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Some hyperstability results for a Cauchy-Jensen type functional equation in 2-Banach spaces

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Abstract:

In this paper, we investigate some stability and hyperstability results for the following Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z)$$

in 2-Banach spaces by using Brzdęk's fixed point approach.

Keywords: Stability; Hyperstability; 2-Banach space; Cauchy-Jensen functional equation.

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1. Introduction

Let E, Y be normed spaces. A function $f : E \to Y$ is Cauchy-Jensen provided it satisfies the functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z) \quad \text{for all } \; x,y,z\in E, \tag{1.1}$$

and we can say that $f : E \to Y$ is Cauchy-Jensen on E_0 if it satisfies (1.1) for all $x, y, z \in E_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$.
Recently, interesting results concerning the Cauchy-Jensen functional equation (1.1) have been obtained in [7] and [20].

Throughout this paper, we will denote the set of natural numbers by $\mathbf{N}$, $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$ and the set of real numbers by $\mathbf{R}$. By $\mathbf{N}_m$, $m \in \mathbf{N}$, we will denote the set of all natural numbers greater than or equal to m.

Let $\mathbf{R}_+ = [0, \infty)$ the set of nonnegative real numbers. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B and we use the notation E_0 for the set $E\backslash\{0\}$.

We need to recall some basic facts concerning 2-normed spaces and some preliminary results (see, for instance, [17]).

Definition 1.1. *let X be a real linear space with $dimX > 1$ and $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbf{R}_+$ be a function satisfying the following properties:*

1. $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*
2. $\|x, y\| = \|y, x\|$,
3. $\|\lambda x, y\| = |\lambda|\|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbf{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space. Sometimes the condition (4) called the triangle inequality.

Example 1.2. *For* $x = (x_1, x_2)$, $y = (y_1, y_2) \in E = \mathbf{R}^2$, *the Euclidean 2-norm* $\|x, y\|_E$ *is defined by*

$$\|x, y\|_E = |x_1 y_2 - x_2 y_1| \, .$$

Definition 1.3. *A sequence* $\{x_k\}$ *in a 2-normed space* X *is called a convergent sequence if there is an* $x \in X$ *such that*

$$\lim_{k \to \infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. *If* $\{x_k\}$ *converges to* x, *write* $x_k \longrightarrow x$ *with* $k \longrightarrow \infty$ *and call* x *the limit of* $\{x_k\}$. *In this case, we also write* $\lim_{k \to \infty} x_k = x$.

Definition 1.4. *A sequence* $\{x_k\}$ *in a 2-normed space* X *is said to be a* Cauchy sequence *with respect to the 2-norm if*

$$\lim_{k,l \to \infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. *If every Cauchy sequence in* X *converges to some* $x \in X$, *then* X *is said to be* complete *with respect to the 2-norm. Any complete 2-normed space is said to be a* 2-Banach space.

Next, it is easily seen that we have the following property.

Lemma 1.5. *If* X *is a linear 2-normed space,* $x, y_1, y_2 \in X$, y_1, y_2 *are linearly independent, and*

$$\|x, y_1\| = \|x, y_2\| = 0,$$

then $x = 0$.

Let us yet recall a lemma from [19].

Lemma 1.6. *If* X *is a linear 2-normed space and* $(x_n)_{\in \mathbf{N}}$ *is a convergent sequence of elements of* X, *then*

$$\lim_{n \to \infty} \|x_n, y\| = \| \lim_{n \to \infty} x_n, y\| = 0, \quad y \in X.$$

The problem of the stability of functional equations was first raised by Ulam [21]. This included the following question concerning the stability of group homomorphisms.

Let $(G_1, *_1)$ *be a group and let* $(G_2, *_2)$ *be a metric group with a metric* $d(.,.)$. *Given* $\varepsilon > 0$, *does there exists a* $\delta > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d\Big(h(x *_1 y), h(x) *_2 h(y)\Big) < \delta$$

for all $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with*

$$d\Big(h(x), H(x)\Big) < \varepsilon$$

for all $x \in G_1$*?*

If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable.

