Proyecciones Journal of Mathematics
Vol. 30, N^o 1, pp. 111–122, May 2011.
Universidad Católica del Norte
Antofagasta – Chile
DOI: 10.4067/S0716-09172011000100010

Some Mathieu–type series for the I–function occuring in the Fokker–Planck equation

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Received: November 2009. Accepted: January 2011

Abstract

Closed form expressions are obtained for a family of convergent Mathieu type $\mathbf{a}$*–series and its alternating variants, whose terms contain an* I*–function which is a generalization of the Fox's* H*–function. The results derived are of general character and provide an elegant generalization for the closed form expressions of these series associated with the* H*–function by Pogány* [9]*, for Fox–Wright functions by Pogány and Srivastava* [10] *and for* ${}_pF_q$ *and Meijer's* G*–function by Pogány and Tomovski* [13]*, and others.*

2000 Mathematics Subject Classification. *Primary* 33C20, 33C60*; Secondary* 40G99, 44A20.

1. **Introduction and preliminaries**

In order to unify and extend the results for the convergent Mathieu–type **a**–series and its alternative form series and alternating Mathieu–type **a**–series whose terms contain the familiar transcendental functions, such as Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric function ${}_pF_q$, the Fox–Wright function ${}_p\Psi_q$, the Meijer's G–function and Fox's H–function, published in a series of papers by Pogány [7, 8, 9], Pogány *et al.* [10, 11, 12, 13], Srivastava and Tomovski [21], Tomovski [23] and Tomovski and Tuan [26], the authors introduce the Mathieu–type a–series and its alternative variant, whose terms contain an I–function. Inequalities for Mathieu–type series are discussed by Cerone [1], Pogány and Tomovski [14], Srivastava and Tomovski [21], Tomovski and Hilfer [24] and Tomovski and Pogány [25]. The results obtained by the authors serve as the key formulas for numerous potentially useful special functions of Science, Engineering and Technology scattered in the literature.

The I–function like the H–function, is defined in terms of a Mellin–Barnes type integral in the following form [19]:

$$
\begin{aligned}
(1.1)\quad I^{\,m,n}_{p_\ell,q_\ell,\kappa}[z] &= I^{\,m,n}_{p_\ell,q_\ell,\kappa}\left[z \,\middle|\, \begin{matrix} (a_j,A_j)_{\overline{1,n}}\,, & (a_{j\ell},A_{j\ell})_{j=\overline{n+1,p_\ell}}^{\ell=\overline{1,\kappa}} \\[2ex] (b_j,B_j)_{\overline{1,m}}\,, & (b_{j\ell},B_{j\ell})_{j=\overline{m+1,q_\ell}}^{\ell=\overline{1,\kappa}} \end{matrix} \right] \\
&:= \frac{1}{2\pi \mathrm{i}} \int_{L_\kappa} \frac{\chi_\kappa(s)}{z^s}\,\mathrm{d}s\,,
\end{aligned}
$$

where $\kappa \in \mathbf{N}$ and $(a_j,A_j)_{\overline{1,n}}$ denotes the parameters sequence $(a_1,A_1), \cdots,$ (a_n,A_n), while $(a_{j\ell},A_{j\ell})_{j=\overline{1,p_\ell}}^{\ell=\overline{1,\kappa}}$ stands for the parameters sequence (a_{11},A_{11}), $\cdots$, $(a_{p_\kappa\kappa},A_{p_\kappa\kappa})$ and

$$
\chi_\kappa(s) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\cdot\prod\limits_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum\limits_{\ell=1}^{\kappa}\prod\limits_{j=n+1}^{p_\ell}\Gamma(a_{j\ell}+A_{j\ell}s)\cdot\prod\limits_{j=m+1}^{q_\ell}\Gamma(1-b_{j\ell}-B_{j\ell}s)}\,.
$$

