Partition of the spectra for the lower triangular double band matrix as generalized difference operator Δ_v over the sequence spaces c and $\boldsymbol{\ell}_p$ (1 < p < ∞)

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Abstract:

Let the sequence (v_k) is assumed to be either constant or strictly decreasing sequence of positive real numbers satisfying $\lim_{k \to \infty} v_k = L > 0$ and $\sup_k v_k \leq 2L$. Then the generalized difference operator Δ_v is $\Delta_v x = \Delta_v (x_n) = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^{\infty}$ with $x_{-1} = v_{-1} = 0$. The aim of this paper is to obtain the approximate point spectrum, the defect spectrum and the compression spectrum of the operator Δ_v and modified of the operator Δ_v on the sequence spaces c and ℓ_p (1 < p < ∞).

1. Introduction

We know that there exists strictly the relationship between matrices and operators. The eigenvalues of matrices have been contained spectrum of an operator. The spectral theory is one of the most useful tools in science. There exist many its applications in mathematics and physics which contain matrix theory, control theory, function theory, differential and integral equations, complex analysis, and quantum physics. For example, atomic energy levels are determined and therefore the frequency of a laser or the spectral signature of a star are obtained by it in quantum mechanics. The resolvent set of the band operators is important for solving in above explanations problems. Band matrices emerge in many areas of mathematics and its applications. Tridiagonal, or more general, banded matrices are used in telecommunication system analysis, finite difference methods for solving partial differential equations, linear recurrence systems with non-constant coefficients, etc, (see [27]).

Quite recently, many authors have studied several types of spectra which have important applications; for example, the approximate point spectrum, defect spectrum, compression spectrum, essential spectrum, etc.

Let $L : X \to Y$ be a bounded linear operator where X and Y are Banach spaces. Denote the range of L,

$R(L) = \{y \in Y : y = Lx, \ x \in X\}$

and

$B(X) = \{L : X \to X : L \text{ is bounded linear operator}\}$.

Assume that X be a Banach space and $L \in B(X)$. The adjoint operator $L^* \in B(X^*)$ of L is defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$ where X^* is the dual space X.

Let X is a complex normed linear space and $D(L) \subset X$ be domain of L where $L : D(L) \to X$ is a linear operator. For $L \in B(X)$ we determine a complex number λ by the operator $(\lambda I - L)$ denoted by L_λ which has the same domain $D(L)$, such that I is the identity operator. Recall that the resolvent operator of L_λ is $L_\lambda^{-1} := (\lambda I - L)^{-1}$.

Let $\lambda \in \mathbf{C}$. If L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X then λ is called a regular value of L.

The set $\rho(L, X)$ of all regular values of L is called the resolvent set of L. $\sigma(L, X) := \mathbf{C} \backslash \rho(L; X)$ is called the spectrum of L where $\mathbf{C}$ is complex plane. Hence those values $\lambda \in \mathbf{C}$ for which L_λ is not invertible are contained in the spectrum $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: The

point (discrete) spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. Further $\lambda \in \sigma_p(L, X)$ is called the eigen value of L. We say that $\lambda \in \mathbf{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L if the resolvent operator L_λ^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbf{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the resolvent operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X; in this case L_λ^{-1} may be bounded or unbounded. Together with the point spectrum, these two subspectra form a disjoint subdivision

$$\text{(1.1)} \qquad \sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X)$$

of the spectrum of L.

Also the spectrum $\sigma(L, X)$ is partitioned into three sets which are not necessarily disjoint as follows:

If there exists a sequence (x_k) in X shuch that $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$ then (x_k) is called Weyl sequence for L.

We call the set

$$\text{(1.2)} \quad \sigma_{ap}(L, X) := \{\lambda \in \mathbf{C} : \text{there exists a Weyl sequence for } \lambda I - L\}$$

the approximate point spectrum of L. Moreover, the subspectrum

$$\text{(1.3)} \qquad \sigma_\delta(L, X) := \{\lambda \in \sigma(L, X) : \lambda I - L \text{ is not surjective}\}$$

is called defect spectrum of L. There exists another subspectrum,

$$\text{(1.4)} \qquad \sigma_{co}(L, X) = \{\lambda \in \mathbf{C} : \overline{R(\lambda I - L)} \neq X\}$$

which is often called compression spectrum in the literature. Clearly, $\sigma_p(L, X) \subseteq \sigma_{ap}(L, X)$ and $\sigma_{co}(L, X) \subseteq \sigma_\delta(L, X)$.

The following Proposition is quitly useful for calculating the separation of the spectrum of linear operator in Banach spaces.

Proposition 1 ([4], Proposition 1.3). *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(L^*, X^*) = \sigma(L, X)$, (b) $\sigma_c(L^*, X^*) \subseteq \sigma_{ap}(L, X)$,
(c) $\sigma_{ap}(L^*, X^*) = \sigma_\delta(L, X)$, (d) $\sigma_\delta(L^*, X^*) = \sigma_{ap}(L, X)$,
(e) $\sigma_p(L^*, X^*) = \sigma_{co}(L, X)$, (f) $\sigma_{co}(L^*, X^*) \supseteq \sigma_p(L, X)$,
(g) $\sigma(L, X) = \sigma_{ap}(L, X) \cup \sigma_p(L^*, X^*) = \sigma_p(L, X) \cup \sigma_{ap}(L^*, X^*)$.

