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# Statistical convergence of complex uncertain sequences defined by Orlicz function

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## Abstract:

*Complex uncertain variables are measurable functions from an uncertainty space to the set of complex numbers and are used to model complex uncertain quantities. This paper introduces the statistical convergence concepts of complex uncertain sequences: statistical convergence almost surely(a.s.), statistical convergence in measure, statistical convergence in mean, statistical convergence in distribution and statistical convergence uniformly almost surely sequences of complex uncertain sequences defined by Orlicz function. In addition, Decomposition Theorems and relationships among them are discussed.*

**Keywords:** Uncertainty theory; Complex uncertain variable; Statistical convergence.

**MSC (2010):** 40A05, 40A35, 46E30, 60B10, 60B12, 60F17.

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## 1. Introduction and Preliminaries

Uncertainty is an extremely important feature of the real world. How do we understand uncertainty? How do we model uncertainty? In order to answer those questions, an uncertainty theory was founded by Liu [8] in 2007 and refined by Liu [9] in 2009. Nowadays uncertainty theory has become a branch of mathematics for modeling human uncertainty.

Let $\omega$ be the family of all real or complex sequences. Any subspace of $\omega$ is called sequence space.

**Definition 1.1.** An Orlicz function is a function $\mathcal{M} : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$ for $x > 0$ and $\mathcal{M}(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $\mathcal{M}$ is replaced by

$$\mathcal{M}(x + y) \leq \mathcal{M}(x) + \mathcal{M}(y),$$

then this function is called Modulus function.

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct the sequence space

$$\ell_{\mathcal{M}} = \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty, \quad for\ some\ \rho > 0 \right\}.$$

The space $\ell_{\mathcal{M}}$ with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space, which is called an Orlicz sequence space. Lindenstrauss and Tzafriri [6] proved that every Orlicz sequence space $\ell_{\mathcal{M}}$ contains a subspace isomorphic to $c_0$ or some $\ell_p$, positively for a class of spaces.

The space $\ell_{\mathcal{M}}$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $\mathcal{M}(x) = x^p; 1 \leq p \leq \infty$.

The concept of Orlicz function has been applied for studying different classes of sequences by Krasnoselskii and Rutitsky [5], Lindenstrauss [7], Et et.al [10], Tripathy and Dutta [14], Tripathy and Dutta [15], Tripathy and Goswami [16], Tripathy and Mahanta [17, 18] and others.

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [3] in 1951, Buck [1] in 1953 and Schoenberg [13] in 1959 independently. Later on it was studied from sequence space point of view and linked with summability theory by Fridy [4], $\bar{S}al\bar{a}t$ [12] and many others.

The notion of statistical convergence depends on the notion of asymptotic density of subsets of the set $N$ of natural numbers.

For any subset $A$ of $N$, we say that $A$ possesses asymptotic density(or, simply density) $\delta(A)$ if $\delta(A) = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\chi_A(k)$ exists, where $\chi_A$ is the characteristic function of $A$.

Clearly all finite subsets of $N$ have zero natural density and $\delta(A^c) = \delta(N - A) = 1 - \delta(A)$.
A given complex sequence $x = (x_k)$ is said to be statistically convergent to $L$, if for any $\varepsilon > 0$, we have $\delta(\{k \in N : |x_k - L| \geq k\}) = 0$. We write $x_k \to^{stat} L$ or $stat - \lim x_k = L$.

In this section, we introduce some concepts and theorems of complex uncertain variables those were first proposed by Peng [11].

As a complex function on uncertainty space, complex uncertain variable is mainly used to model a complex uncertain quantity.

**Definition 1.2.**(Peng [11]) A complex uncertain variable is a measurable function $\zeta$ from an uncertainty space $(\Gamma, L, M)$ to the set of complex numbers, i.e., for any Borel set $B$ of complex numbers, the set

$$\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\}$$

is an event.

**Definition 1.3.**(Peng [11]) The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable $\zeta$ is a function from $C$ to $[0, 1]$ defined by

$$\Phi(c) = M\{Re(\zeta) \leq Re(c), Im(\zeta) \leq Im(c)\}$$

for any complex $c$.