Here m,n,p_ℓ,q_ℓ are nonnegative integers satisfying $0 \le n \le p_\ell, 0 \le m \le q_\ell$, $\ell = \overline{1,\kappa}$ and $(a_j,A_j),(b_j,B_j) \in \mathbf{C}\times\mathbf{R}^+$ such that $A_j(b_h+\ell) \neq B_h(a_j - k-1)$, for $\ell,k \in \mathbb{N}_0; h=\overline{1,m}, j=\overline{1,n}$. The parameters $a_{j\ell},A_{j\ell},b_{j\ell},B_{j\ell} \in \mathbf{C}$. The contour L_κ in the complex s–plane extends from $w-\mathrm{i}\infty$ to $w+\mathrm{i}\infty, w+\max_{1\le h\le m}|\Im\{b_h\}|/B_h > 0$ separating the points $-B_h^{-1}(b_h+\ell), h=\overline{1,m}, \ell \in \mathbf{N}_0$, which are the poles of $\Gamma(b_j+B_js), j=\overline{1,m}$, from the points $-A_j^{-1}(a_j - k-1), j=\overline{1,n}, k\in\mathbf{N}_0$ which are the poles of $\Gamma(1-a_j-A_js), j=\overline{1,n}$.

Existence conditions for the integral (1.1) are given below

$$\alpha_\ell > 0, \quad |\arg\{z\}| < \frac{\pi}{2}\,\alpha_\ell, \quad \ell = \overline{1,\kappa}$$

and

$$\alpha_\ell \geq 0, \quad |\arg\{z\}| \leq \frac{\pi}{2}\,\alpha_\ell, \quad \text{and} \quad \Re\{\beta_\ell\} + 1 < 0, \quad \ell = \overline{1,\kappa}\,,$$

where

$$\alpha_\ell = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p_\ell} A_{j\ell} + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q_\ell} B_{j\ell},\ \ell = \overline{1,\kappa},$$

$$\beta_\ell = \sum_{j=1}^{m} b_j + \sum_{j=n+1}^{q_\ell} b_{j\ell} - \sum_{j=1}^{n} a_j - \sum_{j=n+1}^{p_\ell} a_{j\ell} + \frac{1}{2}\big(p_\ell - q_\ell\big),\ \ell = \overline{1,\kappa}\,.$$

Remark 1. *For* $\kappa = 1$ *in* (1.1)*, the I–function reduces to the familiar H–function, defined by Fox* [4] *in the following manner:*

$$\begin{aligned} I^{\,m,n}_{p_1,q_1,1}[z] \;&=\; H^{m,n}_{p_1,q_1}\left[z \,\middle|\, \begin{matrix} (a_j,A_j)_{\overline{1,n}}\,, & (a_{j1},A_{j1})_{j=\overline{n+1,p_1}} \\ (b_j,B_j)_{\overline{1,m}}\,, & (b_{j1},B_{j1})_{j=\overline{m+1,q_1}} \end{matrix} \right] \\ &:=\; \frac{1}{2\pi\mathrm{i}} \int_{L_1} \frac{\chi_1(s)}{z^s}\,\mathrm{d}s\,. \end{aligned}$$

A comprehensive account of the H–function is available from the monographs written by Mathai and Saxena [5]*, Srivastava* et al. [20]*, Kilbas and Saigo* [3] *and Mathai* et al. [6]*.*

Remark 2. *I–function naturally occurs in certain problems associated with driftless Fokker–Planck equations with power–law diffusion, see Südland* et al. [22]*.*

Note 1. We note that integral operators involving I–function are defined and studied by Saxena and Singh [17]. A basic analogue of the I–function is given by Saxena and Kumar in [15]. Saigo–Maeda operators of the product of I–function and a general class of polynomials are discussed by Saxena *et al.* [16].

