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JEWELL THEOREM FOR HIGHER DERIVATIONS ON C^*-ALGEBRAS

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Abstract

Let $\mathcal{A}$ be an algebra. A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a higher derivation if $d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b)$ for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Jewell [Pacific J. Math. **68** *(1977), 91-98], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\ker(d_0) \subseteq \ker(d_m)$, for all $m \geq 1$. In this paper, under a different approach using C^*-algebraic tools, we prove that each higher derivation $\{d_n\}$ on a C^*-algebra $\mathcal{A}$ is automatically continuous, provided that it is normal, i. e. d_0 is the identity mapping on $\mathcal{A}$.*

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1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the Leibniz rule, i. e. $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. If we define the sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where I is the identity mapping on $\mathcal{A}$, then the Leibniz rule ensures us that d_n's satisfy the condition

$$d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b) \quad (*)$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying $(*)$. Such a sequence is called a higher derivation. Higher derivations were introduced by Hasse and Schmidt [2], and algebraists sometimes call them Hasse-Schmidt derivations. Though, if $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation then $d_n = \frac{\delta^n}{n!}$ is a higher derivation, this is not the only example of a higher derivation.

Regarding to a celebrated theorem of Sakai [11, 12], all derivations defined on a C^*-algebra are automatically continuous. Some results concerning to the theorem are discussed in [8] and [3]. Regarding to the Sakai's theorem we can deduce that the higher derivation $d_n = \frac{\delta^n}{n!}$ defined on a C^*-algebra is automatically continuous in the sense that each d_n is continuous. This poses the problem of automatic continuity of higher derivations. Many mathematicians could find some affirmative answers to the problem in special cases. Loy [7] proved that if $\mathcal{A}$ is an (F)-algebra which is a subalgebra of a Banach algebra $\mathcal{B}$ of power series, then every higher derivation $\{d_n\} : \mathcal{A} \to \mathcal{B}$ is automatically continuous. Jewell [5], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\ker(d_0) \subseteq \ker(d_m)$, for all $m \geq 1$. Villena [14], proved that every higher derivation from a unital Banach algebra $\mathcal{A}$ into $\mathcal{A}/\mathcal{P}$, where $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ with infinite codimension, is continuous. Hejazian and Shatery [4] prove the automatic continuity of higher derivations in the case of JB^*-algebras.

Here, we prove automatic continuity of higher derivations in the domain of C^*-algebras. Though, this is a consequence of the Jewell result in [5], our proof just depends on C^*-algebraic tools. Prior to that, we need some elementary facts concerning higher derivations. For the definition and elementary properties of C^*-algebras we refer the reader to [6, 9] and [10]. One can find a collection of suitable information about automatic continuity and some applications of higher derivations in [1] and [13].

2. Preliminaries

Let $\mathcal{A}$ be an algebra, $\mathbf{Z}_k^+ = \{0, 1, \ldots, k\}$ for $k \in \mathbf{N}$ and $\mathbf{Z}^+ = \{0, 1, 2, \ldots\}$. A higher derivation of order k is a sequence $\{d_n\}_{n \in \mathbf{Z}_k^+}$ of linear mappings from $\mathcal{A}$ to $\mathcal{A}$ such that

$$d_n(ab) = \sum_{j=0}^{n} d_j(a) d_{n-j}(b)$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbf{Z}_k^+$. A sequence $\{d_n\}_{n \in \mathbf{Z}^+}$ is a higher derivation of infinite order if $\{d_n\}_{n \in \mathbf{Z}_k^+}$ is a higher derivation of order k for each $k \in \mathbf{N}$. A higher derivation $\{d_n\}$ is called normal if $d_0 = I$ (the identity mapping on $\mathcal{A}$). As a simple example, for a derivation $\delta : \mathcal{A} \to \mathcal{A}$ we can assume the sequence $d_0 = I$, $d_n = \frac{\delta^n}{n!}$. The Leibniz rule implies that $\{d_n\}$ is a higher derivation.

A higher derivation $\{d_n\}$ is called continuous if each d_n is continuous. It is said to be onto if d_0 is onto.