## 2. Convergence Concepts of Complex Uncertain Sequences

Complex uncertain sequence is a sequence of complex uncertain variables indexed by integers. In this section, we discuss about five convergence concepts of complex uncertain sequence: convergence almost surely(a.s.), convergence in measure, convergence in mean, convergence in distribution and convergence uniformly almost surely(a.s.).

**Definition 2.1.**(Chen, Ning, Wang [2]) The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent almost surely(a.s.) to* $\zeta$ if there exists an event $\Lambda$ with $M\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$. In that case we write $\zeta_n \to \xi$, a.s.

**Definition 2.2.**(Chen, Ning, Wang [2]) The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in measure to* $\zeta$ if

$$\lim_{n\to\infty} M\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0,$$

for every $\varepsilon > 0$.

**Definition 2.3.**(Chen, Ning, Wang [2]) The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in mean to* $\zeta$ if

$$\lim_{n\to\infty} E[\|\zeta_n - \zeta\|] = 0.$$

**Definition 2.4.**(Chen, Ning, Wang [2]) Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. We say the complex uncertain sequence $\{\zeta_n\}$ *converges in distribution to* $\zeta$ if

$$\lim_{n\to\infty} \Phi_n(c) = \Phi(c),$$

for all $c$ at which $\Phi(c)$ is continuous.

**Definition 2.5.**(Chen, Ning, Wang [2]) The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent uniformly almost surely(a.s.) to* $\zeta$ if there exists an sequence of events $\{E'_k\}$, $M\{E'_k\} \to 0$ such that $\{\zeta_n\}$ converges uniformly to $\zeta$ in $\Gamma - E'_k$, for any fixed $k \in N$.

## 3. Main Results

In this section we define the statistical version of sequence spaces given by Orlicz function for uncertain variables.

**Definition 3.1.** The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *statistically convergent almost surely(s.a.s.) to* $\zeta$ is

$$c(\mathcal{M}; s.a.s) = \left\{ \{\zeta_n\} : \zeta_n \to^{s.a.s} \zeta \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0 \right\}.$$

**Definition 3.2.** The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *statistically convergent in measure(s.m) to* $\zeta$ is

$$c(\mathcal{M}; s.m) = \left\{ \{\zeta_n\} : \zeta_n \to^{s.m} \zeta \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0 \right\}.$$

**Definition 3.3.** The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *statistically convergent in mean(s.mean) to* $\zeta$ is

$$c(\mathcal{M}; s.mean) = \left\{ \{\zeta_n\} : \zeta_n \to^{s.mean} \zeta \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0 \right\}.$$

**Definition 3.4.** Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. Then the sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *statistically converges in distribution(s.dis) to* $\zeta$ is

$$c(\mathcal{M}; s.dis) = \left\{ \{\zeta_n\} : \lim_{n\to\infty} \Phi_n(c) = \Phi(c) \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0 \right\}.$$

**Definition 3.5.** The sequence spaces given by Orlicz function for the complex uncertain sequences $\{\zeta_n\}$ which are *statistically convergent uniformly almost surely(s.u.a.s.) to* $\zeta$ is

$$c(\mathcal{M}; u.a.s) = \left\{ \{\zeta_n\} : \zeta_n \to^{u.a.s} \zeta \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0 \right\}.$$

We establish the relationship between the above classes of sequences in this section.

**Theorem 3.1.** $c(\mathcal{M}; s.mean) \subseteq c(\mathcal{M}; s.m)$.

**Proof.** Let $\zeta_n \in c(\mathcal{M}; s.mean)$. Then by definition there exists $\zeta \in c(\mathcal{M}; s.mean)$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : E[\|\zeta_k - \zeta\|] \geq \varepsilon\}| = 0,$$

for every

$$\varepsilon > 0 \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0.$$

It follows from the Markov inequality that for any given $\varepsilon, \delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M(\|\zeta_k - \zeta\| \geq \varepsilon) \geq \delta\}| \leq \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \left(\frac{E(\|\zeta_k - \zeta\|)}{\varepsilon}\right) \geq \delta\}|.$$

Thus $\{\zeta_n\}$ converges in measure to $\zeta$. Hence we get

$$\zeta_n \to^{s.m} \zeta \ and \ \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty, \ for \ some \ \rho > 0.$$

This proves the theorem.