Now, let us define the Mathieu–type **a**–series $\Theta_{\lambda,\mu}$ and its alternating variant $\widetilde{\Theta}_{\lambda,\mu}$ by the following series:

$$\begin{aligned} &\Theta_{\lambda,\mu}\Big\{ I^{m,n+1}_{p_\ell+1,q_\ell,\kappa}; \mathbf{c}, r \Big\} \\ &:= \sum_{j=1}^{\infty} \frac{ I^{m,n+1}_{p_\ell+1,q_\ell}\left[\frac{r}{c_j} \,\middle|\, \begin{matrix} (\alpha,\beta),\ (a_j,A_j)_{\overline{1,n}},\ (a_{j\ell},A_{j\ell})^{\ell=\overline{1,\kappa}}_{j=\overline{n+1,p_\ell}} \\ \\ (b_j,B_j)_{\overline{1,m}}, \quad (b_{j\ell},B_{j\ell})^{\ell=\overline{1,\kappa}}_{j=\overline{m+1,q_\ell}} \end{matrix} \right] }{ c_j^{\lambda}(c_j+r)^{\mu} }\,, \end{aligned} \tag{1.2}$$

$$\widetilde{\Theta}_{\lambda,\mu}\Big\{I_{p+1,q}^{m,n+1};\mathbf{c},r\Big\}$$
$$:= \sum_{j=1}^\infty \frac{(-1)^{j-1}\, I_{p_\ell+1,q_\ell}^{m,n+1}\left[\frac{r}{c_j}\,\middle|\, \begin{matrix} (\alpha,\beta),\, (a_j,A_j)_{\overline{1,n}},\, (a_{j\ell},A_{j\ell})_{j=n+1,p_\ell}^{\ell=\overline{1,\kappa}} \\ (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})_{j=m+1,q_\ell}^{\ell=\overline{1,\kappa}}\end{matrix}\right]}{c_j^\lambda(c_j+r)^\mu},$$

where the following convention is followed that the real sequence $\mathbf{c} = \big(c_n\big)_{n\in\mathbf{N}}$ increases and tends to infinity; equivalently

$$\mathbf{c}\colon 0 < c_1 < c_2 < \cdots < c_n \uparrow \infty\,. \tag{1.3}$$

2. **Integral representations**

The Laplace transform of the I–function is given by

$$\begin{aligned}&\int_0^\infty x^{\lambda-1}\mathrm{e}^{-sx} I_{p_\ell,q_\ell,\kappa}^{\;m,n}\big[\omega x^\rho\big]\mathrm{d}x \\ &= s^{-\lambda}\, I_{p_\ell+1,q_\ell,\kappa}^{m,n+1}\left[\frac{\omega}{s^\rho}\,\middle|\, \begin{matrix} (1-\lambda,\rho),\, (a_j,A_j)_{\overline{1,n}},\, (a_{j\ell},A_{j\ell})_{j=n+1,p_\ell}^{\ell=\overline{1,\kappa}} \\ (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})_{j=m+1,q_\ell}^{\ell=\overline{1,\kappa}}\end{matrix}\right],\end{aligned} \tag{2.1}$$

where $\lambda, s, \omega \in \mathbf{C}$; $\Re\{s\} > 0$, $\rho > 0$, and

$$\Re\{\lambda\} + \rho \min_{1\le j\le m} \frac{\Re\{b_j\}}{B_j} > 0, \qquad |\arg \omega| < \frac{\pi}{2}\,\Omega_\ell\,,$$

being

$$\Omega_\ell = \sum_{j=1}^m B_j + \sum_{j=1}^n A_j - \sum_{j=m+1}^{q_\ell} B_{j\ell} - \sum_{j=n+1}^{p_\ell} A_{j\ell} > 0 \qquad \ell = \overline{1,\kappa}\,.$$

The formula (2.1) can be easily established with the help of the definition (1.1) of I–function and using the gamma function formula

$$\Gamma(\mu)\zeta^{-\mu} = \int_0^\infty x^{\mu-1}\,\mathrm{e}^{-\zeta x}\mathrm{d}x \qquad \min\Big(\Re\{\mu\},\Re\{\zeta\}\Big) > 0\,. \tag{2.2}$$