Goldberg's Classification of Spectrum

If $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$, (II) $\overline{R(T)} = X$, but $R(T) \neq X$, (III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous, (2) T^{-1} exists but discontinuous, (3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [16]).

If λ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$, then $\lambda \in \rho(L, X)$. All scalar values of λ not in $\rho(L, X)$ comprise the spectrum of L. The further classification of $\sigma(L, X)$ gives rise to the fine spectrum of L. That is, $\sigma(L, X)$ can be divided into the subsets $I_2\sigma(L, X) = \emptyset$, $I_3\sigma(L, X)$, $II_2\sigma(L, X)$, $II_3\sigma(L, X)$, $III_1\sigma(L, X)$, $III_2\sigma(L, X)$, $III_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L, X)$.

By the definitions given above, we can write following table

		1 L_λ^{-1} exists and is bounded	2 L_λ^{-1} exists and is unbounded	3 L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L, X)$	$-$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L, X)$	$\lambda \in \sigma_c(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$

Table 1

Let us denote the set of all sequences; the space of all null sequences; space of all convergent sequences; space of all sequences such that $\sum_k |x_k|^p < \infty$ by w; c_0; c; ℓ_p; respectively.

Lemma 1 ([16], Theorem II 3.11). *The adjoint operator T^* is onto if and only if T has a bounded inverse.*

Lemma 2 ([16], Theorem II 3.7). *A linear operator T has a dense range if and only if the adjoint operator T^* is one to one.*

Lemma 3 ([17], Sections 28 Theorem 2). *The sequence of the factors in a convergent infinite product always tends 1.*

2. Results and discussion

The matrices which are the infinite element or finite difference problems are frequently banded in numerical analysis. We define the relationship between the problem variables helping by these matrices. The bandedness is confirmed with variables which are not conjugate in arbitrarily large distances. We can furthermore divide these matrices. For example, there are banded matrices with every element in the band is nonzero. We generally encounter these matrices while we are separating one-dimensional problems.

In addition, there are also band matrices in higher dimensional problems. Herein the bands are thiner. For example, the matrix which its bandwidth is the square root of the matrix dimension, correspond to partial differential equation defined in a square domain where the five diagonals are not zero in the band. Unfortunately, if we apply Gaussian elimination to this matrix, we obtain matrix which has the band with many non-zero elements. Therefore the resolvent set of the band operators is important for solving such problems (see [20]).

In the last years, several authors have investigated spectral divisions of generalized difference matrices. For example, Akhmedov and El-Shabrawy, [1, 2] have investigated the spectrum and fine spectrum of the generalized lower triangle double-band matrix Δ_v over the sequence spaces c_0, c and ℓ_p, where $1 < p < \infty$. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c, has investigated by Altay and Başar [3] etc.

The above-mentioned articles are concerned with the decomposition of the spectrum which defined by Goldberg. However, in [9] Durna and Yildirim have investigated subdivision of the spectra for factorable matrices on c_0 and in [5] Başar, Durna and Yildirim have investigated subdivisions of the spectra for generalized difference operator over certain sequence spaces. In [22], the norm and spectrum of the Cesàro matrix considered as a bounded operator on $\overline{bv_0} \cap \ell_\infty$ were studied by Tripathy and Saikia. In

[23], Tripathy and Paul examined the spectra of the operator $D(r, 0, 0, s)$ on sequence spaces c_0 and c. In [24], the spectra of the Rhaly operator on the class of bounded statistically null bounded variation sequence space was determined by Tripathy and Das. In [19], Paul and Tripathy investigated the fine spectrum of the operator $D(r, 0, 0, s)$ over a sequence space bv_0. In [25], Tripathy and Das determined the spectrum and subdivisions of the spectrum of the upper triangular matrix $U(r, s)$ on the sequence space cs. In [6], the spectrum and fine spectrum of the lower triangular matrix $B(r, s, t)$ on the sequence space cs were studied by Das and Tripathy. In [8], the fine spectrum of the lower triangular matrix $B(r, s)$ over the Hahn sequence space was investigated by Das. In [10], [11] Durna has studied subdivision of the spectra for the generalized difference operators over the sequence spaces c_0, c and ℓ_p, $(1 < p < \infty)$. In [18], Paul and Tripathy studied the spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p and bv_p. In [7], Das has calculated the spectrum and fine spectrum of the upper triangular matrix $U(r_1, r_2; s_1, s_2)$ over the sequence space c_0. In [15], El-Shabrawy and Abu-Janah determined spectra and the fine spectra of generalized difference operator $B(r, s)$ on the sequence spaces bv_0 and h. In [28], Yildirim and Durna examined the spectrum and some subdivisions of the spectrum of discrete generalized Cesaro operators on ℓ_p, $(1 < p < \infty)$. In [26], the fine spectrum of the upper triangular matrix $U(r, 0, 0, s)$ over the squence spaces c_0 and c was studied by Tripathy and Das. In [12], Durna et al. studied partition of the spectra for the generalized difference operator $B(r, s)$ on the sequence space cs, in [13], Durna studied subdivision of spectra for some lower triangular doule-band matrices as operators on c_0.