Lemma 2.1. *If $\{d_n\}$ is a normal higher derivation on a unital C^*-algebra with unit ι, then $d_n(\iota) = 0$ for $n \geq 1$.*

Proof. Since $\{d_n\}$ is normal, d_1 is a dervation and so $d_1(\iota) = 0$. Let $d_j(\iota) = 0$ for $1 \leq j \leq n-1$. Then we have

$$d_n(\iota) = d_n(\iota.\iota) = \iota.d_n(\iota) + \sum_{j=1}^{n-1} d_j(\iota) d_{n-j}(\iota) + d_n(\iota).\iota = d_n(\iota) + d_n(\iota)$$

Hence $d_n(\iota) = 0$. $\square$

From now on, we assume that $\mathcal{A}$ is a unital C^*-algebra. In fact, if $\mathcal{A}$ has no identity, we shall consider the C^*-unitization $\mathcal{A}_1$ of $\mathcal{A}$, and define $d_n(\iota) = 0$ for each n.

Recall that if T is a linear mapping and we define T^* by $T^*(a) = T(a^*)^*$ for all $a \in \mathcal{A}$, then T^* is a linear mapping on $\mathcal{A}$.

Lemma 2.2. *Let $\{d_n\}$ be a higher derivation on a C^*-algebra $\mathcal{A}$. Then $\{d_n^*\}$ is also a higher derivation on $\mathcal{A}$.*

Proof. For each $a, b \in \mathcal{A}$ and $n \in \mathbf{Z}^+$ we have

$$d_n^*(ab) = (d_n(b^*a^*))^* = \left(\sum_{j=0}^{n} d_j(b^*)d_{n-j}(a^*)\right)^* = \sum_{j=0}^{n} d_{n-j}^*(a)d_j^*(b)$$

$=\sum_{k=0}^{n} d_k^*(a)d_{n-k}^*(b).$

Thus $\{d_n^*\}$ is a higher derivation. □

It is known that the derivation $d : \mathcal{C}^1([0,1]) \to \mathcal{C}([0,1])$ defined by $d(f) = f'$ on the dense subalgebra $\mathcal{C}^1([0,1])$ of $\mathcal{C}([0,1])$ is not continuous. So the higher derivation $\{\frac{d^n}{n!}\}$ is an example of a discontinuous densely defined normal higher derivation in the C^*-algebra $\mathcal{C}([0,1])$. In the next section, we will show that this is not the case for everywhere defined higher derivations on C^*-algebras.

3. The Result

Theorem 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra. Then every normal higher derivation $\{d_n\}$ on $\mathcal{A}$ is continuous.*

Proof. For each $n \in \mathbf{Z}^+$ we can write

$$d_n(ab) = \frac{d_n^* + d_n}{2} + i\frac{id_n^* - id_n}{2}.$$

Put $d_n^1 = \frac{d_n^*+d_n}{2}$ and $d_n^2 = \frac{id_n^*-id_n}{2}$. Then d_n^1's and d_n^2's are $*$-mappings and $d_n^1(\iota) = d_n^2(\iota) = 0$ for all $n \in \mathbf{N}$. We also have

$$d_n^1(ab) = ad_n^1(b) + d_n^1(a)b + \frac{1}{2}\sum_{j=1}^{n-1} d_j(a)d_{n-j}(b) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(a)d_{n-j}^*(b),$$

$$d_n^2(ab) = ad_n^2(b) + d_n^2(a)b - \frac{i}{2}\sum_{j=1}^{n-1} d_j(a)d_{n-j}(b) + \frac{i}{2}\sum_{j=1}^{n-1} d_j^*(a)d_{n-j}^*(b).$$

It suffices to show that d_n^1 and d_n^2 are continuous for all $n \in \mathbf{Z}^+$. At first we prove continuity of d_n^1's by induction:

Since $d_0^1 = I$, d_0^1 is continuous. Suppose that d_j^1 is continuous for $j \leq n-1$. Let a be a self-adjoint element of $\mathcal{A}$ and φ be a state on $\mathcal{A}$ such that $|\varphi(a)| = \|a\|$. We may assume that $\varphi(a) = \|a\|$ (If $-\varphi(a) = \|a\|$ then we can write $\varphi(-a) = \|-a\|$ and choose the self-adjoint element $-a$ instead of a). Put $\|a\|\iota - a = h^2$ ($h \geq 0, h \in \mathcal{A}$. Then $\varphi(h^2) = 0$ and