Theorem. *Let* $\lambda > 0, \mu > 0, r > 0, \alpha = 1-\lambda, \beta = \rho = 1$ *and let the sequence* $\mathbf{c}$ *satisfies* (1.3). *Then there holds the following results:*

$$\Theta_{\lambda,\mu}\Big\{I_{p_\ell+1,q_\ell,\kappa}^{\;m,n+1};\mathbf{c},r\Big\} = \mathbf{I}_\mathbf{c}^I(\lambda+1,\mu) + \mu\mathbf{I}_\mathbf{c}^I(\lambda,\mu+1) \tag{2.3}$$

$$\widetilde{\Theta}_{\lambda,\mu}\Big\{I_{p_\ell+1,q_\ell,\kappa}^{\;m,n+1};\mathbf{c},r\Big\} = \widetilde{\mathbf{I}}_\mathbf{c}^I(\lambda+1,\mu) + \mu\widetilde{\mathbf{I}}_\mathbf{c}^I(\lambda,\mu+1)\,, \tag{2.4}$$

where

$$\begin{aligned}
&\mathbf{I}_{\mathbf{c}}^{I}(u,v)\\
&:= \int_{c_1}^{\infty} I_{p_\ell+1,q_\ell,\kappa}^{\,m,n+1}\left[\frac{r}{x}\,\middle|\, \begin{matrix} (1-u,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j\ell},A_{j\ell})_{j=\overline{n+1,p_\ell}}^{\ell=\overline{1,\kappa}} \\ (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})_{j=\overline{m+1,q_\ell}}^{\ell=\overline{1,\kappa}} \end{matrix}\right]\\
&\qquad\qquad \times \frac{[c^{-1}(x)]}{x^u(r+x)^v}\,\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\mathbf{I}}_{\mathbf{c}}^{I}(u,v)\\
&:= \int_{c_1}^{\infty} I_{p_\ell+1,q_\ell,\kappa}^{\,m,n+1}\left[\frac{r}{x}\,\middle|\, \begin{matrix} (1-u,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j\ell},A_{j\ell})_{j=\overline{n+1,p_\ell}}^{\ell=\overline{1,\kappa}} \\ (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})_{j=\overline{m+1,q_\ell}}^{\ell=\overline{1,\kappa}} \end{matrix}\right]\\
&\qquad\qquad \times \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)\mathrm{d}x}{x^u(r+x)^v}.
\end{aligned}$$

Here $c\colon \mathbf{R}_+ \mapsto \mathbf{R}_+$ *is an increasing function such that* $c(x)\big|_{x\in\mathbf{N}} = \mathbf{c}, c^{-1}(x)$ *denotes the inverse of* $c(x)$, $\left[c^{-1}(x)\right]$ *stands for the integer part of the quantity* $c^{-1}(x)$.

Proof. Taking $\zeta = c_n + r$ in (2.2), setting $s = c_j$; $\rho = 1, \omega = r$ and inserting $\lambda = 1-\alpha, \beta = 1$ in (1.2), that is *a fortiori* in (2.1), we find that

$$\begin{aligned}
&\Theta_{\lambda,\mu}\Big\{I_{p_\ell+1,q_\ell,\kappa}^{\,m,n+1};\mathbf{c},r\Big\}\\
&= \sum_{j=1}^{\infty}\frac{I_{p_\ell+1,q_\ell,\kappa}^{\,m,n+1}\left[\frac{r}{c_j}\,\middle|\, \begin{matrix} (1-\lambda,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j\ell},A_{j\ell})_{j=\overline{n+1,p_\ell}}^{\ell=\overline{1,\kappa}} \\ (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})_{j=\overline{m+1,q_\ell}}^{\ell=\overline{1,\kappa}} \end{matrix}\right]}{c_j^{\lambda}(c_j+r)^{\mu}}\\
&= \sum_{j=1}^{\infty}\int_0^{\infty} s^{\lambda-1}\mathrm{e}^{-c_j s} I_{p_\ell,q_\ell,\kappa}^{\,m,n}[rs]\mathrm{d}s \int_0^{\infty}\frac{t^{\mu-1}}{\Gamma(\mu)}\,\mathrm{e}^{-(c_j+r)t}\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{\Gamma(\mu)}\int_0^{\infty}\int_0^{\infty}\left(\sum_{j=1}^{\infty}\mathrm{e}^{-c_j(s+t)}\right)s^{\lambda-1}t^{\mu-1}\mathrm{e}^{-rt} I_{p_\ell,q_\ell,\kappa}^{\,m,n}[rs]\mathrm{d}s\mathrm{d}t\,, \tag{2.5}$$

