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SOME DIFFERENCE SEQUENCES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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Abstract

The idea of difference sequence spaces was introduced by Kizmaz [6], and this concept was generalized by Bektas and Colak [1]. In this paper, we define the sequence spaces $c_0(F, \Delta_u^m x)$ *and* $l_\infty(F, \Delta_u^m x)$*, where* $F = (f_k)$ *is a sequence of modulus functions, and examine some inclusion relations and properties of these spaces.*

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1. Definitions and notations

Let w denote the set of all complex sequences $x = (x_k)$, and l_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by $\| x \|= \sup_k | x_k |$, where $k \in \mathbf{N}$, the set of positive integers.

Kizmaz [6] defined the sequence spaces

$l_\infty(\Delta) = \{x \in w : \Delta x \in l_\infty\}$,

$c(\Delta) = \{x \in w : \Delta x \in c\}$, and

$c_0(\Delta) = \{x \in w : \Delta x \in c_0\}$, where for any sequence $x = (x_k)$, the difference sequence Δx is defined by

$\Delta x = (\Delta x_k)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$,

$\Delta^0 x = (\Delta^0 x_k)_{k=1}^{\infty} = (x_k)_{k=1}^{\infty}$.

Kizmaz [6] proved that these are Banach spaces with the norm $\| x \|=| x_1 | + \| x \|_\infty$. Also, he showed that $E \subset E(\Delta)$, where $E = \{l_\infty, c, c_0\}$, since there exists a sequence $x = (x_k)$ such that $x_k = k$, for each k,that is , $x = (1, 2, 3, \cdots)$ for which $\Delta x = (-1, -1, -1, \cdots)$, so that although x is not convergent, but it is $\Delta-$convergent.

If m is a nonnegative integer and $x = (x_k)$ is any sequence, then the difference sequences $\Delta x, \Delta^2 x, \cdots, \Delta^m x$ are defined by

$\Delta x = (\Delta x_k)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$,

$\Delta^2 x = \Delta(\Delta x) = \Delta x_k - \Delta x_{k+1}$,

$\vdots$

$\Delta^m x = \Delta(\Delta^{m-1} x) = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$,

so that

$\Delta^m x_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} x_{k+r}$.

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for each k. Then Gnanaseelan and Srivastava [4] defined and studied the sequence spaces

$E(u,\Delta) = \{x \in w : u\Delta x \in E\}$,

where $E = \{l_\infty, c, c_0\}$ and $u\Delta x = u_k x_k - u_{k+1} x_{k+1}$.

After then, Et. and Colak [2] defined the sequence spaces

$E(\Delta^m) = \{x \in w : \Delta^m x \in E\}$,

where $E = \{l_\infty, c, c_0\}$.

They proved that these are Banach spaces with the norm $\| x \|= \sum_{i=1}^{m} | x_i | + \| \Delta^m x \|_\infty$.

We recall that a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Since $\mid f(x) - f(y) \mid \leq f(\mid x - y \mid)$, it follows from conditions (ii) and (iv) that f is continuous everywhere on $[0, \infty)$.

A modulus function may be bounded or unbounded. For example, $f(t) = \frac{t}{t+1}$ is bounded but $f(t) = t^p$ $(0 < p \leq 1)$ is unbounded. Furthermore, we have $f(nx) \leq nf(x)$, from condition (ii), and so $f(x) = f(nx\frac{1}{n}) \leq nf(\frac{x}{n})$, hence $\frac{1}{n}f(x) \leq f(\frac{x}{n})$, for all $n \in \mathbf{N}$.

Ruckle [13] used the concept of modulus function to define the sequence space

L(f)=$\{$x$\in w : \sum_{k=1}^{\infty} f(\mid x_k \mid) < \infty\}$.

If we take $f(x) = x^p$, then $L(f)$ reduces to the familiar space l_p which is given by

$l_p = \{x \in w : \sum_{k=1}^{\infty} \mid x_k \mid^p < \infty\}$.

In particular, if $f(x) = x$, then $L(f) = l_1$ which is given by

$l_1 = \{x \in w : \sum_{k=1}^{\infty} \mid x_k \mid < \infty\}$.

Several authors including Maddox ([8]-[11]), Ozturk and Bilgin [12], and some others studied some sequence spaces defined by a modulus function. Let X be a sequence space. Then the sequence space $X(f)$ is defined by

X(f)=$\{$x$\in w : f(\mid x_k \mid) \in X\}$.

Kolk [7] gave an extension of $X(f)$ by considering a sequence of modulus functions $F = (f_k)$ and defined the space

X(F)=$\{$x$\in w : f_k(\mid x_k \mid) \in X\}$.

Gaur and Mursaleen [3] defined and studied the following sequence spaces

$l_\infty(F, \Delta) = \{x \in w : \Delta x \in l_\infty(F)\}$,

and

$c_0(F, \Delta) = \{x \in w : \Delta x \in c_0(F)\}$.