The first partial answer to Ulam's question was given by Hyers [18] and he established the stability result as follows:

Theorem 1.7. *[18] Let* E_1 *and* E_2 *be two Banach spaces and* $f : E_1 \to E_2$ *be a function such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ *and for all* $x, y \in E_1$. *Then the limit*

$$A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, *and* $A : E_1 \to E_2$ *is the unique additive function such that*

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$. *Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$, *then the function* A *is linear.*

Later, T. Aoki [4] and D. G. Bourgin [8] considered the problem of stability with unbounded Cauchy differences. Th. Rassias [21] attempted to weaken the condition for the bound of the norm of Cauchy difference

$$\|f(x+y)-f(x)-f(y)\|$$

and proved a generalization of Theorem 1.7 using a direct method (cf. Theorem 1.8):

Theorem 1.8. *[24] Let E_1 and E_2 be two Banach spaces. If $f: E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\Big(\|x\|^p+\|y\|^p\Big)$$

for some $\theta \geq 0$, for some $p \in \mathbf{R}$ with $0 \leq p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A: E_1 \to E_2$ such that

$$\|f(x)-A(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Later, Th. Rassias [22],[23] motivated Theorem 1.8 as follows:

Theorem 1.9. *[22],[23] Let E_1 be a normed space, E_2 be a Banach space, and $f: E_1 \to E_2$ be a function. If f satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\Big(\|x\|^p+\|y\|^p\Big) \tag{1.2}$$

for some $\theta \geq 0$, for some $p \in \mathbf{R}$ with $p \neq 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A: E_1 \to E_2$ such that

$$\|f(x)-A(x)\| \leq \frac{2\theta}{|2-2^p|}\|x\|^p \tag{1.3}$$

for each $x \in E_1 - \{0_{E_1}\}$.

Note that Theorem 1.9 reduces to Theorem 1.7 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, J. Brzdęk [9] showed that estimation (1.3) is optimal for $p \geq 0$ in the general case.

Recently, J. Brzdęk [14] showed that Theorem 1.9 can be significantly improved; namely, in the case $p < 0$, each $f: E_1 \to E_2$ satisfying (1.2) must actually be additive, and the assumption of completeness of E_2 is not necessary. Unfortunately, this result does not remain valid if we restrict the domain of f (see the further detail in [16]). On the other hand, several mathematicians showed that the fixed point method is an another very efficient and convenient tool for proving the Hyers-Ulam stability for a quite wide class of functional equations (see [15]). J. Brzdęk et al. [10] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:

By using this theorem, Brzdęk [13] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular Theorem 1.9). During the past few years many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [1, 2, 3, 5, 6, 15, 16, 12] and references therein).

Now, we present the fixed point theorem concerning 2-Banach spaces given in [11]. First, we need the following hypotheses:

(H1) E is a nonempty set, $\Big(Y, \|\cdot,\cdot\|\Big)$ is a 2-Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbf{N}$, $f_i : E \to E$, $g_i : Y_0 \to Y_0$, and $L_i : E \times Y_0 \to \mathbf{R}_+$ for $i = 1, ..., j$;

(H2) $\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality

$$\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\Big\| \leq \sum_{i=1}^{j} L_i(x,y)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big), g_i(y)\Big\|, \ \xi, \mu \in Y^E, x \in E, y \in Y_0; \tag{1.4}$$

(H3) $\Lambda : \mathbf{R}_+^{E\times Y_0} \to \mathbf{R}_+^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta\Big(f_i(x), g_i(y)\Big), \quad \delta \in \mathbf{R}_+^{E\times Y_0}, \ x \in E, y \in Y_0. \tag{1.5}$$

Theorem 1.10. *[11] Let hypotheses (H1)-(H3) hold and functions $\varepsilon : E \times Y_0 \to R_+$ and $\varphi : E \to Y$ fulfill the following two conditions:*