where $\Re\{\mu\} > 0$ is already assumed. The inside Dirichlet series

$$\mathcal{D}_{\mathbf{c}}(s+t) = \sum_{j=1}^{\infty}\mathrm{e}^{-c_j(s+t)}$$

has a Laplace type integral representation [7, 9] such that it can be expressed in the form

$$\begin{aligned}\mathcal{D}_{\mathbf{c}}(s+t) &= (s+t)\int_0^\infty \mathrm{e}^{-(s+t)x}\Bigg(\sum_{j\colon c_j\le x} 1\Bigg)\mathrm{d}x \\ &= (s+t)\int_{c_1}^\infty \mathrm{e}^{-(s+t)x}\big[c^{-1}(x)\big]\mathrm{d}x\,, \end{aligned} \tag{2.6}$$

with $\big[c^{-1}(x)\big]=0$ for $x\in[0,c_1)$. By virtue of (2.6), the equation (2.5) becomes

$$\begin{aligned}&\Theta_{\lambda,\mu}\Big\{I^{\,m,n+1}_{p_\ell+1,q_\ell,\kappa};\mathbf{c},r\Big\} \\ &\quad= \frac{1}{\Gamma(\mu)}\int_0^\infty\int_0^\infty\int_{c_1}^\infty s^{\lambda}t^{\mu-1}\mathrm{e}^{-(r+x)t-xs}I^{\,m,n}_{p_\ell,q_\ell,\kappa}[rs]\big[c^{-1}(x)\big]\mathrm{d}s\mathrm{d}t\mathrm{d}x \\ &\qquad+\frac{1}{\Gamma(\mu)}\int_0^\infty\int_0^\infty\int_{c_1}^\infty s^{\lambda-1}t^{\mu}\mathrm{e}^{-(r+x)t-xs}I^{\,m,n}_{p_\ell,q_\ell,\kappa}[rs]\big[c^{-1}(x)\big]\mathrm{d}s\mathrm{d}t\mathrm{d}x\,.\end{aligned}$$

Since

$$\begin{aligned}J_s &= \int_{c_1}^\infty\left(\int_0^\infty s^{\lambda}\mathrm{e}^{-sx}I^{\,m,n}_{p_\ell,q_\ell,\kappa}[rs]\mathrm{d}s\right)\left(\int_0^\infty t^{\mu-1}\mathrm{e}^{-(r+x)t}\mathrm{d}t\right)\frac{\big[c^{-1}(x)\big]}{\Gamma(\mu)}\mathrm{d}x \\ &= \int_{c_1}^\infty I^{\,m,n+1}_{p_\ell+1,q_\ell,\kappa}\left[\frac{r}{x}\;\middle|\;\begin{matrix}(-\lambda,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j\ell},A_{j\ell})^{\ell=\overline{1,\kappa}}_{j=\overline{n+1,p_\ell}}\\[2ex] (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})^{\ell=\overline{1,\kappa}}_{j=\overline{m+1,q}}\end{matrix}\right] \\ &\qquad\times\frac{\big[c^{-1}(x)\big]}{x^{\lambda+1}(r+x)^{\mu}}\,\mathrm{d}x\,,\end{aligned}$$