2.1. The fine spectrum of the operator Δ_v on c and ℓ_p, $1 < p < \infty$

In [21] Srivastava and Kumar have defined the generalized difference operator Δ_v as follows:

Let the sequence (v_k) is assumed to be either constant or strictly decreasing sequence of positive real numbers satisfying

$$(2.1) \qquad \lim_{k \to \infty} v_k = L > 0$$

and

$$(2.2) \qquad \sup_k v_k \le 2L.$$

Then the generalized difference operator
Δ_v is $\Delta_v x = \Delta_v (x_n) = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^{\infty}$ with $x_{-1} = v_{-1} = 0$.

The Δ_v's matrix representation is

$$(2.3) \qquad \Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \cdots \\ -v_0 & v_1 & 0 & \cdots \\ 0 & -v_1 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $v_k = L \neq 0$ for all $k \in \mathbf{N}$ is a constant sequence, then the operator Δ_v is the operator $B(r, s)$ with $r = L$, $s = -L$ and the results for the subdivisions of the spectra for generalized difference operator Δ_v over c_0, c, ℓ_p and bv_p have been studied in [5].

2.1.1. Partition of the spectrum of Δ_v on c

The fine spectrum of the operator Δ_v has been investigated by Akhmedov and El-Shabrawy [1] and [2] on the sequence space c. In this study, let us assume that $v_0 \neq 2L$. Herein we mention the main results.

Theorem 1 ([1], Theorem 2.2). $\sigma(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\}$.

Theorem 2 ([1], Theorem 2.3). $\sigma_p(\Delta_v, c) = \emptyset$.

The following lemma is useful for finding the adjoint of a linear transformation on the sequence space c.

Lemma 4. [29, p.267]If $T : c \to c$ is a linear transformation and $T^* : \ell_1 \to \ell_1$, $T^* g = g \circ T$, $g \in c^* \cong \ell_1$, then T and T^* have matrix representations, also $T^* : \ell_1 \to \ell_1$ is given by

$$T^* = A^* = \begin{pmatrix} \chi(\lim A) & (\vartheta_n)_{n=0}^{\infty} \\ (a_k)_{k=0}^{\infty} & A^t \end{pmatrix}$$

$$= \begin{pmatrix} \chi(\lim A) & \vartheta_0 & \vartheta_1 & \vartheta_2 & \cdots \\ a_0 & a_{00} & a_{10} & a_{20} & \cdots \\ a_1 & a_{01} & a_{11} & a_{21} & \cdots \\ a_2 & a_{02} & a_{12} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\chi(\lim A) = \lim Ae - \sum_{k=0}^{\infty} \lim Ae_k = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk}$$

$$\vartheta_n = \chi(P_n \circ T) = (P_n \circ T)e - \sum_k a_{nk},$$

$$a_{nk} = P_n(T(e_k)) = (T(e_k))_n.$$

From Lemma 4 the adjoint of $\Delta_v : c \to c$ is the matrix

$$\Delta_v^* = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_v^t \end{pmatrix}$$

and $\Delta_v^* \in B(\ell_1)$.

Theorem 3 ([2], Theorem 2.7). $\sigma_p(\Delta_v^*, c^*) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \cup H \cup \{0\}$,

where

$$H = \left\{ \lambda \in \mathbf{C} : |\lambda - L| = L, \ \sum_{k=2}^{\infty} \left| \prod_{i=0}^{k-2} \frac{\lambda - v_i}{v_i} \right| < \infty \right\}.$$

Theorem 4 ([2], Theorem 2.9). $\sigma_r(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \cup H \cup \{0\}$.

Theorem 5 ([2], Theorem 2.11). $\sigma_c(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| = L\} \setminus (H \cup \{0\})$.

Lemma 5. *For $p, r \in \mathbf{N}$,*

$$\sum_{n=p}^{\infty} \left(\sum_{k=r}^{n-r} a_k b_{nk} \right) = \sum_{k=r}^{\infty} a_k \left(\sum_{n=p}^{\infty} b_{nk} \right)$$

where (a_k) and (b_{nk}) are nonnegative real numbers and $p \geq 2r$.

Proof. $\sum_{n=p}^{\infty} \left(\sum_{k=r}^{n-r} a_k b_{nk} \right) = \sum_{k=r}^{p-r} a_k b_{nk} + \sum_{k=r}^{p+1-r} a_k b_{nk} + \sum_{k=r}^{p+2-r} a_k b_{nk} + \sum_{k=r}^{p+3-r} a_k b_{nk} +$