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x), y\Big\| \leq \varepsilon(x,y) \quad x \in E, y \in Y_0, \tag{1.6}$$

$$\varepsilon^*(x,y) := \sum_{n=0}^{\infty} \Big(\Lambda^n \varepsilon\Big)(x,y) < \infty \quad x \in E, y \in Y_0. \tag{1.7}$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ for which

$$\Big\|\varphi(x) - \psi(x), y\Big\| \leq \varepsilon^*(x,y) \quad x \in E, y \in Y_0. \tag{1.8}$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x) \quad x \in E. \tag{1.9}$$

2. Main results

In this section, we prove some stability and hyperstability results for the Cauchy-Jensen equation (1.1) in 2-Banach spaces by using Theorem 1.10. In what follows $(Y, \|\cdot,\cdot\|)$ is a real 2-Banach space.

Theorem 2.1. *Let $h_1, h_2, h_3 : E_0 \times Y_0 \to \mathbf{R}_+$ be three functions such that*

$$\mathcal{U} := \{n \in \mathbf{N}\colon \alpha_n := \lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n) + \lambda_1(n)\lambda_2(n)\lambda_3(n) < 1\} \neq \phi \tag{2.1}$$

where

$$\lambda_i(n) := \inf\{t \in \mathbf{R}_+\colon h_i(nx,w) \leq t\, h_i(x,w), \quad x \in E_0, w \in Y_0\} \tag{2.2}$$

for all $n \in \mathbf{N}$, where $i = 1,2,3$. Assume that $f : E \to Y$ satisfies the inequality

$$\left\|f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x), w\right\| \leq h_1(x,w)h_2(y,w)h_3(z,w), \tag{2.3}$$

for all $x, y, z, \in E_0, w \in Y_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$. Then there exists a unique Cauchy-Jensen function $F : E \to Y$ such that

$$\left\| f(x) - F(x), w \right\| \le \lambda_0 h_1(x,w) h_2(x,w) h_3(x,w) \tag{2.4}$$

for all $x \in E_0, w \in Y_0$*, where*

$$\lambda_0 := \inf_{n \in \mathcal{U}} \left\{ \frac{\lambda_1(n)\lambda_2(n)}{1 - \lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n) - \lambda_1(n)\lambda_2(n)\lambda_3(n)} \right\}.$$

Proof. Replacing y and x with mx and z with x, where $x \in E_0$ and $m \in \mathbf{N}$, in inequality (2.3) we get

$$\left\| f\Big((1+m)x\Big) - f(mx) - f(x), w \right\| \le h_1(mx,w) h_2(mx,w) h_3(x,w) \tag{2.5}$$

for all $x \in E_0, w \in Y_0$. For each $m \in \mathbf{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m \xi(x) := \xi\Big((1+m)x\Big) - \xi(mx), \quad \xi \in Y^{E_0}, x \in E_0. \tag{2.6}$$

Further put

$$\varepsilon_m(x,w) := h_1(mx,w) h_2(mx,w) h_3(x,w), \quad x \in E_0, w \in Y_0, \tag{2.7}$$

and observe that

$$\varepsilon_m(x,w) = h_1(mx,w) h_2(mx,w) h_3(x,w) \le \lambda_1(m)\lambda_2(m) h_1(x,w) h_2(x,w) h_3(x,w), \tag{2.8}$$

for all $x \in E_0, w \in Y_0, m \in \mathbf{N}$. Then the inequality (2.5) takes the form

$$\left\| f(x) - \mathcal{T}_m f(x), w \right\| \le \varepsilon_m(x,w), \quad x \in E_0, w \in Y_0. \tag{2.9}$$

Furthermore, for every $x \in E_0, w \in Y_0$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned} \left\| \mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x), w \right\| &= \left\| \xi\Big((1+m)x\Big) - \xi(mx) - \mu\Big((1+m)x\Big) + \mu(mx), w \right\| \\ &\le \left\| (\xi - \mu)\Big((1+m)x\Big), w \right\| + \left\| (\xi - \mu)(mx), w \right\|. \end{aligned}$$

So, (H2) is valid for $\mathcal{T}_m$.

