k - 1)$ in ascending order, in the order in which they are visited. Clearly, the induced edge label of T_i; $1 \le i \le k$, are in strictly increasing order. Hence $\bigcup_{i=1}^{k} T_i$ is square sum. $\square$

Theorem 2.5. *Cycles are square sum.*

Proof. Let C_n be a cycle of length n and let $C_n : u_1, u_2, \ldots, u_n, u_1$. $E(C_n) = \{u_i u_{i+1}; 1 \le i \le n-1\} \cup u_n u_1$.
Define $f : V(C_n) \to \{0, 1, 2, \ldots, n-1\}$ by

$f(u_1) = 0$ and

when n is odd

$$f(u_i) = \begin{cases} 2i - 3 & \text{if } 2 \le i \le \lceil \frac{n}{2} \rceil, \\ 2(n+1-i) & \text{if } \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

when n is even

$$f(u_i) = \begin{cases} 2i - 3 & \text{if } 2 \leq i \leq \dfrac{n}{2} + 1, \\ 2(n + 1 - i) & \text{if } \dfrac{n}{2} + 1 < i \leq n. \end{cases}$$

Clearly f is injective. There are three classes of edges.

1) Edges connecting consecutive distinct odd numbers,
$\{u_i u_{i+1}; \ 2 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

2) Edges connecting consecutive distinct even numbers,
$\{u_i u_{i+1}; \ \lfloor \frac{n}{2} + 1 \rfloor < i \leq n - 1\} \cup u_n u_1$ and

3) Edges $u_1 u_2$ and $u_{\lfloor \frac{n}{2} + 1 \rfloor} u_{\lfloor \frac{n}{2} + 2 \rfloor}$, having end vertices with the labels in which one of them is even and the other is odd, which receive odd edge labels.

Clearly the labelings of class 1 and class 2 are distinct. In class 3, $f^*(u_1 u_2) = 1$ and $f^*(u_{\lfloor \frac{n}{2} + 1 \rfloor} u_{\lfloor \frac{n}{2} + 2 \rfloor})$ is strictly greater than 1. Hence $f^*(u_1 u_2) \neq f^*(u_{\lfloor \frac{n}{2} + 1 \rfloor} u_{\lfloor \frac{n}{2} + 2 \rfloor})$. Since the labels of the edges connecting consecutive even numbers and consecutive odd numbers are distinct, the edge labelings of class 1 edges are distinct from edge labeling of class 2 edges. Since the edge labelings of class 3 edges are distinct odd numbers, they are distinct from edge labeling of class 1 edges and class 2 edges of distinct even numbers. Hence all the induced edge labels are distinct and f is square sum labeling of C_n. $\square$

Theorem 2.6. *Unicyclic graphs are square sum.*

Proof.   Let G be a unicyclic graph with unique cycle $C_n : u_1, u_2, u_3, \ldots, u_n, u_1$ and define the function $f : V(G) \rightarrow \{0, 1, 2, \ldots, n - 1\}$ as follows.

Case (1): $G \cong C_n$. Choose an arbitrary vertex say u_1 of C_n. Starting from u_1, visit all the vertices of G using BFS algorithm and label the vertices $u_i; \ 1 \leq i \leq n$ in the order in which they are visited. By Theorem 2.5, f is a square sum labeling of G.

Case (2): Let $G \not\cong C_n$. We prove the theorem in following two steps. Choose any vertex u_1 of C_n such that $deg(u_1) \geq 3$.

Subcase (i): n is even. Starting from u_1, visit all the vertices of G using BFS algorithm subject to the following restrictions. First visit all the neighbors of u_1 not on the cycle so that u_n and u_2 are visited consecutively. Without loss of generality, assume that u_n is visited first. In general, if $0 \leq i \leq \frac{n}{2}$ while we visit the neighbors of u_{n-i}, first visit the neighbors that are not on cycle and for neighbors of u_{i+2}, first visit its neighbor on the cycle. Label the vertices of G with the consecutive numbers $0, 1, 2, \ldots, n-1$ in the order in which we visit them. Clearly $f(V) = \{0, 1, 2, \ldots, n-1\}$. Since G is a unicyclic graph, BFS algorithm gives exactly one back edge $u_k u_{k+1}$, where $k = \dfrac{n}{2}$. Now let e_1 and e_2 be any two edges of G. If either e_1 and e_2 are adjacent or $e_1, e_2 \neq u_k u_{k+1}$, then obviously $f^*(e_1) \neq f^*(e_2)$.

Suppose $e_1 = u_k u_{k+1}$. e_1 and e_2 are non-adjacent and $e_2 = v_1 v_2$ with $f(v_1) < f(v_2)$. If $f(v_2) < f(u_k)$ then $f(v_1) < f(v_2) < f(u_k) < f(u_{k+1})$. If $f(u_k) < f(v_2) < f(u_{k+1})$, then $f(v_1) < f(u_{k+2}) < f(u_k) < f(v_2) < f(u_{k+1})$. if $f(u_k), f(u_{k+1}) < f(v_2)$ then $f(v_1) > f(u_k)$ so that $f(u_k) < f(v_1) < f(u_{k+1}) < f(v_2)$ or $f(u_k) < f(u_{k+1}) < f(v_1) < f(v_2)$. In all cases we have $f^*(e_1) \neq f^*(e_2)$.