$\cdots$

$= (a_r b_{pr} + a_{r+1} b_{p,r+1} + a_{r+2} b_{p,r+2} + \cdots + a_{p-r} b_{p,p-r})$
$+ (a_r b_{p+1,r} + a_{r+1} b_{p+1,r+1} + a_{r+2} b_{p+1,r+2} + \cdots + a_{p+1-r} b_{p+1,p+1-r})$
$+ (a_r b_{p+2,r} + a_{r+1} b_{p+2,r+1} + a_{r+2} b_{p+2,r+2} + \cdots + a_{p+2-r} b_{p+2,p+2-r}) + \cdots$
$= a_r (b_{pr} + b_{p+1,r} + b_{p+2,r} + \cdots) + a_{r+1} (b_{p,r+1} + b_{p+1,r+1} + b_{p+2,r+1} + \cdots)$
$+ a_{r+2} (b_{p,r+2} + b_{p+1,r+2} + b_{p+2,r+2} + \cdots) + \cdots$
$= a_r \sum_{n=p}^{\infty} b_{nr} + a_{r+1} \sum_{n=p}^{\infty} b_{n,r+1} + a_{r+2} \sum_{n=p}^{\infty} b_{n,r+2} + \cdots$
$= \sum_{k=r}^{\infty} a_k \left(\sum_{n=p}^{\infty} b_{nk} \right)$

Theorem 6. $III_1 \sigma(\Delta_v, c) = H \cup \{v_k : k \in \mathbf{N}\}$.

Proof. Let us investigate whether the operator $(\lambda I - \Delta_v)^* = \lambda I - \Delta_v^*$ is surjective or not. Does there exist $x \in \ell_1$ for all $y \in \ell_1$ such that $(\lambda I - \Delta_v^*) x = y$? Firstly, we assume that $\lambda = 0$. In this case, there is no $x \in \ell_1$ for $y = e_0 = (1, 0, 0, \ldots) \in \ell_1$ such that $(-\Delta_v^*) x = y$. Therefore Δ_v^*

is not surjective. Hence from Lemma 1, we have $0 \notin III_1\sigma(\Delta_v, c)$. Now, we assume that $\lambda \neq 0$. In this case, if $(\lambda I - \Delta_v^*)x = y$ for all $y \in \ell_1$, then we obtain that

$$
\begin{aligned}
\lambda x_0 &= y_0 \\
(\lambda - v_0)x_1 + v_0 x_2 &= y_1 \\
(\lambda - v_1)x_2 + v_1 x_3 &= y_2 \\
&\vdots \\
(\lambda - v_{n-1})x_n + v_{n-1}x_{n+1} &= y_n \\
&\vdots
\end{aligned}
$$

Hence

$$
x_0 = \frac{1}{\lambda}y_0
$$

$$
x_1 = x_1
$$

$$
x_2 = \frac{v_0 - \lambda}{v_0}x_1 + \frac{1}{v_0}y_1
$$

$$
x_3 = \frac{v_1 - \lambda}{v_1}x_2 + \frac{1}{v_1}y_2 = \frac{v_0 - \lambda}{v_0}\frac{v_1 - \lambda}{v_1}x_1 + \frac{1}{v_0}\frac{v_1 - \lambda}{v_1}y_1 + \frac{1}{v_1}y_2
$$

$$
\begin{aligned}
x_4 = \frac{v_2 - \lambda}{v_2}x_3 + \frac{1}{v_2}y_3 &= \frac{v_0 - \lambda}{v_0}\frac{v_1 - \lambda}{v_1}\frac{v_2 - \lambda}{v_2}x_1 + \frac{v_1 - \lambda}{v_1}\frac{v_2 - \lambda}{v_2}y_1 \\
&+ \frac{1}{v_1}\frac{v_2 - \lambda}{v_2}y_2 + \frac{1}{v_2}y_3
\end{aligned}
$$

$$
\begin{aligned}
x_5 = \frac{v_3 - \lambda}{v_3}x_4 + \frac{1}{v_3}y_4 &= \frac{v_0 - \lambda}{v_0}\frac{v_1 - \lambda}{v_1}\frac{v_2 - \lambda}{v_2}\frac{v_3 - \lambda}{v_3}x_1 + \frac{v_1 - \lambda}{v_1}\frac{v_2 - \lambda}{v_2}\frac{v_3 - \lambda}{v_3}y_1 \\
&+ \frac{1}{v_1}\frac{v_2 - \lambda}{v_2}\frac{v_3 - \lambda}{v_3}y_2 + \frac{1}{v_2}\frac{v_3 - \lambda}{v_3}y_3 + \frac{1}{v_3}y_4
\end{aligned}
$$

$$\cdots$$

Thus

$$
x_n = x_1 \prod_{k=0}^{n-2}\frac{v_k - \lambda}{v_k} + y_1 \prod_{k=1}^{n-2}\frac{v_k - \lambda}{v_k} + \sum_{k=2}^{n-2}\frac{y_k}{v_{k-1}}\prod_{i=k}^{n-2}\frac{v_i - \lambda}{v_i} + \frac{1}{v_{n-2}}y_{n-1}, \ n \geq 4.
$$

(2.4)