This brings us to define the operator $\Lambda_m : \mathbf{R}_+^{E_0 \times Y_0} \to \mathbf{R}_+^{E_0 \times Y_0}$ by

$$\Lambda_m \delta(x,w) := \delta\Big((1+m)x, w\Big) + \delta(mx,w), \quad \delta \in \mathbf{R}_+^{E_0 \times Y_O}, x \in E_0, w \in Y_0.$$

(2.10)

For each $m \in \mathbf{N}$, the above operator has the form described in (H3) with $f_1(x) = (1+m)x,\ f_2(x) = mx,\ g_1(w) = g_2(w) = w$ and $L_1(x) = L_2(x) = 1$ for all $x \in E_0$. By induction, we will show that for each $x \in E_0, w \in Y_0$, $n \in \mathbf{N}_0$, and $m \in \mathcal{U}$ we have

$$(\Lambda_m^n \varepsilon_m)(x,w) \le \lambda_1(m)\lambda_2(m)\alpha_m^n h_1(x,w)h_2(x,w)h_3(x,w) \tag{2.11}$$

where

$$\alpha_m = \lambda_1(1+m)\lambda_2(1+m)\lambda_3(1+m) + \lambda_1(m)\lambda_2(m)\lambda_3(m).$$

From (2.7) and (2.8), we obtain that the inequality (2.11) holds for $n = 0$. Next, we will assume that (2.11) holds for $n = k$, where $k \in \mathbf{N}$. Then we have

$$\begin{aligned}(\Lambda_m^{k+1}\varepsilon_m)(x,w) &= \Lambda_m\Big((\Lambda_m^k\varepsilon_m)(x,w)\Big)\\ &= (\Lambda_m^k\varepsilon_m)\Big((1+m)x,w\Big) + (\Lambda_m^k\varepsilon_m)(mx,w)\\ &\le \lambda_1(m)\lambda_2(m)\alpha_m^k h_1((1+m)x,w)h_2((1+m)x,w)h_3((1+m)x,w)\\ &\quad +\lambda_1(m)\lambda_2(m)\alpha_m^k h_1(mx,w)h_2(mx,w)h_3(mx,w)\\ &\le \lambda_1(m)\lambda_2(m)\alpha_m^{k+1} h_1(x,w)h_2(x,w)h_3(x,w)\end{aligned}$$

for all $x \in E_0, w \in Y_0,\ m \in \mathcal{U}$. This shows that (2.11) holds for $n = k+1$. Now we can conclude that the inequality (2.11) holds for all $n \in \mathbf{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,w) &= \textstyle\sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,w)\\ &\le \textstyle\sum_{n=0}^{\infty}\lambda_1(m)\lambda_2(m)\alpha_m^n h_1(x,w)h_2(x,w)h_3(x,w)\\ &= \tfrac{\lambda_1(m)\lambda_2(m)}{1-\alpha_m} h_1(x,w)h_2(x,w)h_3(x,w) < \infty\end{aligned}$$

for all $x \in E_0, w \in Y_0,\ m \in \mathcal{U}$. Therefore, according to Theorem 1.10 with $\varphi = f$, we get that the limit

$$F_m(x) := \lim_{n\to\infty}\Big(\mathcal{T}_m^n f\Big)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

$$\Big\|f(x) - F_m(x), w\Big\| \le \frac{\lambda_1(m)\lambda_2(m)h_1(x,w)h_2(x,w)h_3(x,w)}{1-\alpha_m},\ \ x \in E_0, w \in Y_0,\ m \in \mathcal{U}. \tag{2.12}$$

