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On the instability of solutions of an eighth order nonlinear differential equation of retarded type

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Abstract

In this paper, we give some sufficient conditions on the instability of the zero solution of a kind of eighth order nonlinear differential equations of retarded type by using the Lyapunov direct method. The obtained sufficient conditions improve an existing result in the literature.

Key words : *Instability; the Lyapunov direct method; delay differential equation; eighth order.*

AMS Classification numbers : *34K20.*

1. Introduction

Since 1991 by now, the instability of solutions for various nonlinear scalar and vector differential equations of eighth order without delay has only been investigated by a few authors, see, Bereketoglu [1], Tunç [6, 7], C. Tunç and E. Tunç [8] and the references contained therein. In addition, in 1996, Iyase [4] proved a result on the nonexistence of nontrivial periodic solutions to the nonlinear eighth order scalar differential equation without delay

$$x^{(8)} + a_1x^{(7)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5x''' + f_6(x')x'' + f_7(x)x' + f_8(x) = 0. \quad (1.1)$$

Consider the linear constant coefficient differential equation of eighth order

$$x^{(8)} + a_1x^{(7)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5x''' + a_6x'' + a_7x' + a_8x = 0. \quad (1.2)$$

It is well known that the auxiliary equation of Eq. (1.2) is given by

$$\psi(\lambda) \equiv \lambda^8 + a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 + a_6\lambda^2 + a_7\lambda + a_8 = 0.$$

If β is an arbitrary real number, then the real part of $\psi(i\beta)$ is given by

$$\phi(\beta) = \beta^8 - a_2\beta^6 + a_4\beta^4 - a_6\beta^2 + a_8.$$

It is also well known that if

$$a_2 \leq 0, a_4 \geq 0, a_6 \leq 0, a_8 > 0$$

in which case $\phi(\beta) > 0$, then the auxiliary equation cannot have any purely imaginary root whatever. It therefore follows from general theory that Eq. (1.2) does not have a periodic solution except $x = 0$. An analogous consideration of the imaginary part of $\psi(i\beta)$ also leads to conditions on a_1 , a_3 , a_5 and a_7 for the nonexistence of any periodic solution of Eq. (1.2) other than $x = 0$.

In this paper, instead of Eq. (1.1), we consider the eighth order nonlinear functional differential equation of delay type

$$(1.3) \quad x^{(8)} + a_1 x^{(7)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 x''' + f_6(x')x'' + f_7(x)x' + f_8(x, x(t-r), x', x'(t-r), \dots, x^{(7)}, x^{(7)}(t-r)) = 0.$$

We write Eq. (1.4) in system form as

$$(1.4) \quad \begin{aligned} x'_1 &= x_2, x'_2 = x_3, x'_3 = x_4, x'_4 = x_5, x'_5 = x_6, x'_6 = x_7, x'_7 = x_8, \\ x'_8 &= -a_1 x_8 - a_2 x_7 - a_3 x_6 - a_4 x_5 - a_5 x_4 - f_6(x_2)x_3 - f_7(x_1)x_2 \\ &\quad - f_8(x_1, x_1(t-r), x_2, x_2(t-r), \dots, x_8, x_8(t-r)), \end{aligned}$$

which was obtained as usual by setting $x = x_1$, $x' = x_2$, $x'' = x_3$, $x''' = x_4$, $x^{(4)} = x_5$, $x^{(5)} = x_6$, $x^{(6)} = x_7$ and $x^{(7)} = x_8$ in (1.3), where r is a positive constant, a_1, a_2, a_3, a_4 and a_5 are some constants, the primes in Eq. (1.3) denote differentiation with respect to t , $t \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty)$; f_6, f_7 and f_8 are continuous functions on $\mathbb{R}, \mathbb{R}, \mathbb{R}^{16}$, respectively, with $f_8(0, x_1(t-r), \dots, x_8(t-r)) = 0$, and satisfy a Lipschitz condition in their respective arguments. Hence, the existence and uniqueness of the solutions of Eq. (1.3) are guaranteed (see El'sgol'ts [2, pp.14, 15]). We assume in what follows that $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t), x_7(t)$ and $x_8(t)$ are abbreviated as $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 , respectively.