Now, we must show that $x \in \ell_1$. That is, is the series $\sum\limits_{n=0}^{\infty}|x_n|$ convergent? We have

$$
\sum_{n=0}^{\infty}|x_n| = |x_0| + |x_1| + |x_2| + |x_3| + \sum_{n=4}^{\infty}|x_n|
$$

$$
\leq |x_0| + |x_1| + |x_2| + |x_3| + |x_1|\sum_{n=4}^{\infty}\left|\prod_{k=0}^{n-2}\frac{v_k - \lambda}{v_k}\right| + |y_1|\sum_{n=4}^{\infty}\left|\prod_{k=1}^{n-2}\frac{v_k - \lambda}{v_k}\right|
$$

$$+ \sum_{n=4}^{\infty} \left| \sum_{k=2}^{n-2} \frac{y_k}{v_{k-1}} \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right| + \sum_{n=4}^{\infty} \left| \frac{1}{v_{n-2}} y_{n-1} \right|.$$

Let $\sum_1 = \sum_{n=4}^{\infty} \left| \prod_{k=0}^{n-2} \frac{v_k - \lambda}{v_k} \right|$, $\sum_2 = \sum_{n=4}^{\infty} \left| \prod_{k=1}^{n-2} \frac{v_k - \lambda}{v_k} \right|$, $\sum_3 = \sum_{n=4}^{\infty} \left| \sum_{k=2}^{n-2} \frac{y_k}{v_{k-1}} \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right|$

and $\sum_4 = \sum_{n=4}^{\infty} \left| \frac{1}{v_{n-2}} y_{n-1} \right|$. Since for all $n \in \mathbf{N}$, $\frac{1}{v_n} \leq \frac{1}{L}$, the series

$$\sum_4 = \sum_{n=4}^{\infty} \left| \frac{1}{v_{n-2}} y_{n-1} \right| \leq \frac{1}{L} \sum_{n=4}^{\infty} |y_{n-1}| \leq \frac{1}{L} \|y\|_{\ell_1}$$

is convergent. If $|\lambda - L| < L$, then $\lim_{k \to \infty} \left| \frac{v_k - \lambda}{v_k} \right| = \left| \frac{L - \lambda}{L} \right| < 1 \neq 1$ and

from Lemma 3, the product $\prod_i \frac{v_i - \lambda}{v_i}$ is divergent. Hence for $\lambda \in \sigma_r(\Delta_v, c)$,

the series $\sum_1$ and $\sum_2$ are convergent if and only if $\lambda \in H \cup \{v_k : k \in \mathbf{N}\}$.
Now, let us investigate the series $\sum_3$ to be convergent. If $\lambda \in \{v_k : k \in \mathbf{N}\}$,
then it is clear that the series $\sum_3$ is convergent. Let $\lambda \in H$. Then, we get

$$\sum_3 = \sum_{n=4}^{\infty} \left| \sum_{k=2}^{n-2} \frac{y_k}{v_{k-1}} \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right| \leq \frac{1}{L} \sum_{n=4}^{\infty} \left[\sum_{k=2}^{n-2} |y_k| \left| \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right| \right].$$

Thus, if we take $p = 4$, $r = 2$, $a_k = |y_k|$ and $b_{nk} = \left| \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right|$ in Lemma

5, we get

$$\sum_3 \leq \frac{1}{L} \sum_{k=2}^{\infty} \left[|y_k| \sum_{n=4}^{\infty} \left| \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right| \right].$$

Since $\lambda \in H$, $\sum_{n=4}^{\infty} \left| \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right|$ is convergent. Setting $M := \sum_{n=4}^{\infty} \left| \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} \right|$,

we obtain that

$$\sum_3 \leq \frac{M}{L} \sum_{k=2}^{\infty} |y_k| \leq \frac{M}{L} \|y\|_{\ell_1}$$

and $\sum_3$ is convergent. That is, for $\lambda \in \sigma_r(\Delta_v, c)$, the operator $(\lambda I - \Delta_v)^*$
is surjective if and only if $\lambda \in H \cup \{v_k : k \in \mathbf{N}\}$. Thus from Lemma 1,
$\lambda I - \Delta_v$ has bounded inverse.

Corollary 1. $III_2\sigma(\Delta_v, c) = (\{\lambda \in \mathbf{C} : |\lambda - L| < L\} \setminus \{v_k : k \in \mathbf{N}\}) \cup \{0\}$.

Proof.   It is clear from Theorem 4 and Theorem 6 since
$III_2\sigma(\Delta_v, c) = \sigma_r(\Delta_v, c) \setminus III_1\sigma(\Delta_v, c)$.

Theorem 7. *(a)* $\sigma_{ap}(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\} \setminus (H \cup \{v_k : k \in \mathbf{N}\})$,
(b) $\sigma_{\delta}(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\}$,
(c) $\sigma_{co}(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \cup H \cup \{0\}$.

Proof.   (a) It is clear from Theorem 1 and Theorem 6 since $\sigma_{ap}(\Delta_v, c) = \sigma(\Delta_v, c) \setminus III_1\sigma(\Delta_v, c)$ from Table 1. (b) From Theorem 2, $I_3\sigma(\Delta_v, c) = II_3\sigma(\Delta_v, c) = III_3\sigma(\Delta_v, c) = \emptyset$ since $\sigma_p(\Delta_v, c) = I_3\sigma(\Delta_v, c) \cup II_3\sigma(\Delta_v, c) \cup III_3\sigma(\Delta_v, c)$ from Table 1. Also, since from Table 1, $\sigma_\delta(\Delta_v, c) = \sigma(\Delta_v, c) \setminus I_3\sigma(\Delta_v, c)$, the proof is finished from Theorem 1. (c) It is clear from Theorem 4 since from Table 1, $\sigma_{co}(\Delta_v, c) = III_1\sigma(\Delta_v, c) \cup III_2\sigma(\Delta_v, c) \cup III_3\sigma(\Delta_v, c) = \sigma_r(\Delta_v, c) \cup III_3\sigma(\Delta_v, c)$.