introducing the auxiliary integral

$$\begin{aligned}\mathbf{I}^{I}_{\mathbf{c}}(u,v) &:= \int_{c_1}^\infty I^{\,m,n+1}_{p_\ell+1,q_\ell,\kappa}\left[\frac{r}{x}\;\middle|\;\begin{matrix}(1-u,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j\ell},A_{j\ell})^{\ell=\overline{1,\kappa}}_{j=\overline{n+1,p_\ell}}\\[2ex] (b_j,B_j)_{\overline{1,m}},\quad (b_{j\ell},B_{j\ell})^{\ell=\overline{1,\kappa}}_{j=\overline{m+1,q}}\end{matrix}\right] \\ &\qquad\times\frac{\big[c^{-1}(x)\big]}{x^{u}(r+x)^{v}}\mathrm{d}x\,,\end{aligned}$$

it readily follows that

$$J_s=\mathbf{I}^{I}_{\mathbf{c}}(\lambda+1,\mu)\qquad\text{and}\qquad J_t=\mu\cdot\mathbf{I}^{I}_{\mathbf{c}}(\lambda,\mu+1)\,.$$

This establishes the first assertion of the Theorem.

The proof of (2.4) is similar to that of (2.3), if we employ the definition of the new alternating inner Dirichlet series $\widetilde{\mathcal{D}}_{\mathbf{c}}(\cdot)$ [11, p. 77, Section 4] given below:

$$
\begin{aligned}
\widetilde{\mathcal{D}}_{\mathbf{c}}(s+t) &= \sum_{j=1}^{\infty}(-1)^{j-1}\mathrm{e}^{-c_j(s+t)} \\
&= (s+t)\int_0^{\infty}\mathrm{e}^{-(s+t)x}\sum_{j\,:\,c_j\le x}(-1)^{j-1}\mathrm{d}x \\
&= \frac{s+t}{2}\int_0^{\infty}\mathrm{e}^{-(s+t)x}\Big(1-(-1)^{[c^{-1}(x)]}\Big)\mathrm{d}x \\
&= (s+t)\int_{c_1}^{\infty}\mathrm{e}^{-(s+t)x}\sin^2\Big(\frac{\pi}{2}\big[c^{-1}(x)\big]\Big)\mathrm{d}x\,. \qquad (2.7)
\end{aligned}
$$

The application of (2.7) completes the proof of (2.4). □

3. **Special cases**

As I–function is most generalized function, numerous special cases associated with various transcendental functions, such Mittag–Leffler functions, Bessel functions, Whittaker functions, hypergeometric functions, generalized hypergeometric function, Meijer's G–function, Fox–Wright Ψ function and Fox's H–function and their special cases can be deduced by making use of the special cases of the I–function given in the book [18].

Now, some interesting special case results are presented here.

3.1.

For $\kappa = 1$, the I–function reduces to Fox's H-function (see Remark 1.) and we obtain the following result given by Pogány [9, Theorem 1].

Corollary 1. *Assume that*

$$\lambda, \mu, r > 0, a_{p_1+1,1} = 1-\lambda, A_{p_1+1,1} = 1$$

and let the sequence $\mathbf{c}$ *satisfies the condition* (1.3). *Then there hold the following formulas:*

$$
\begin{aligned}
&\Theta_{\lambda,\mu}\Big\{H^{m,n+1}_{p_1+1,q_1};\mathbf{c},r\Big\} \\
&\quad := \sum_{j=1}^{\infty}\frac{H^{m,n+1}_{p_1+1,q_1}\left[\dfrac{r}{c_j}\middle|\begin{array}{c}(a_j,A_j)_{\overline{1,n}},\,(a_{j1},A_{j1})_{\overline{n+1,p_1+1}}\\(b_j,B_j)_{\overline{1,m}},\,(b_{j1},B_{j1})_{\overline{m+1,q_1}}\end{array}\right]}{c_j^{\lambda}(c_j+r)^{\mu}} \\
&\quad = \mathbf{I}^H_{\mathbf{c}}(\lambda+1,\mu)+\mu\mathbf{I}^H_{\mathbf{c}}(\lambda,\mu+1)\,,
\end{aligned}
$$