$$\begin{aligned}
& |-\varphi(d_n^1(a)) - \varphi(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h)d_{n-j}(h) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h)d_{n-j}^*(h))| \\
= \ & |\varphi(d_n^1(\|a\|\iota - a)) - \varphi(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h)d_{n-j}(h) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h)d_{n-j}^*(h))| \\
= \ & |\varphi(d_n^1(h^2) - \varphi(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h)d_{n-j}(h) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h)d_{n-j}^*(h))| \\
= \ & |\varphi(hd_n^1(h)) + \varphi(d_n^1(h)h)| \\
\leq \ & \varphi(h^2)^{1/2}\varphi(d_n^1(h)^2)^{1/2} + \varphi(d_n^1(h)^2)^{1/2}\varphi(h^2)^{1/2} \\
= \ & 0.
\end{aligned}$$

Hence $\varphi(d_n^1(a)) = -\varphi(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h)d_{n-j}(h) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h)d_{n-j}^*(h))$. Suppose that $\{a_m\}$ is a sequence of self-adjoint elements in $\mathcal{A}$ such that $a_m \to 0$ and $d_n^1(a_m) \to b(\neq 0)$. Let φ_m be a state on $\mathcal{A}$ such that $|\varphi_m(b + a_m)| = \|b + a_m\|$, and let φ_0 be an accumulation point of $\{\varphi_m\}$ in the state space of $\mathcal{A}$. Then we have

$$\begin{aligned}
|\varphi_{m_k}(b + a_{m_k}) - \varphi_0(b)| &= |\varphi_{m_k}(b + a_{m_k}) - \varphi_{m_k}(b) + \varphi_{m_k}(b) - \varphi_0(b)| \\
&\leq |\varphi_{m_k}(b + a_{m_k}) - \varphi_{m_k}(b)| + |\varphi_{m_k}(b) - \varphi_0(b)| \\
&\leq \|b + a_{m_k} - b\| + |\varphi_{m_k}(b) - \varphi_0(b)| \to 0
\end{aligned}$$

for some subsequence $\{m_k\}$ of $\{m\}$. Hence $|\varphi_0(b)| = \|b\|$ and so

$$\varphi_0(d_n^1(b)) = -\varphi_0\left(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h_b)d_{n-j}(h_b) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h_b)d_{n-j}^*(h_b)\right),$$

where $h_b = (\|b\|\iota - b)^{1/2}$. Similarly one can show that

$$|\varphi_{m_k}(d_n^1(a_{m_k})) - \varphi_0(b)| \to 0.$$

Also if $(h_{b+a_{m_k}})^2 = \|b + a_{m_k}\|\iota - (b + a_{m_k})$ then $h^2_{b+a_{m_k}} \to h_b^2$ and since $h_{b+a_{m_k}}$'s and h_b are positive, $h_{b+a_{m_k}} \to h_b$. So continuity of $d_0^1, d_1^1, \ldots, d_{n-1}^1$ implies that

$$-\varphi_0\left(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h_b)d_{n-j}(h_b) + \frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h_b)d_{n-j}^*(h_b)\right)$$

$=\lim\limits_{m_k\to\infty} -\varphi_{m_k}\left(\frac{1}{2}\sum\limits_{j=1}^{n-1} d_j\left(h_{b+a_{m_k}}\right) d_{n-j}\left(h_{b+a_{m_k}}\right)\right.$
$\left.+\frac{1}{2}\sum_{j=1}^{n-1} d_j^*\left(h_{b+a_{m_k}}\right) d_{n-j}^*\ h_{b+a_{m_k}}\right)$
$=\lim_{m_k\to\infty}\varphi_{m_k}(d_n^1(b+a_{m_k}))$
$=\lim_{m_k\to\infty}\varphi_{m_k}(d_n^1(b)+d_n^1(a_{m_k}))$
$=\varphi_0(d_n^1(b)+\varphi_0(b))$
$=-\varphi_0\left(\frac{1}{2}\sum_{j=1}^{n-1} d_j(h_b)d_{n-j}(h_b)+\frac{1}{2}\sum_{j=1}^{n-1} d_j^*(h_b)d_{n-j}^*(h_b)\right)+\varphi_0(b).$

Hence $\varphi_0(b)=0$, which is a contradiction. So the closed graph theorem guarantees that d_n^1 is continuous.

Similarly we can show that d_n^2's are continuous. Whence the continuity of the higher derivation $\{d_n\}$ is deduced. □

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