The aim of this paper is to improve the result established in [4, Theorem 1] to the eighth order nonlinear delay differential equation (1.3) for the instability of its zero solution. Our motivation comes from the papers [1, 3, 4, 5, 6, 7, 8, 9].

As known, in general, it is not possible to solve all linear and nonlinear differential equations, except numerically. Therefore, it is very important obtaining information for the qualitative behaviors of solutions when there is no analytical expression for solutions of any differential equation under investigation. It should be noted that the Lyapunov's second (or direct) method is the most effective tool to discuss the instability of solutions of nonlinear differential equations of higher order, when there is no analytical expression for solutions. The technique is also called the direct method, because it can be applied directly to the differential equation under study, without any knowledge of solutions, provided the person using the method is clever to construct or define an appropriate the right auxiliary function, the Lyapunov function or functional. Here, by means of the Lyapunov direct method, we prove a theorem for the zero solution of Eq. (1.3) to

be unstable. That is to say that, by defining a Lyapunov function, we will accomplish the result to be established in this work.

Let $r \geq 0$ be given, and let $C = C([-r, 0], \mathbb{R}^n)$ with

$$\|\phi\| = -r \leq s \leq 0 \max |\phi(s)|, \quad \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, a] \rightarrow \mathbb{R}^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \rightarrow \mathbb{R}^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(\cdot)$ so that $x_0(\phi) = \phi$.

Definition. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

Theorem A. Suppose there exists a Lyapunov function $V : G \rightarrow \mathbb{R}_+$ such that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. If either

(i) $\dot{V}(\phi) > 0$ for all ϕ in G for which

$$V[\phi(0)] = -s \leq t \leq 0 \max V[\phi(s)] > 0$$

or

(ii) $\dot{V}(\phi) > 0$ for all ϕ in G for which

$$V[\phi(0)] = -s \leq t \leq 0 \min V[\phi(s)] > 0,$$

then $x = 0$ of $\dot{x} = F(x_t)$ is unstable (see Haddock and Ko [3]).

2. The main results

The main result of this paper is given by the following theorem.

Theorem. Together with all the assumptions imposed to the functions f_6 , f_7 and f_8 in Eq. (1.3), we assume that there exist constants a_1 , a_2 and a_4 such that the conditions

$$a_1 > 0, a_2 < 0, a_4 > 0, f_6(x_2) \leq 0$$

$$x_1 f_8(x_1, \dots, x_8(t-r)) > 0, (x_1 \neq 0),$$

hold for arbitrary $x_1, x_1(t-r), \dots, x_8(t-r)$. Then, the zero solution, $x = 0$, of Eq. (1.3) is unstable.

Remark. Note that there is not any restriction on the function f_7 . Next, the proof of the above theorem is based on the instability criteria of Krasovskii [5]. According to these criteria, it is necessary to show here that there exists a Lyapunov function $V \equiv V(x_1, x_2, \dots, x_8)$ which has the following three properties, Krasovskii properties, say (K_1) , (K_2) and (K_3) :

(K_1) In every neighborhood of $(0, 0, 0, 0, 0, 0, 0, 0)$, there exists a point $(\xi_1, \xi_2, \dots, \xi_8)$ such that $V(\xi_1, \xi_2, \dots, \xi_8) > 0$,

(K_2) the time derivative $\dot{V} \equiv \frac{d}{dt}(x_1, x_2, \dots, x_8)$ along solution paths of (1.4) is positive semi-definite,

(K_3) the only solution $(x_1, x_2, \dots, x_8) = (x_1(t), x_2(t), \dots, x_8(t))$ of (1.4) which satisfies

$\frac{d}{dt}V(x_1, x_2, \dots, x_8) = 0, (t \geq 0)$, is the trivial solution $(0, 0, 0, 0, 0, 0, 0, 0)$.