Subcase(ii): n is odd. Starting from u_1, visit all the neighbors of u_1, using BFS algorithm, such that we first visit u_n, then the neighbors of u_1 that are not on the cycle and then u_2. Next visit u_{n-1} and then the other neighbors of u_n so that u_2 and u_{n-1} are visited consecutively. Now proceeding as in Subcase (i) we obtain a square sum labeling of G. $\square$

Theorem 2.7. *For integers m and n, the graph mC_n is square sum for all $m \geq 1$ and for all $n \geq 3$.*

Proof. Let G denote m copies of C_n. When $m = 1$ by Theorem 2.5, C_n is square sum. Let $m \geq 2$.

Let $V(mC_n) = \{u_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n\}$ and
$E(mC_n) = \{u_{i,j}u_{i,j+1}; \ 1 \leq i \leq m; \ 1 \leq j \leq n-1 \cup u_{i,n}u_{i,1}; \ 1 \leq i \leq m\}$

Define $f : V(mC_n) \rightarrow \{0, 1, \ldots, mn-1\}$ as follows.
$f(u_{i,1}) = n(i-1); \ 1 \leq i \leq m$
and when n is odd
$$f(u_{i,j}) = \begin{cases} f(u_{i,1}) + 2j - 3 & \text{if } 1 \leq i \leq m; \ 2 \leq j \leq \lceil \frac{n}{2} \rceil, \\ f(u_{i,1}) + 2(n+1-j) & \text{if } 1 \leq i \leq m; \ \lceil \frac{n}{2} \rceil < j \leq n. \end{cases}$$

when n is even

$$f(u_{i,j}) = \begin{cases} f(u_{i,1}) + 2j - 3 & \text{if } 1 \leq i \leq m; \ \ 2 \leq j \leq \dfrac{n}{2} + 1, \\ f(u_{i,1}) + 2(n+1-j) & \text{if } 1 \leq i \leq m; \ \ \dfrac{n}{2} + 1 < j \leq n. \end{cases}$$

Clearly f is injective. The induced edge labels of mC_n; $m \geq 1$ are strictly increasing and all the edge labels are distinct. Hence mC_n is square sum for all $m \geq 1$.

$\square$

Definition 2.8. *Path union of graphs: Let graphs $G_1, G_2, G_3, \ldots, G_n$, $n \geq 2$ be all copies of a fixed graph G. Adding an edge between G_i to $G_i + 1$ for $i = 1, 2, \ldots, n-1$ is called path union of G.*

Theorem 2.9. *The graph obtained by joining two copies of cycle C_n by a path P_k is square sum.*

Proof. Let C_{n_i}; $1 \leq i \leq 2$ be two copies of cycle C_n. Let G be the graph obtained by joining two copies of cycle C_n with path P_k. Let u_i; $1 \leq i \leq n$ be the vertices of c_{n_1} and v_i; $1 \leq i \leq n$ be the vertices of C_{n_2}. Let w_i; $1 \leq i \leq k$ be the vertices of path P_k with $u_1 = w_1$ and $v_1 = w_k$. Here we note that $|V(G)| = 2n + k - 2$. Define $f : V(G) \to \{0, 1, 2, \ldots, 2n + k - 3\}$ as follows. Start from u_1 of C_{n_1} visit all the vertices of C_{n_1}, using BFS algorithm and label the vertices u_i; $1 \leq i \leq n$ as $n - 1, n - 2, \ldots, 2, 1, 0$ in descending order, the order in which they are visited. By Theorem 2.5, f is a square sum labeling of C_{n_1}. Define $f : V(P_k) \to \{n - 1, n, n + 1, n + 2 \ldots, (n + k - 3), (n + k - 2)\}$ by $f(w_i) = f(u_1) + i - 1$; $1 \leq i \leq k$. Clearly the edge labels of path P_k are strictly increasing sequence of odd numbers, and hence they are distinct. Starting from $w_k = v_1$, of C_{n_2} visit all the vertices of C_{n_2} using BFS algorithm, and label the vertices v_i; $1 \leq i \leq n$ as $(n + k - 2), (n + k - 1), (n + k), \ldots, (2n + k - 3)$, in the order in which they are visited. The edge labeling of path P_k are strictly greater than edge labels of C_{n_1}. Also edge labels of C_{n_2} are strictly greater than that of P_k. Hence induced edge labels of C_{n_i}; $1 \leq i \leq 2$ with path P_k are distinct and f is a square sum labeling of G. $\square$

Theorem 2.10. *The path union of k copies of C_n, where $k \geq 1$ when the path $P_n \cong P_2$ are square sum.*

Proof. Let C_{n_i}; $1 \leq i \leq k$ be the k copies of C_n. Let G be a graph obtained by joining k copies of C_n by path P_2. Here $|V(G)| = kn$. Let