To prove that F_m satisfies the functional equation (1.1), just prove the following inequality

$$\left\|(\mathcal{T}_m^n f)\left(\frac{x+y}{2}+z\right)+(\mathcal{T}_m^n f)\left(\frac{x-y}{2}+z\right)-2(\mathcal{T}_m^n f)(z)-(\mathcal{T}_m^n f)(x),w\right\| \leq \alpha_m^n h_1(x,w)h_2(y,w)h_3(z,w) \tag{2.13}$$

for every $x, y, z \in E_0$ such that $\frac{x+y}{2}+z \neq 0$ and $\frac{x-y}{2}+z \neq 0$, $w \in Y_0$, $n \in \mathbf{N}_0$, and $m \in \mathcal{U}$. Since the case $n = 0$ is just (2.3), take $k \in \mathbf{N}$ and assume that (2.13) holds for $n = k$ and every $x, y, z \in E_0$ such that $\frac{x+y}{2}+z \neq 0$ and $\frac{x-y}{2}+z \neq 0$, $w \in Y_0$, $m \in \mathcal{U}$. Then, for each $x, y, z \in E_0, w \in Y_0$ and $m \in \mathcal{U}$, we get

$$\begin{aligned}
&\left\|(\mathcal{T}_m^{k+1} f)\left(\frac{x+y}{2}+z\right)+(\mathcal{T}_m^{k+1} f)\left(\frac{x-y}{2}+z\right)-2(\mathcal{T}_m^{k+1} f)(z)-(\mathcal{T}_m^{k+1} f)(x),w\right\| \\
&= \Big\|(\mathcal{T}_m^k f)f\left((1+m)\left(\frac{x+y}{2}+z\right)\right)-(\mathcal{T}_m^k f)f\left(m\left(\frac{x+y}{2}+z\right)\right) \\
&+(\mathcal{T}_m^k f)f\left((1+m)\left(\frac{x-y}{2}+z\right)\right)-(\mathcal{T}_m^k f)f\left(m\left(\frac{x-y}{2}+z\right)\right) \\
&-2(\mathcal{T}_m^k f)f\Big((1+m)z\Big)+2(\mathcal{T}_m^k f)f(mz)-(\mathcal{T}_m^k f)f\Big((1+m)x\Big)+(\mathcal{T}_m^k f)f(mx),w\Big\| \\
&\leq \Big\|(\mathcal{T}_m^k f)f\left((1+m)\left(\frac{x+y}{2}+z\right)\right)+(\mathcal{T}_m^k f)f\left((1+m)\left(\frac{x-y}{2}+z\right)\right) \\
&-2(\mathcal{T}_m^k f)f\Big((1+m)z\Big)-(\mathcal{T}_m^k f)f\Big((1+m)x\Big),w\Big\|+ \\
&\left\|(\mathcal{T}_m^k f)f\left(m\left(\frac{x+y}{2}+z\right)\right)+(\mathcal{T}_m^k f)f\left(m\left(\frac{x-y}{2}+z\right)\right)-2(\mathcal{T}_m^k f)f(mz)-(\mathcal{T}_m^k f)f(mx),w\right\| \\
&\leq \alpha_m^k h_1\Big((1+m)x,w\Big)h_2\Big((1+m)y,w\Big)h_3\Big((1+m)z,w\Big)+ \\
&\alpha_m^k h_1\Big(mx,w\Big)h_2\Big(my,w\Big)h_3\Big(mz,w\Big) \\
&\leq \alpha_m^{k+1} h_1(x,w)h_2(y,w)h_3(z,w)
\end{aligned}$$

Thus, by induction, we have shown that (2.13) holds for every $x, y, z \in E_0$ such that $\frac{x+y}{2}+z \neq 0$ and $\frac{x-y}{2}+z \neq 0$, $w \in Y_0$, $n \in \mathbf{N}_0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.13), we obtain the equality

$$F_m\left(\frac{x+y}{2}+z\right)+F_m\left(\frac{x-y}{2}+z\right)=F_m(x)+2F_m(z), \tag{2.14}$$

for all $x, y, z \in E_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0, m \in \mathcal{U}$. This implies that $F_m : E_0 \to Y$, defined in this way, is a solution of the equation

$$F(x) = F\Big((1+m)x\Big) - F(mx), \quad x \in E_0, m \in \mathcal{U}. \tag{2.15}$$