Corollary 2. (a) $\sigma_{ap}(\Delta_v^*, \ell_1) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\}$,
(b) $\sigma_\delta(\Delta_v^*, \ell_1) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\} \setminus (H \cup \{v_k : k \in \mathbf{N}\})$.

Proof.   It is clear from Theorem 7 and Proposition 1 (c) and (d).

2.1.2. Partition of the spectrum of Δ_v on ℓ_p, $1 < p < \infty$

El-Shabrawy [14] and Akhmedov, El-Shabrawy [2] have examined the fine spectrum of the operator Δ_v on the sequence space ℓ_p, $(1 < p < \infty)$. Herein we mention the main results.

Theorem 8 ([14], Theorem 2.2). $\sigma(\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\}$.

Theorem 9 ([14], Theorem 2.3). $\sigma_p(\Delta_v, \ell_p) = \emptyset$.

Let $T : \ell_p \to \ell_p$ $(1 < p < \infty)$ be a bounded linear operator and A be its matrix representation. We know that the adjoint operator $T^* : \ell_p^* \to \ell_p^*$ is a bounded linear operator and A^t is its matrix representation. We notice that the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_q with $p^{-1} + q^{-1} = 1$.

Theorem 10 ([2], Theorem 2.20). $\sigma_p\left(\Delta_v^*, \ell_p^*\right) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \cup H_1$, where

$$H_1 = \left\{ \lambda \in \mathbf{C} : |\lambda - L| = L, \ \sum_{k=1}^{\infty} \left| \prod_{i=0}^{k-1} \frac{\lambda - v_i}{v_i} \right|^q < \infty \right\}.$$

Theorem 11 ([2], Theorem 2.21). $\sigma_r(\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \cup H_1$.

Theorem 12 ([2], Theorem 2.22). $\sigma_c(\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| = L\} \setminus H_1$.

Theorem 13. $III_1\sigma(\Delta_v, \ell_p) = H_1 \cup \{v_k : k \in \mathbf{N}\}$.

Proof. Let us investigate whether the operator $(\lambda I - \Delta_v)^* = \lambda I - \Delta_v^*$ is surjective or not. Does there exist $x \in \ell_q$ for all $y \in \ell_q$ such that $(\lambda I - \Delta_v^*) x = y$? If for all $y \in \ell_q$, $(\lambda I - \Delta_v^*) x = y$, then we get

$$
\begin{aligned}
(\lambda - v_0)\, x_0 + v_0 x_1 &= y_0 \\
(\lambda - v_1)\, x_1 + v_1 x_2 &= y_1 \\
(\lambda - v_2)\, x_2 + v_2 x_3 &= y_2 \\
&\vdots \\
(\lambda - v_n)\, x_n + v_n x_{n+1} &= y_n \\
&\vdots
\end{aligned}
$$

Therefore

$$
x_1 = \frac{v_0 - \lambda}{v_0} x_0 + \frac{1}{v_0} y_1
$$

$$
x_2 = \frac{v_1 - \lambda}{v_1} x_1 + \frac{1}{v_1} y_1 = \frac{v_0 - \lambda}{v_0}\frac{v_1 - \lambda}{v_1} x_0 + \frac{1}{v_0}\frac{v_1 - \lambda}{v_1} y_0 + \frac{1}{v_1} y_1
$$

$$
x_3 = \frac{v_2 - \lambda}{v_2} x_2 + \frac{1}{v_2} y_2 = \frac{v_0 - \lambda}{v_0}\frac{v_1 - \lambda}{v_1}\frac{v_2 - \lambda}{v_2} x_0 + \frac{1}{v_0}\frac{v_1 - \lambda}{v_1}\frac{v_2 - \lambda}{v_2} y_0
$$
$$
+ \frac{1}{v_1}\frac{v_2 - \lambda}{v_2} y_1 + \frac{1}{v_2} y_2
$$
$$
\cdots
$$

Thus

$$
(2.5) \qquad x_n = x_0 \prod_{k=0}^{n-1} \frac{v_k - \lambda}{v_k} + \sum_{k=1}^{n-1} \frac{y_{k-1}}{v_{k-1}} \prod_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} + \frac{y_{n-1}}{v_{n-1}}, \; n \geq 2.
$$