and

$$\begin{aligned}
&\widetilde{\Theta}_{\lambda,\mu}\Big\{H^{m,n+1}_{p_1+1,q_1};\mathbf{c},r\Big\} := \\
&\sum_{j=1}^\infty (-1)^{j-1}\frac{H^{m,n+1}_{p_1+1,q_1}\Big[\frac{r}{c_j}\Big|\begin{array}{c}(a_j,A_j)_{\overline{1,n}},\,(a_{j1},A_{j1})_{\overline{n+1,p_1+1}}\\(b_j,B_j)_{\overline{1,m}},\,(b_{j1},B_{j1})_{\overline{m+1,q_1}}\end{array}\Big]}{c_j^\lambda(c_j+r)^\mu}\\
&\qquad = \widetilde{\mathbf{I}}^H_{\mathbf{c}}(\lambda+1,\mu)+\mu\widetilde{\mathbf{I}}^H_{\mathbf{c}}(\lambda,\mu+1)\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{I}^H_{\mathbf{c}}(u,v) &:= \int_{c_1}^\infty H^{m,n+1}_{p_1+1,q_1}\Big[\frac{r}{x}\Big|\begin{array}{c}(1-u,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j1},A_{j1})_{\overline{n+1,p_1}}\\(b_j,B_j)_{\overline{1,m}},\,(b_{j1},B_{j1})_{\overline{m+1,q_1}}\end{array}\Big]\\
&\qquad\times\frac{\big[c^{-1}(x)\big]}{x^u(r+x)^v}\,\mathrm{d}x,\\
\widetilde{\mathbf{I}}^H_{\mathbf{c}}(u,v) &:= \int_{c_1}^\infty H^{m,n+1}_{p_1+1,q_1}\Big[\frac{r}{x}\Big|\begin{array}{c}(1-u,1),\,(a_j,A_j)_{\overline{1,n}},\,(a_{j1},A_{j1})_{\overline{n+1,p_1}}\\(b_j,B_j)_{\overline{1,m}},\,(b_{j1},B_{j1})_{\overline{m+1,q_1}}\end{array}\Big]\\
&\qquad\times\frac{\sin^2\big(\frac{\pi}{2}[c^{-1}(x)]\big)}{x^u(r+x)^v}\,\mathrm{d}x,
\end{aligned}$$

and $c:\mathbb{R}_+\mapsto\mathbb{R}_+$ *is an increasing function such that* $c(x)\big|_{x\in\mathbb{N}}=\mathbf{c}, c^{-1}(x)$ *is the inverse of* $c(x)$, $\big[c^{-1}(x)\big]$ *stands for the integer part of* $c^{-1}(x)$.

3.2. If we employ further the identity

$$ {}_p\Psi_q\Big[\begin{array}{c}(a_p,A_p)\\(b_q,B_q)\end{array}\Big|-z\Big] = H^{1,p}_{p,q+1}\Big[z\Big|\begin{array}{c}(1-a_p,A_p)\\(0,1),(1-b_q,B_q)\end{array}\Big]\,, \tag{3.1}$$

where ${}_p\Psi_q$ is the Fox–Wright function defined by [5, p. 11]

$$ {}_p\Psi_q\Big[\begin{array}{c}(a_p,\alpha_p)\\(b_q,\beta_q)\end{array}\Big|z\Big] := \sum_{n=0}^\infty\frac{\prod_{j=1}^p\Gamma\big(a_j+A_jn\big)}{\prod_{j=1}^q\Gamma\big(b_j+B_jn\big)}\,\frac{z^n}{n!}\,.$$

Here $a_j,b_j\in\mathbf{C}$, $A_j,B_j>0$, $(i=\overline{1,p}, j=\overline{1,q})$ and $\sum\limits_{j=1}^q B_j-\sum\limits_{j=1}^p A_j>-1$.

Remark 3. *The relation connecting Fox–Wright* ${}_p\Psi_q$ *function and the* H*–function has been given by Mathai and Saxena for the first time in the book* [5, p. 11, Eq. (1.7.8)].