Proof. Consider the function $V = V(x_1, x_2, \dots, x_8)$ defined by

$$\begin{aligned} V &= -x_1 x_8 - a_1 x_1 x_7 - a_2 x_1 x_6 - a_3 x_1 x_5 - a_4 x_1 x_4 - a_5 x_1 x_3 \\ &\quad + x_2 x_7 + a_1 x_2 x_6 + a_2 x_2 x_5 + a_3 x_2 x_4 + a_4 x_2 x_3 \\ &\quad + \frac{1}{2} a_5 x_2^2 - x_3 x_6 - a_1 x_3 x_5 - a_2 x_3 x_4 - \frac{1}{2} a_3 x_3^2 \\ (2.1) \quad &\quad + x_4 x_5 + \frac{1}{2} a_1 x_4^2 - x_1 \int_0^{x_2} f_6(s) ds - \int_0^{x_1} f_7(s) s ds. \end{aligned}$$

It follows that

$$V(0, 0, 0, 0, 0, 0, 0, 0) = 0$$

and

$$V(0, 0, 0, \varepsilon, 0, 0, 0, 0) = \frac{1}{2} a_1 \varepsilon^2 > 0$$

for all sufficiently small ε so that every neighborhood of the origin in the (x_1, x_2, \dots, x_8) -space contains points $(\xi_1, \xi_2, \dots, \xi_8)$ such that $V(\xi_1, \xi_2, \dots, \xi_8) > 0$, which verifies the property (K_1) of Krasovskii [5].

Let

$$(x_1, x_2, \dots, x_8) = (x_1(t), x_2(t), \dots, x_8(t))$$

be an arbitrary solution of (1.4).

Along solutions of the system (1.4), we obtain the time derivative of the function V in (2.1) as the following:

$$\dot{V} \equiv \frac{dV}{dt} = x_5^2 - a_2 x_4^2 + a_4 x_3^2 + f_8(x_1, \dots, x_8(t-r))x_1 - x_2 \int_0^{x_2} f_6(s)ds.$$

In view of the assumptions $a_2 < 0$, $a_4 > 0$, $f_6(x_2) \leq 0$ and $x_1 f_8(\cdot) > 0$, ($x_1 \neq 0$), it follows that

$$\dot{V} = x_5^2 - a_2 x_4^2 + a_4 x_3^2 + f_8(x_1, \dots, x_8(t-r))x_1 - x_2 \int_0^{x_2} f_6(s)ds > 0.$$

Thus, if the assumptions of the theorem hold then \dot{V} is positive semi-definite, which verifies the property (K_2) of Krasovskii [5].

On the other hand, $\dot{V} = 0$ for all $t \geq 0$ necessarily implies that $x_1 = 0$ and therefore also that

$$x_1 = x = 0, x_2 = x' = 0, x_3 = x'' = 0, x_4 = x''' = 0, x_5 = x^{(4)} = 0,$$

$$x_6 = x^{(5)} = 0, x_7 = x^{(6)} = 0, x_8 = x^{(7)} = 0$$

for all $t \geq 0$. Hence,

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0, (t \geq 0).$$

Furthermore, in view of $\frac{d}{dt}V(x_1, x_2, \dots, x_8) = 0$ and the system (1.4), we can also easily obtain $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0$ since $f_8(\xi_1, \dots, \xi_8(t-r)) = 0$ if and only if $\xi_1 = 0$, which verifies the property (K_3) of Krasovskii [5]. It now follows that the Lyapunov function V thus has all the requisite Krasovskii properties subject to the conditions in the theorem. By the above discussion, we conclude that the zero solution of Eq. (1.3) is unstable. The proof of the theorem is completed.

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