Now, we must show that $x \in \ell_q$. That is, is the series $\sum_{n=0}^{\infty} |x_n|^q$ convergent? From Minkowski inequality, we have

$$
\left(\sum_{n=2}^{\infty} |x_n|^q \right)^{1/q} = \left(\sum_{n=2}^{\infty} \left| x_0 \prod_{k=0}^{n-1} \frac{v_k-\lambda}{v_k} + \sum_{k=1}^{n-1} \frac{y_{k-1}}{v_{k-1}} \prod_{i=k}^{n-1} \frac{v_i-\lambda}{v_i} + \frac{y_{n-1}}{v_{n-1}} \right|^q \right)^{1/q}
$$

$$
\leq |x_0| \left(\sum_{n=2}^{\infty} \left| \prod_{k=0}^{n-1} \frac{v_k-\lambda}{v_k} \right|^q \right)^{1/q} + \left[\sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \left| \frac{y_{k-1}}{v_{k-1}} \right|^q \left| \prod_{i=k}^{n-1} \frac{v_i-\lambda}{v_i} \right|^q \right) \right]^{1/q}
$$

$$
+ \left(\sum_{n=2}^{\infty} \left| \frac{y_{n-1}}{v_{n-1}} \right|^q \right)^{1/q}.
$$

Let $\sum_5 = \sum_{n=2}^{\infty} \left| \prod_{k=0}^{n-1} \frac{v_k-\lambda}{v_k} \right|^q$, $\sum_6 = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \left| \frac{y_{k-1}}{v_{k-1}} \right|^q \left| \prod_{i=k}^{n-1} \frac{v_i-\lambda}{v_i} \right|^q \right)$, $\sum_7 = \sum_{n=2}^{\infty} \left| \frac{y_{n-1}}{v_{n-1}} \right|^q$. Since for all $n \in \mathbf{N}$, $\frac{1}{v_n} \leq \frac{1}{L}$, the series

$$\sum\nolimits_{7} = \sum_{n=2}^{\infty} \left| \frac{y_{n-1}}{v_{n-1}} \right|^q \le \frac{1}{L^q} \sum_{n=4}^{\infty} |y_{n-1}|^q \le \frac{1}{L^q} \|y\|_{\ell_q}^q$$

is convergent. If $|\lambda - L| < L$, then $\lim\limits_{k \to \infty} \left| \frac{v_k - \lambda}{v_k} \right| = \left| \frac{L - \lambda}{L} \right| < 1 \neq 1$ and from Lemma 3, the product $\prod\limits_{i} \frac{v_i - \lambda}{v_i}$ is divergent. Thus for $\lambda \in \sigma_r (\Delta_v, c)$, the series $\sum\nolimits_{5}$ is convergent if and only if $\lambda \in H_1 \cup \{v_k : k \in \mathbf{N}\}$. Now, let us investigate the series $\sum\nolimits_{6}$ to be convergent. If $\lambda \in \{v_k : k \in \mathbf{N}\}$, then it is clear that the series $\sum\nolimits_{6}$ is convergent. Let $\lambda \in H_1$. We get

$$\sum\nolimits_{6} = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \left| \frac{y_{k-1}}{v_{k-1}} \right|^q \left| \prod_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} \right|^q \right) \le \frac{1}{L^q} \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n-1} |y_k|^q \left| \prod_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} \right|^q \right].$$

Therefore, if we take $p = 2$, $r = 1$, $a_k = |y_k|^q$ and $b_{nk} = \left| \prod\limits_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} \right|^q$ in Lemma 5, then we have

$$\sum\nolimits_{6} \le \frac{1}{L^q} \sum_{k=1}^{\infty} \left[|y_k|^q \sum_{n=2}^{\infty} \left| \prod_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} \right|^q \right].$$

Since $\lambda \in H_1$, $\sum\limits_{n=2}^{\infty} \left| \prod\limits_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} \right|^q$ is convergent. Setting $M_1^q := \sum\limits_{n=2}^{\infty} \left| \prod\limits_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} \right|^q$, we obtain that

$$\sum\nolimits_{6} \le \left(\frac{M_1}{L} \right)^q \sum_{n=2}^{\infty} |y_k|^q \le \left(\frac{M_1}{L} \|y\|_{\ell_q} \right)^q$$

and so $\sum\nolimits_{6}$ is convergent. That is, for $\lambda \in \sigma_r (\Delta_v, c)$, the operator $(\lambda I - \Delta_v)^*$ is surjective if and only if $\lambda \in H_1 \cup \{v_k : k \in \mathbf{N}\}$. Thus from Lemma 1, $\lambda I - \Delta_v$ has a bounded inverse.

Corollary 3. $III_2\sigma (\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \setminus \{v_k : k \in \mathbf{N}\}$.

Proof.   It is clear from Theorem 11 and Theorem 13, since $III_2\sigma (\Delta_v, c) = \sigma_r (\Delta_v, c) \setminus III_1\sigma (\Delta_v, c)$.

Theorem 14. (a) $\sigma_{ap} (\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| \le L\} \setminus (H_1 \cup \{v_k : k \in \mathbf{N}\})$,
(b) $\sigma_\delta (\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| \le L\}$,
(c) $\sigma_{co} (\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| < L\} \cup H_1 \cup \{0\}$.