By virtue of Theorem and the above result (3.1), one can deduce results for Fox–Wright function. Suppose

$$\Theta_{\lambda,\mu}\big\{{}_{p+1}\Psi_q;\mathbf{c},r\big\} := \sum_{j=1}^{\infty} \frac{{}_{p+1}\Psi_q\Big[\begin{array}{c}(\alpha,\beta),\,(a_p,A_p)\\(b_q,B_q)\end{array}\Big| -\dfrac{r}{c_j}\Big]}{c_j^{\lambda}(c_j+r)^{\mu}}\,,$$

and

$$\widetilde{\Theta}_{\lambda,\mu}\big\{{}_{p+1}\Psi_q;\mathbf{c},r\big\} := \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\,{}_{p+1}\Psi_q\Big[\begin{array}{c}(\alpha,\beta),\,(a_p,A_p)\\(b_q,B_q)\end{array}\Big| -\dfrac{r}{c_j}\Big]}{c_j^{\lambda}(c_j+r)^{\mu}}\,.$$

Now one can easily arrive at

Corollary 2. *Let $\lambda \notin \mathbf{N}, \mu > 0, r > 0, (\alpha,\beta) = (1-\lambda,1), (b_q,B_q) = (1,1)$ and let the sequence $\mathbf{c}$ satisfies* (1.3). *Then we have*

$$\begin{aligned}\Theta_{\lambda,\mu}\big\{{}_{p+1}\Psi_q;\mathbf{c},r\big\} &= \mathbf{I}_{\mathbf{c}}^{\Psi}(\lambda+1,\mu)+\mu\mathbf{I}_{\mathbf{c}}^{\Psi}(\lambda,\mu+1)\\ \widetilde{\Theta}_{\lambda,\mu}\big\{{}_{p+1}\Psi_q;\mathbf{c},r\big\} &= \widetilde{\mathbf{I}}_{\mathbf{c}}^{\Psi}(\lambda+1,\mu)+\mu\widetilde{\mathbf{I}}_{\mathbf{c}}^{\Psi}(\lambda,\mu+1)\,,\end{aligned}$$

where

$$\mathbf{I}_{\mathbf{c}}^{\Psi}(u,v) := \int_{c_1}^{\infty} \frac{\big[c^{-1}(x)\big]}{x^u(r+x)^v}\,{}_{p+1}\Psi_q\Big[\begin{array}{c}(1-u,1),(a_p,A_p)\\(1,1),(b_{q-1},B_{q-1})\end{array}\Big| -\frac{r}{x}\Big]\,\mathrm{d}x$$

and

$$\widetilde{\mathbf{I}}_{\mathbf{c}}^{\Psi}(u,v) := \int_{c_1}^{\infty} \frac{\sin^2\big(\frac{\pi}{2}[c^{-1}(x)]\big)}{x^u(r+x)^v}\,{}_{p+1}\Psi_q\Big[\begin{array}{c}(1-u,1),(a_p,A_p)\\(1,1),(b_{q-1},B_{q-1})\end{array}\Big| -\frac{r}{x}\Big]\,\mathrm{d}x\,.$$

Here $c(x), c^{-1}(x), \big[c^{-1}(x)\big]$ retain the same meanings than in previous corollary.

Remark 4. *Finally, it is interesting to observe that, by virtue of the relation*

$$E_{\alpha,\beta}(z) = H_{1,2}^{1,1}\Big[-z\Big|\begin{array}{c}(0,1)\\(0,1),\,(1-\beta,\alpha)\end{array}\Big]\,,$$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function [2, Chapter 18] *and* [4, p. 80], *defined by*

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)},$$

where

$$\alpha,\beta \in \mathbf{C}; \Re\{\alpha\}, \Re\{\beta\} > 0,$$

the results for the Mittag–Leffler function can be easily deduced from Corollary 1.

Acknowledgements

The present investigation was supported for the first author by the *Ministry of Science, Education and Sports of Croatia* under Research Project No. 112-2352818-2814.

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