**Remark 3.1.** Converse of above theorem is not true. i.e. $c(\mathcal{M}; s.m) \subset c(\mathcal{M}; s.mean)$ (strict inclusion). Following example illustrate this.

**Example 3.1.** Consider the uncertainty space $(\Gamma, L, M)$ to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{1}{(n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{1}{(n+1)} < 0.5 \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{1}{(n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{1}{(n+1)} < 0.5; \\ 0.5; & \text{otherwise,} \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n; \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. For some small number $\varepsilon, \delta > 0$ and $n \geq 2$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}|$$
$$= \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right) \geq \delta\}|$$
$$= \lim_{n\to\infty} \frac{1}{n}|\{k \in N : M\{\gamma_n\} \geq \delta\}| = 0.$$

thus, the sequence $\{\zeta_n\}$ statistically converges in measure to $\zeta$. However, for each $n \geq 2$, we have the uncertainty distribution of uncertain variable $\|\zeta_n - \zeta\| = \|\zeta_n\|$ is

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{n+1}, & \text{if } 0 \leq x < n+1; \\ 1, & x \geq n+1. \end{cases}$$

So for each $n \geq 2$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : E[\|\zeta_n - \zeta\| - 1]\}| = \left[\int_0^{n+1} 1 - (1 - \frac{1}{n+1})dx\right] - 1 = 0.$$

That is, the sequence $\{\zeta_n\}$ does not statistically converge in mean to $\zeta$.

Hence the result follows.

**Theorem 3.2.** $c(\mathcal{M}; s.m) \subseteq c(\mathcal{M}; s.dis)$.

**Proof.** Let $c = a+ib$ be a given continuity point of the complex uncertainty distribution $\Phi$. On the one hand, for any $\alpha > a, \beta > b$, we have

$$\{\xi_n \leq a, \eta_n \leq b\} = \{\xi_n \leq a, \eta_n \leq b, \xi \leq \alpha, \eta \leq \beta\} \cup \{\xi_n \leq a, \eta_n \leq b, \xi > \alpha, \eta > \beta\}$$
$$\cup \{\xi_n \leq a, \eta_n \leq b, \xi \leq \alpha, \eta > \beta\} \cup \{\xi_n \leq a, \eta_n \leq b, \xi > \alpha, \eta \leq \beta\}$$
$$\subset \{\xi \leq \alpha, \eta \leq \beta\} \cup \{|\xi_n - \xi| \geq \alpha - a\} \cup \{|\eta_n - \eta| \geq \beta - b\}.$$

It follows from the subadditivity axiom that

$$\Phi_n(c) = \Phi_n(a+ib) \leq \Phi(\alpha+i\beta) + M\{|\xi_n - \xi| \geq \alpha - a\} + M\{|\eta_n - \eta| \geq \beta - b\}.$$

Since $\{\xi_n\}$ and $\{\eta_n\}$ statistically converges in measure to $\xi$ and $\eta$, respectively, so for any small number $\varepsilon > 0$ we have

$\lim_{n\to\infty}\frac{1}{n}|\{k \le n : M(\|\xi_k - \xi\| \ge \alpha - a) \ge \varepsilon\}| = 0$ and $\lim_{n\to\infty}\frac{1}{n}|\{k \le n : M(\|\xi_k - \xi\| \ge \beta - b) \ge \varepsilon\}| = 0$.

Thus we obtain $\limsup_{n\to\infty}\Phi_n(c) \le \Phi(\alpha + i\beta)$ for any $\alpha > a, \beta > b$. Taking $\alpha + i\beta \to a + ib$, we get

$$\limsup_{n\to\infty}\Phi_n(c) \le \Phi(c). \tag{3.1}$$

On the other hand, for any $x < a, y < b$ we have

$$\begin{aligned}\{\xi \le x, \eta \le y\} &= \{\xi_n \le a, \eta_n \le b, \xi \le x, \eta \le y\} \cup \{\xi_n \le a, \eta_n \le b, \xi \le x, \eta \le y\}\\ &\cup \{\xi_n > a, \eta_n \le b, \xi \le x, \eta \le y\} \cup \{\xi_n > a, \eta_n > b, \xi \le x, \eta \le y\}\\ &\subset \{\xi_n \le a, \eta_n \le b\} \cup \{|\xi_n - \xi| \ge a - x\} \cup \{|\eta_n - \eta| \ge b - y\}.\end{aligned}$$