$C_{n_1} : u_1, u_2, \ldots, u_n, u_1$ and $C_{n_2} : v_1, v_2, \ldots, v_n, v_1$ be the first two copies of C_n, where $u_1 v_1$ be the path P_2 joining C_{n_1} and C_{n_2}. Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, kn - 1\}$ as follows. Start from vertex u_1 of C_{n_1}, visit all the vertices of C_{n_1} using BFS algorithm and label the vertices u_i; $1 \leq i \leq n$ of C_{n_1} as $n - 1, n - 2, \ldots, 1, 0$ in descending order, the order in which they are visited. By Theorem 2.5, this labeling is a square sum labeling of C_{n_1}. Starting from v_1 visit all the vertices of C_{n_2} using BFS algorithm and label the vertices of C_{n_2} as $n, n + 1, \ldots, 2n - 1$ in ascending order, the order in which they are visited. Clearly label of edge $u_1 v_1$, joining C_{n_1} and C_{n_2} is strictly greater than the edge labels of C_{n_1} and edge label of C_{n_2} is strictly greater than that of $u_1 v_1$. Hence edge labels of C_{n_1}, C_{n_2} and path P_2 are distinct. Label the end vertex of path P_2 joining C_{n_2} and C_{n_3} by $2n - 1$ and $2n$ respectively. Start from vertex with label $2n$ on C_{n_3}, visit all the vertices using BFS algorithm and label the vertices in ascending order. Continue this process and label the vertices of $C_{n_4}, \ldots, C_{n_k}$ respectively using BFS algorithm and lastly, label the end vertices of path P_2 joining $C_{n_{k-1}}$ and C_{n_k} by $(k - 1)n - 1$ and $(k - 1)n$ respectively. Clearly all the edge labels of path are increasing sequence of odd numbers and are distinct from the edge labels of cycles. Hence f is square sum labeling of G. $\square$

Theorem 2.11. *Finite union* $\bigcup\limits_{i=1}^{k} C_i$ *of cycles* C_i *are square sum.*

Proof. Let $\bigcup\limits_{i=1}^{k} C_i$ be finite union of cycles. Let $C_1 : u_1, u_2, \ldots, u_{n_1}, u_1$ be a cycle of length n_1, $C_2 : v_1, v_2, \ldots, v_{n_2}, v_1$ be a cycle of length n_2, and $C_k : w_1, w_2, \ldots, w_{n_k}, w_1$ be a cycle of length n_k. Start from vertex u_1 of C_1, visit all the vertices of C_1 using BFS algorithm, label the vertices u_i; $1 \leq i \leq n_1$ of C_1 as $0, 1, 2, \ldots, n_1 - 1$, in the order in which they are visited. By Theorem 2.5, f is square sum labeling of C_1. Choose an arbitrary vertex v_1 of C_2. Start from vertex v_1 of C_2, visit all the vertices of C_2 using BFS algorithm, label the vertices v_i; $1 \leq i \leq n_2$ of C_2 as $n_1, n_1 + 1, \ldots, n_1 + n_2 - 1$, in the order in which they are visited. Clearly $C_1 \cup C_2$ is square sum. Now label the vertices of C_3 in a similar way. Proceeding in this way we label the vertices of C_k as follows. Choose an arbitrary vertex w_1 of C_k. Start from the vertex w_1 of C_k, visit all the vertices of C_k using BFS algorithm, and label the vertices w_i; $1 \leq i \leq n_k$ of C_k as $(n_1 + n_2 + \ldots + n_{k-1}), (n_1 + n_2 + \ldots + n_{k-1} + 1), \ldots, (n_1 + n_2 + \ldots + n_{k-1} + n_k - 1)$, in the order in which they are visited. Since the induced

edge labels of $\bigcup_{i=1}^{k} C_i$; $1 \le i \le k$ are strictly increasing, all the edge labels are distinct. Hence $\bigcup_{i=1}^{k} C_i$; $1 \le i \le k$ is square sum. $\square$

Theorem 2.12. *Cycle with a chord is square sum.*

Proof.   By theorem 2.5, $C_n : u_1, u_2, \ldots, u_n, u_1$ is square sum. Join the vertices u_2 and u_n by a chord. The edge label of $u_2 u_n$ is clearly distinct from all the induced edge labels of C_n. Hence cycle with a chord is square sum. $\square$

Before concluding this section let us point out an interesting problem which will have a strong link with number theory. Consider an edge labeled graph with distinct labels for the edges. It would be worthwhile to find out suitable labeling for the vertices such that sum of the squares of the labels on the end vertices become the given edge labeling. This in turn is a reformulation of the Problem 1.1.

Scope for further investigation

One may define a particular case of square sum by labeling the vertices of the (p, q)-graph G, with $\{0, 1, 2, \ldots, p - 1\}$ so as to have the induced edge labelings as distinct prime numbers. Another version of square sum graphs can be defined as labeling the vertices of the graph G using labels from $\{0, 1, 2, \ldots, q - 1\}$ so as to have the induced edge labelings as distinct integers (prime numbers). Structural properties of graphs which admit such labelings seems to be interesting for further investigation.

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