Next, we will prove that each Cauchy-Jensen function $F : E \to Y$ satisfying the inequality

$$\Big\|f(x) - F(x), w\Big\| \leq L\, h_1(x,w)h_2(x,w)h_3(x,w), \quad x \in E_0, w \in Y_0 \tag{2.16}$$

with some $L > 0$, is equal to F_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $F : E \to Y$ satisfying (2.16). From (2.12), for each $x \in E_0$, we get

$$\begin{aligned} \Big\|F(x) - F_{m_0}(x), w\Big\| &\leq \Big\|F(x) - f(x), w\Big\| + \Big\|f(x) - F_{m_0}(x), w\Big\| \\ &\leq L\, h_1(x,w)h_2(x,w)h_3(x,w) + \varepsilon^*_{m_0}(x,w) \\ &\leq L_0\, h_1(x,w)h_2(x,w)h_3(x,w) \sum_{n=0}^{\infty} \alpha^n_{m_0}, \end{aligned} \tag{2.17}$$

where

$L_0 := (1 - \alpha_{m_0})L + \lambda_1(m_0)\lambda_2(m_0) > 0$ and we exclude the case that $h_1(x,w) \equiv 0$, $h_2(x,w) \equiv 0$ or $h_3(x,w) \equiv 0$ which is trivial. Observe that F and F_{m_0} are solutions to equation (2.15) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbf{N}_0$, we have

$$\Big\|F(x) - F_{m_0}(x), w\Big\| \leq L_0\, h_1(x,w)h_2(x,w)h_3(x,w) \sum_{n=j}^{\infty} \alpha^n_{m_0}, \quad x \in E_0, w \in Y_0. \tag{2.18}$$

The case $j = 0$ is exactly (2.17). We fix $k \in \mathbf{N}$ and assume that (2.18) holds for $j = k$. Then, in view of (2.17), for each $x \in E_0, w \in Y_0$, we get

$$\begin{aligned}\Big\|F(x)-F_{m_0}(x),w\Big\| &= \Big\|F\Big((1+m_0)x\Big)-F(m_0x)\\ &-F_{m_0}\Big((1+m_0)x\Big)+F_{m_0}(m_0x),w\Big\|\\ &\le \Big\|F\Big((1+m_0)x\Big)-F_{m_0}\Big((1+m_0)x\Big),w\Big\|\\ &\quad+\Big\|F(m_0x)-F_{m_0}(m_0x),w\Big\|\\ &\le L_0\ h_1\Big((1+m_0)x,w\Big)h_2\Big((1+m_0)x,w\Big)h_3\Big((1+m_0)x,w\Big)\textstyle\sum_{n=k}^{\infty}\alpha_{m_0}^n\\ &\quad+L_0\ h_1\Big(m_0x,w\Big)h_2\Big(m_0x,w\Big)h_3\Big(m_0x,w\Big)\textstyle\sum_{n=k}^{\infty}\alpha_{m_0}^n\\ &= L_0\ \Big(h_1\Big((1+m_0)x,w\Big)h_2\Big((1+m_0)x,w\Big)h_3\Big((1+m_0)x,w\Big)\\ &\quad+h_1\Big(m_0x,w\Big)h_2\Big(m_0x,w\Big)h_3\Big(m_0x,w\Big)\Big)\textstyle\sum_{n=k}^{\infty}\alpha_{m_0}^n\\ &\le L_0\ \alpha_{m_0}h_1(x,w)h_2(x,w)h_3(x,w)\textstyle\sum_{n=k}^{\infty}\alpha_{m_0}^n\\ &= L_0\ h_1(x,w)h_2(x,w)h_3(x,w)\textstyle\sum_{n=k+1}^{\infty}\alpha_{m_0}^n.\end{aligned}$$

This shows that (2.18) holds for $j=k+1$. Now we can conclude that the inequality (2.18) holds for all $j\in\mathbf{N}_0$. Now, letting $j\to\infty$ in (2.18), we get

$$F=F_{m_0}. \tag{2.19}$$

Thus, we have also proved that $F_m=F_{m_0}$ for each $m\in\mathcal{U}$, which (in view of (2.12)) yields

$$\Big\|f(x)-F_{m_0}(x),w\Big\|\le\frac{\lambda_1(m)\lambda_2(m)h_1(x,w)h_2(x,w)h_3(x,w)}{1-\alpha_m},\ x\in E_0,w\in Y_0,m\in\mathcal{U}. \tag{2.20}$$