Which implies

$$\Phi(x + iy) \le \Phi_n(a + ib) + M\{|\xi_n - \xi| \ge a - x\} + M\{|\eta_n - \eta| \ge b - y\}.$$

Since $\lim_{n\to\infty}\frac{1}{n}|\{k \le n : M(\|\xi_k - \xi\| \ge a - x) \ge \varepsilon\}| = 0$ and $\lim_{n\to\infty}\frac{1}{n}|\{k \le n : M(\|\xi_k - \xi\| \ge b - y) \ge \varepsilon\}| = 0$, we obtain
$\Phi(x + iy) \le \liminf_{n\to\infty}\Phi_n(a + ib)$ for any $x < a, y < b$. Taking $x + iy \to a + ib$, we get

$$\Phi(c) \le \liminf_{n\to\infty}\Phi_n(c). \tag{3.2}$$

It follows from (1) and (2) that $\Phi_n(c) \to \Phi(c)$ as $n \to \infty$. That is the complex uncertain sequence $\{\zeta_n\}$ is statistically convergent in distribution to $\zeta = \xi + i\eta$. Hence the result follows.

**Remark 3.2.** Converse of the above theorem is not necessarily true. i.e. $c(\mathcal{M}; s.dis) \subset c(\mathcal{M}; s.m)$ (strict inclusion). Following example illustrate this.

**Example 3.2.** Consider the uncertainty space $(\Gamma, L, M)$ to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}$. We define a complex uncertain variable as

$$\zeta(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1;\\ -i, & \text{if } \gamma = \gamma_2.\end{cases}$$

We also define $\zeta_n = -\zeta$ for $n = 1, 2, ....$ Then $\zeta_n$ and $\zeta$ have the same distribution

$$\Phi_n(c) = \Phi_n(a+ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty \\ 0, & \text{if } a \geq 0, b < -1 \\ \frac{1}{2}, & \text{if } a \geq 0, -1 \leq b < 1 \\ 1, & \text{if } a \geq 0, b \geq 1. \end{cases}$$

Then $\{\zeta_n\}$ Statistical convergence in distribution to $\zeta$. However, for a given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M(\|\zeta_k - \zeta\| \geq \varepsilon) \geq 1\}|$$
$$= \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M\left(\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right) \geq 1\}| = 0.$$

That is the sequence $\{\zeta_n\}$ does not statistically converge in measure to $\zeta$. By Theorem 5.2, the real part and imaginary part of $\{\zeta_n\}$ also not statistically convergent in measure.
In addition, since $\zeta_n = -\zeta$ for $n = 1, 2, ....,$ the sequence $\{\zeta_n\}$ does not statistically converge a.s to $\zeta$.

$\{\zeta_n\} \in c(\mathcal{M}; s.a.s)$ does not imply $\{\zeta_n\} \in c(\mathcal{M}; s.m)$.

**Example 3.3.** Consider the uncertainty space $(\Gamma, L, M)$ to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{n}{(2n+1)} < 0.5 \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{n}{(2n+1)} < 0.5 \\ 0.5, & \text{otherwise,} \end{cases}$$

Then we define a complex uncertain variables by

$$\zeta_n(\gamma) = \begin{cases} in, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. Then the sequence $\{\zeta_n\}$ Statistically convergence a.s to $\zeta$. However for some small number $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \frac{1}{2}\}|$$
$$= \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M\left(\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right) \geq \frac{1}{2}\}|$$

$$= \lim_{n\to\infty} \frac{1}{n} |\{k \in N : M\{\gamma_n\} \geq \frac{1}{2}\}| = 0.$$

as $n \to \infty$. That is the sequence $\{\zeta_n\}$ does not statistically converge in measure to $\zeta$.