For a nonnegative integer m, Bektas and Colak [1] extended the above mentioned spaces to

$l_\infty(F, \Delta^m) = \{x \in w : \Delta^m x \in l_\infty(F)\}$,

and

$c_0(F, \Delta^m) = \{x \in w : \Delta^m x \in c_0(F)\}$.

We further give an extension of the spaces of Bektas and Colak [1] and define the sequence spaces

$l_\infty(F, \Delta_u^m) = \{x \in w : \Delta_u^m x \in l_\infty(F)\}$,
and
$c_0(F, \Delta_u^m) = \{x \in w : \Delta_u^m x \in c_0(F)\}$,
where $u = (u_k)$ is any sequence such that $u_k \neq 0$ for each k,and
$\Delta_u^0 x = u_k x_k$,
$\Delta_u^1 x = u_k x_k - u_{k+1} x_{k+1}$,
$\Delta_u^2 x = \Delta(\Delta_u^1 x)$,
$\vdots$
$\Delta_u^m x = \Delta(\Delta_u^{m-1} x)$,
so that
$\Delta_u^m x = \Delta_{u_k}^m x_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} u_{k+r} x_{k+r}$.

If $u = e = (1, 1, 1, \cdots)$, then these spaces will give the spaces of Bektas and Colak [1] as special cases.

2. Main results

For a sequence $F = (f_k)$ of modulus functions, we will give the necessary and sufficient conditions for the inclusion between $X(\Delta_u^m)$ and $Y(F, \Delta_u^m)$, where $X, Y = l_\infty$ or c_0.

We need the following Lemmas (see Kolk [7]) :

Lemma 2.1. *The condition* $\sup_k f_k(t) < \infty, t > 0$ *holds if and only if there exists a point* $t_0 > 0$ *such that* $\sup_k f_k(t_0) < \infty$.

Lemma 2.2. *The condition* $\inf_k f_k(t) > 0, t > 0$ *holds if and only if there exists a point* $t_0 > 0$ *such that* $\inf_k f_k(t_0) > 0$.

Now, we prove the following theorems

Theorem 2.3. *For a sequence* $F = (f_k)$ *of modulus functions, the following statements are equivalent :*

(i) $l_\infty(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$
(ii) $c_0(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$
(iii) $\sup_k f_k(t) < \infty, t > 0$

Proof. (i) implies (ii) is obvious

(ii) implies (iii) : Let $c_0(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$. Suppose that (iii) is not true. Then by Lemma 2.1 , we see that $\sup_k f_k(t) = \infty$, for all $t > 0$, and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(\frac{1}{i}) > i \text{ for } i = 1, 2, 3, \cdots \tag{2.1}$$

Define $x = (x_k)$ as follows

$$x_k = \begin{cases} \frac{1}{i}, & k = k_i \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in c_0(\Delta_u^m)$, but by (2.1), $x \notin l_\infty(F, \Delta_u^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) implies (i) : Let (iii) be satisfied and $x \in l_\infty(\Delta_u^m)$. If we suppose that $x \notin l_\infty(F, \Delta_u^m)$, then $\sup_k f_k(\mid \Delta_u^m x_k \mid) = \infty$ for $\Delta_u^m x \in l_\infty$. Now take $t = \mid \Delta_u^m x \mid$, then $\sup_k f_k(t) = \infty$ which contradicts (iii). Hence $l_\infty(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$. This completes the proof of the theorem. □

Theorem 2.4. *For a sequence $F = (f_k)$ of modulus functions, the following statements are equivalent :*

(i) $c_0(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$
(ii) $c_0(F, \Delta_u^m) \subseteq l_\infty(\Delta_u^m)$
(iii) $\inf_k f_k(t) > 0$, $(t > 0)$

Proof. (i) implies (ii) is obvious

(ii) implies (iii) : Let $c_0(F, \Delta_u^m) \subseteq l_\infty(\Delta_u^m)$. Suppose that (iii) does not hold. Then by Lemma 2.2, we see that

$$\inf_k f_k(t) = \infty, (t > 0) \tag{2.2}$$

and therefore there is a sequence (k_i) of positive integers such that
$f_{k_i}(i^2) < \frac{1}{i}$ for $i = 1, 2, 3, \cdots$.
Define $x = (x_k)$ as follows

$$x_k = \begin{cases} i^2, & k = k_i \text{ for } i = 1, 2, 3, \cdots \\ 0, & \text{otherwise} \end{cases} .$$

Then by (2.2), $x \in c_0(F, \Delta_u^m)$ but $x \notin l_\infty(\Delta_u^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) implies (i) : Let (iii) be satisfied and $x \in c_0(F, \Delta_u^m)$ that is $\lim_k f_k(\mid \Delta_u^m x_k \mid) = 0$. Suppose that $x \notin c_0(\Delta_u^m)$, then for some number $\varepsilon_0 > 0$ and positive integer k_0, we have $\mid \Delta_u^m x_k \mid \varepsilon_0 > 0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq$