This implies (2.4) with $F=F_{m_0}$ and (2.19) confirms the uniqueness of F. □

The following theorem concerns the η-hyperstability of (1.1) in 2-Banach spaces. Namely, We consider functions $f:E\to Y$ fulfilling (1.1) approximately, i.e., satisfying the inequality

$$\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-2f(z)-f(x),w\right\|\le\eta(x,y,z,w), \tag{2.21}$$

for all $x,y,z\in E_0$ such that $\frac{x+y}{2}+z\neq0$ and $\frac{x-y}{2}+z\neq0,w\in Y_0$, with $\eta:E_0\times E_0\times E_0\times Y_0\to\mathbf{R}_+$ is a given mapping. Then we find a unique Cauchy-Jensen function $F:E\to Y$ which is close to f. Then, under some additional assumptions on η, we prove that the conditional functional equation (1.1) is η-hyperstable in the class of functions $f : E \to Y$, i.e., each $f : E \to Y$ satisfying inequality (2.21), with such η, must fulfil equation (1.1).

Theorem 2.2. *Let h_1, h_2, h_3 and $\mathcal{U}$ be as in Theorem 2.1. Assume that*

$$\begin{cases} \lim_{n\to\infty} \lambda_1(n)\lambda_2(n) = 0, \\ \lim_{n\to\infty} \lambda_1(n)\lambda_2(n)\lambda_3(n) = 0. \end{cases} \tag{2.22}$$

Then every $f : E \to Y$ satisfying (2.3) is a solution of (1.1).

Proof. Suppose that $f : E \to Y$ satisfies (2.3). Then, by Theorem 2.1, there exists a mapping $F : E \to Y$ satisfies (1.1)
and

$$\|f(x) - F(x), w\| \le \lambda_0 h_1(x,w)h_2(x,w)h_3(x,w) \tag{2.23}$$

for all $x \in E_0, w \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{\lambda_1(n)\lambda_2(n)}{1 - \lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n) - \lambda_1(n)\lambda_2(n)\lambda_3(n)} \right\}.$$

Since, in view of (2.22), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) = f(x) + 2f(z),$$

for all $x, y, z \in E_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$, which implies that f satisfies the functional equation (1.1) on E. □

Corollary 2.3. *Let $\theta \ge 0$, $s \ge 0$, $p, q, r \in \mathbf{R}$ such that $p + q + r < 0$. Suppose that $f : E \to Y$ such that $f(0) = 0$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x), w \right\| \le \theta \|x\|^p \,\|y\|^q \,\|z\|^r \,\|w\|^s, \tag{2.24}$$

for all $x, y, z \in E_0$ such that $\frac{x+y}{2} + z \neq 0$ and $\frac{x-y}{2} + z \neq 0$, $w \in Y_0$. Then f is Cauchy-Jensen on E_0.

Proof. The proof follows from Theorem 2.1 by defining

$h_1, h_2, h_3 : E_0 \times Y_0 \to \mathbf{R}_+$ by $h_1(x,w) = \theta_1\|x\|^p\|w\|^{s_1}$, $h_2(y,w) = \theta_2|y|^q|w|^{s_2}$, $h_3(z,w) = \theta_3\|z\|^r\|w\|^{s_3}$ and $h_1(0,w) = h_2(0,w) = h_3(0,w) = 0$ with $\theta_1, \theta_2, \theta_3 \in \mathbf{R}_+$, $s_1, s_2, s_3 \in \mathbf{R}_+$ and $p, q, r \in \mathbf{R}$ such that $\theta_1\theta_2\theta_3 = \theta$, $s_1 + s_2 + s_3 = s$ and $p + q + r < 0$.