In addition the complex uncertainty distributions of $\zeta_n$ are given by

$$\Phi_n(c) = \phi_n(a+ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1 - \frac{n}{2n+1}, & \text{if } a \geq 0, 0 \leq b < n; \\ 1, & a \geq 0, b \geq n. \end{cases}$$

for $n = 1, 2, ...,$ respectively. The complex uncertainty distribution of $\zeta$ is given by

$$\Phi(c) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1, & a \geq 0, b \geq 0. \end{cases}$$

Clearly $\Phi_n(c)$ does not converge to $\Phi(c)$ at $a \geq 0, b \geq 0$. That is, the sequence $\{\zeta_n\}$ does not converge to $\zeta$ in distribution.

**Remark 3.3.** $c(\mathcal{M}; s.m)$ also does not imply $c(\mathcal{M}; s.a.s)$.

**Example 3.4.** Consider the uncertainty space $(\Gamma, L, M)$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer $n$, there is an integer $m$ such that $n = 2^m + k$ where $k$ is an integer between 0 and $2^m - 1$. Then we define a complex uncertain variable by

$$\zeta_n(\gamma) = \begin{cases} i, & \text{if } \frac{k}{2^m} \leq \gamma \leq \frac{(k+1)}{2^m}; \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. For some small number $\varepsilon, \delta > 0$ and $n \geq 2$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}|$$

$$= \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : M\left(\gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right) \geq \delta\}|$$

$$= \lim_{n\to\infty} \frac{1}{n} |\{k \in N : M\{\gamma_n\} \geq \delta\}| = 0.$$

as $n \to \infty$. So the sequence $\{\zeta_n\}$ statistically converges in measure to $\zeta$. In addition for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : E[\|\zeta_k - \zeta\|] \geq \varepsilon\}| = 0$$

as $n \to \infty$. Thus the sequence $\{\zeta_n\}$ also statistically converges in mean to $\zeta$.
However, for any $\gamma \in [0,1]$, there is an infinite number of intervals of the form $[\frac{k}{2^m}, \frac{k+1}{2^m}]$ containing $\gamma$. Thus $\zeta_n(\gamma)$ does not statistically converge to 0. In other words, the sequence $\{\zeta_n\}$ does not statistically converge a.s to $\zeta$.

**Theorem 3.3.** $\{\zeta_n\} \in c(\mathcal{M}; s.a.s)$ does not imply $\{\zeta_n\} \in c(\mathcal{M}; s.mean)$.

Following example illustrate this.

**Example 3.5.** Consider the uncertainty space $(\Gamma, L, M)$ to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n}.$$

The complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} i2^n, & \text{if } \gamma = \gamma_n; \\ 0, & \text{otherwise}, \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. Then the sequence $\{\zeta_n\}$ is statistically convergence a.s to $\zeta$. However, the uncertainty distributions of $\zeta_n$ are given by

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{2^n}, & \text{if } 0 \leq x < 2^n; \\ 1, & x \geq 2^n. \end{cases}$$

for $n = 1, 2, ...$, respectively. Then we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : E[\|\zeta_k - \zeta\|] \geq 1\}| = 0$$

So the sequence $\{\zeta_n\}$ does not statistically converge in mean to $\zeta$.

From Example 3.4, we can obtain that statistically convergence in mean does not imply statistically convergence a.s..

**Proposition 3.4.** Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then $\{\zeta_n\}$ statistically converges a.s to $\zeta$ if and only if for any $\varepsilon, \delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}| = 0.$$

**Proof.** By the definition of statistical convergence a.s., we have that there exists an event $\Lambda$ with $M\{\Lambda\} = 1$ such that $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \|\zeta_k - \zeta\| \geq \varepsilon\}| = 0$ for every $\varepsilon > 0$. Then for any $\varepsilon > 0$, there exists $k$ such that $\|\zeta_n - \zeta\| < \varepsilon$ where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| < \varepsilon\right) \geq 1\}| = 0.$$

It follows from the duality axiom of uncertain measure that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}| = 0.$$

**Proposition 3.5.** Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then $\{\zeta_n\}$ statistically converges uniformly a.s to $\zeta$ if and only if for any $\varepsilon, \delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}| = 0.$$