$f_k(\mid \Delta_u^m x_k \mid)$ for $k \geq k_0$ and hence $\lim_k f_k(\varepsilon_0) = 0$ which contradicts (iii). Thus $c_0(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$ and this completes the proof of the theorem. □

Theorem 2.5. *The inclusion $l_\infty(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$ holds if and only if*

$$\lim_k f_k(t) = \infty \text{ for } t > 0 \tag{2.3}$$

Proof. Let $l_\infty(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$ and suppose that (2.3) does not hold . Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integers such that

$$f_{k_i}(t_0) \leq L < \infty \tag{2.4}$$

Define the sequence $x = (x_k)$ by

$$\mathrm{x}_k = \begin{cases} t_0, & k = k_i \text{ for } i = 1, 2, 3, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in l_\infty(F, \Delta_u^m)$ by (2.4) but $x \notin c_0(\Delta_u^m)$ so that (2.4) must hold if $l_\infty(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$.

Conversely, Suppose that (2.3) is satisfied. If $x \in l_\infty(F, \Delta_u^m)$, then $f_k(\mid \Delta_u^m x_k \mid) \leq L < \infty$ for $k = 1, 2, 3, \cdots$

Now suppose that $x \notin c_0(\Delta_u^m)$. Then for some $\varepsilon_0 > 0$ and positive integer k_0, we have $\mid \Delta_u^m x_k \mid > \varepsilon_0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq f_k(\mid \Delta_u^m x_k \mid) \leq L$ for $k \geq k_0$ which contradicts (2.3). Hence $x \in c_0(\Delta_u^m)$. □

Theorem 2.6. *The inclusion $l_\infty(\Delta_u^m) \subseteq c_0(F, \Delta_u^m)$ holds if and only if*

$$\lim_k f_k(t) = 0 \text{ for } t > 0 \tag{2.5}$$

Proof. Let $l_\infty(\Delta_u^m) \subseteq c_0(F, \Delta_u^m)$ and suppose that (2.5) does not hold . Then

$$\lim_k f_k(t) = l \neq 0 \tag{2.6}$$

for some $t_0 > 0$.

Define the sequence $x = (x_k)$ by

$x_k = t_0 \sum_{r=0}^{m} (-1)^m \binom{m+k-u-1}{k-u}$ for $k = 1, 2, 3, \cdots$.

Then $x \notin c_0(F, \Delta_u^m)$ by (2.6). Hence (2.5) must hold.

Conversely, Suppose that (2.5) is satisfied. If $x \in l_\infty(\Delta_u^m)$, then $|\Delta_u^m x_k| \leq L < \infty$ for $k = 1, 2, 3, \cdots$.

Therefore $f_k(|\Delta_u^m x_k|) \leq f_k(L)$ for $k = 1, 2, 3, \cdots$ and $\lim_k f_k(|\Delta_u^m x_k|) \leq \lim_k f_k(L) = 0$, by (2.5). Hence $x \in c_0(F, \Delta_u^m)$. $\square$

References

[1] Bektas, Cigdem, A. and Colak, R., Generalized difference sequence spaces defined by a sequence of modulli, Soochow J. Math., 29(2), pp. 215-220, (2003).

[2] Et., M. and Colak, R., On some generalized difference sequence spaces, Soochow J. Math., 21, pp. 377-386, (1995).

[3] Gaur, A. K. and Mursaleen, Difference sequence spaces, Internat. J. Math. & Math. Sci., 21(4), pp. 701-706, (1998).

[4] Gnanaseelan, C. and Srivastava, P. D., The $\alpha-, \beta-, \gamma-$duals of some generalized difference sequence spaces, Indain J. Math., 38(2), pp. 111-120, (1996).

[5] Kamthan, P. K. and Gupta, M., Sequence spaces and series, Lecture Notes in Pure and Applied Mathematics, vol. 65 , Marcel Dekker, New York, (1981).

[6] Kizmaz, H., On certain sequence spaces, Canad. Math. Bull. 24, pp. 169-176, (1981).

[7] Kolk, E., Inclusion theorems for some sequence spaces defined by a sequence of modulli, Acta Comment. Univ. Tartu, 970, pp. 65-72, (1994).

[8] Maddox, I. J., Some properties of paranormed sequence spaces, J. London Math. Soc. 1, pp. 316-322, (1969).

[9] Maddox, I. J. , Paranormed sequence spaces generated by infinite matrices, Proc. Camb. Phil. Soc., 64, pp. 431-435, (1969).

[10] Maddox, I. J., Elements of Functional Analysis, 2nd Edition, Cambridge University Press, (1970).

[11] Maddox, I. J., Sequence spaces defined by a modulus, Math. Proc. Camb. Phil. Soc. 100, pp. 161-166, (1986).

[12] Ōztürk, E. and Bilgen, T., Strongly summable sequence spaces defined by a modulus, Indian J. Pure Appl. Math. 25(6), pp. 621-625, (1994).

[13] Ruckle, W. H., FK spaces in which the sequence of coordinate vectors is bounded, Canad. J. Math. 25, pp. 973-978, (1973).

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