For each $n \in \mathbf{N}$, we have

$$\begin{aligned}\lambda_1(n) &= \inf\{t \in \mathbf{R}_+ : h_1(nx,w) \le t\, h_1(x,w),\ \ x \in E_0, w \in Y_0\}\\ &= \inf\{t \in \mathbf{R}_+ : \theta_1\|nx\|^p\|w\|^{s_1} \le t\, \theta_1\|x\|^p\|w\|^{s_1},\ \ x \in E_0, w \in Y_0\}\\ &= n^p.\end{aligned}$$

Also, we have $\lambda_2(n) = n^q$ and $\lambda_3(n) = n^r$ for all $n \in \mathbf{N}$. Clearly, we can find $n_0 \in \mathbf{N}$ such that

$$\lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n)+\lambda_1(n)\lambda_2(n)\lambda_3(n) = (1+n)^{p+q+r}+n^{p+q+r} < 1, \quad n \ge n_0. \tag{2.25}$$

According to Theorem 2.1, there exists a unique Cauchy-Jensen function $F : E \to Y$ such that

$$\Big\|f(x) - F(x), w\Big\| \le \theta\lambda_0 h_1(x,w)h_2(x,w)h_3(x,w) \tag{2.26}$$

for all $x \in E_0, w \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}}\left\{\frac{\lambda_1(n)\lambda_2(n)}{1-\lambda_1(1+n)\lambda_2(1+n)\lambda_3(1+n)-\lambda_1(n)\lambda_2(n)\lambda_3(n)}\right\}.$$

On the other hand, Since $p + q + r < 0$, It is sufficient to consider that $p + q < 0$. Then

$$\begin{cases}\lim\limits_{n\to\infty} \lambda_1(n)\lambda_2(n) = \lim_{n\to\infty} n^{p+q} = 0,\\ \lim\limits_{n\to\infty} \lambda_1(n)\lambda_2(n)\lambda_3(n) = \lim_{n\to\infty} n^{p+q+r} = 0.\end{cases} \tag{2.27}$$

Thus by Theorem 2.2, we get the desired results. □

The next corollary prove the hyperstability results for the inhomogeneous Cauchy-Jensen functional equation.

Corollary 2.4. *Let $\theta, p, q, r, s \in \mathbf{R}$ such that $\theta \ge 0$ and $p + q + r < 0$. Assume that $G : E^3 \to Y$ and $f : E \to Y$ such that $f(0) = 0$ and satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-2f(z)-f(x)-G(x,y,z),w\right\| \leq \theta\|x\|^p\ \|y\|^q\ \|z\|^r\ \|w\|^s, \tag{2.28}$$

for all x,y,z$\in E_0$ such that $\frac{x+y}{2}+z\neq 0$ and $\frac{x-y}{2}+z\neq 0, w\in Y_0$. If the functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=2f(z)+f(x)+G(x,y,z), \tag{2.29}$$

for all $x,y,z\in E_0$ such that $\frac{x+y}{2}+z\neq 0$ and $\frac{x-y}{2}+z\neq 0$, has a solution $f_0 : E\to Y$, then f is a solution to (2.29).

Proof. From (2.28) we get that the function $K : E\to Y$ defined by $K := f - f_0$ satisfies (2.24). Consequently, Corollary 2.3 implies that K is a solution to Cauchy-Jensen functional equation (1.1). Therefore,

$$\begin{aligned} f\left(\tfrac{x+y}{2}+z\right)+f\left(\tfrac{x-y}{2}+z\right)-2f(z)-f(x)-G(x,y,z) &= K\left(\tfrac{x+y}{2}+z\right) \\ +f_0\left(\tfrac{x+y}{2}+z\right)+K\left(\tfrac{x-y}{2}+z\right)+f_0\left(\tfrac{x-y}{2}+z\right) \\ -2K(z)-2f_0(z)-K(x)-f_0(x)-G(x,y,z) \\ =0, \end{aligned}$$

for all $x,y,z\in E_0$ such that $\frac{x+y}{2}+z\neq 0$ and $\frac{x-y}{2}+z\neq 0$, which means f is a solution to (2.29). □

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