**Proof.** If $\{\zeta_n\}$ statistically converges uniformly a.s to $\zeta$, then for any $\delta > 0$ there exists $B$ such that $M\{B\} < \delta$ and $\{\zeta_n\}$ statistically uniformly converges to $\zeta$ on $\Gamma - B$. Thus, for any $\varepsilon > 0$, there exists $k > 0$ such that $\|\zeta_n - \zeta\| < \varepsilon$ where $n \geq k$ and $\gamma \in \Gamma - B$. That is

$$\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\} \subset B.$$

It follows from the subadditivity axiom of uncertain measure that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : M\left(\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \geq \varepsilon\right)\}| \leq \delta(M\{B\}) < \delta.$$

Then

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:\lim_{n\to\infty}M\left(\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\}|=0.$$

On the contrary, if $\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:\lim_{n\to\infty}M\left(\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\}|=0.$ for any $\varepsilon>0$, then for given $\delta>0$ and $m\geq 1$, there exists $m_k$ such that

$$\delta\left(M\left(\bigcup_{n=m_k}^{\infty}\{\|\zeta_n-\zeta\|\geq\frac{1}{m}\}\right)\right)<\frac{\delta}{2^m}.$$

Let $B=\cup_{m=1}^{\infty}\cup_{n=m_k}^{\infty}\{\|\zeta_n-\zeta\|\geq\frac{1}{m}\}$.

Then

$$\delta(M\{B\})\leq\sum_{m=1}^{\infty}\delta\left(M(\bigcup_{n=m_k}^{\infty}\{\|\zeta_n-\zeta\|\geq\frac{1}{m}\})\right)\leq\sum_{m=1}^{\infty}\frac{\delta}{2^m}.$$

Furthermore, we have

$$\sup_{\gamma\in\Gamma-B}\|\zeta_n-\zeta\|<\frac{1}{m}$$

for any $m=1,2,...$ and $n>m_k$. The proposition is thus proved.

**Theorem 3.6.** If$\{\zeta_n\}\in c(\mathcal{M};s.u.a.s)$, then $\{\zeta_n\}\in c(\mathcal{M};s.a.s)$.

**Proof.** It follows from above Proposition that if $\{\zeta_n\}$ statistically converges uniformly a.s to $\zeta$, then

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:\lim_{n\to\infty}M\left(\bigcup_{n=k}^{\infty}\|\zeta_k-\zeta\|\geq\varepsilon\right)\geq\delta\}|=0.$$

Since

$$\delta\left(M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n-\zeta\|\geq\varepsilon\}\right)\right)\leq\delta\left(M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n-\zeta\|\geq\varepsilon\}\right)\right),$$

taking the limit as $n\to\infty$ on both side of above inequality, we obtain

$$\delta\left(M\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n-\zeta\|\geq\varepsilon\}\right)\right)=0.$$

By Proposition 1, $\{\zeta_n\}$ statistically converges a.s to $\zeta$.

**Theorem 3.7.** If a complex uncertain sequence $\{\zeta_n\} \in c(\mathcal{M}; s.u.a.s)$, then $\{\zeta_n\} \in c(\mathcal{M}; s.m)$.

**proof.** If $\{\zeta_n\}$ statistically converges uniformly a.s. to $\zeta$, then from Proposition above we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \lim_{n\to\infty} M\left(\bigcup_{n=k}^{\infty} \|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}| = 0,$$

and

$$\delta\left(M\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) \leq \left(M\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right)\right).$$

On taking $n \to \infty$, we can obtain $\{\zeta_n\}$ statistically converges in measure to $\zeta$.

As it is seen from Example 3.4, $\{\zeta_n\}$ statistically converges in measure to $\zeta$. However, it does not statistically converges a.s. to $\zeta$. It follows from above Theorem that $\{\zeta_n\}$ does not statistically converges uniformly a.s. to $\zeta$.

## Conclusion

In this paper we have discussed the statistical version of five sequence spaces given by Orlicz function for uncertain variables. This is a very little approach in this direction. Further all the sequence spaces for real or complex and their related properties can be extended in this direction.

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