Proof.   (a) It is clear from Theorem 8 and Theorem 13.   (b) From Theorem 9, $I_3\sigma (\Delta_v, \ell_p) = II_3\sigma (\Delta_v, \ell_p) = III_3\sigma (\Delta_v, \ell_p) = \emptyset$ since from Table 1, $\sigma_p (\Delta_v, \ell_p) = I_3\sigma (\Delta_v, \ell_p) \cup II_3\sigma (\Delta_v, \ell_p) \cup III_3\sigma (\Delta_v, \ell_p)$. Also, the proof is finished from Theorem 8, since from Table 1, $\sigma_\delta (\Delta_v, \ell_p) = \sigma (\Delta_v, \ell_p) \setminus I_3\sigma (\Delta_v, \ell_p)$. (c) It is clear from Theorem 11, since from Table 1, $\sigma_{co} (\Delta_v, \ell_p) = III_1\sigma (\Delta_v, \ell_p) \cup III_2\sigma (\Delta_v, \ell_p) \cup III_3\sigma (\Delta_v, \ell_p) = \sigma_r (\Delta_v, \ell_p) \cup III_3\sigma (\Delta_v, \ell_p)$.

Corollary 4. *(a)* $\sigma_{ap}\left(\Delta_v^*, \ell_q\right) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\}$,
(b) $\sigma_\delta\left(\Delta_v^*, \ell_q\right) = \{\lambda \in \mathbf{C} : |\lambda - L| \leq L\} \setminus (H_1 \cup \{v_k : k \in \mathbf{N}\})$.

Proof.   It is clear from Theorem 14 and Proposition 1 (c) and (d).

2.2. The fine spectrum of the modified operator Δ_v on c and ℓ_p, $1 < p < \infty$

Akhmedov and El-Shabrawy [2] have modified the generalized difference operator Δ_v which is represented by the matrix

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \cdots \\ -v_0 & v_1 & 0 & \cdots \\ 0 & -v_1 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

They have eliminated the condition: the sequence (v_k) is strictly decreasing sequence of positive real numbers. Also they have put another condition instead of condition (2.2). That is throughout this section, the sequence (v_k) is assumed to be a sequence of nonzero real numbers which is either constant or satisfying the conditions

$$(2.6) \qquad\qquad\qquad \lim_{k \to \infty} v_k = L > 0$$

and

$$(2.7) \qquad\qquad\qquad \sup_k v_k \leq L.$$

Hereafter the sequence (v_k) satisfies these properties adopted by Akhmedov and El-Shabrawy in [2].

2.2.1. Partition of the spectrum of the modified operator Δ_v on ℓ_p, $1 < p < \infty$

Akhmedov and El-Shabrawy [2] have examined the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the modified operator Δ_v over the sequence space ℓ_p, $(1 < p < \infty)$. Herein we mention the main results.

Theorem 15 ([2], Theorem 3.2). *Let* $D = \{\lambda \in \mathbf{C} : |\lambda - L| \leq |L|\}$ *and* $E = \{v_k : k \in \mathbf{N}, \ |v_k - L| > |L|\}$. *Then* $\sigma\left(\Delta_v, \ell_p\right) = D \cup E$.

Theorem 16 ([2], Theorem 3.3). $\sigma_p\left(\Delta_v, \ell_p\right) = E$.

Theorem 17 ([2], Theorem 3.4). $\sigma_p\left(\Delta_v^*, \ell_p^*\right) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\} \cup \{v_k : k \in \mathbf{N}\}$.

Theorem 18 ([2], Theorem 3.6). $\sigma_r\left(\Delta_v, \ell_p\right) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\}$.

Theorem 19 ([2], Theorem 3.8). $\sigma_c\left(\Delta_v, \ell_p\right) = \{\lambda \in \mathbf{C} : |\lambda - L| = |L|\}$.

Theorem 20. $III_1\sigma\left(\Delta_v, \ell_p\right) = \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$.

Proof. Let us investigate whether the operator $(\lambda I - \Delta_v)^* = \lambda I - \Delta_v^*$ is surjective or not. Does there exist $x \in \ell_q$ for all $y \in \ell_q$ such that $(\lambda I - \Delta_v^*)\, x = y$? If for all $y \in \ell_q$, $(\lambda I - \Delta_v^*)\, x = y$, then from (2.5),

$$x_n = x_0 \prod_{k=0}^{n-1} \frac{v_k - \lambda}{v_k} + \sum_{k=1}^{n-1} \frac{y_{k-1}}{v_{k-1}} \prod_{i=k}^{n-1} \frac{v_i - \lambda}{v_i} + \frac{y_{n-1}}{v_{n-1}}, \ n \geq 2.$$

Now, we must show that $x \in \ell_q$. That is, is the series $\sum\limits_{n=0}^{\infty} |x_n|^q$ convergent?

Since

$$\lim_{n\to\infty} x_n = x_0 \prod_{k=0}^{\infty} \frac{v_k - \lambda}{v_k} + \sum_{k=1}^{\infty} \frac{y_{k-1}}{v_{k-1}} \prod_{i=k}^{\infty} \frac{v_i - \lambda}{v_i} + \lim_{n\to\infty} \frac{y_{n-1}}{v_{n-1}},$$

if for all $k \in \mathbf{N}$, $\lambda \neq v_k$, then the limit of general term of infinite product $\prod\limits_{k=0}^{\infty} \frac{v_k - \lambda}{v_k}$ is $\lim\limits_{n\to\infty} \left|\frac{v_k - \lambda}{v_k}\right| = \left|\frac{v_k - \lambda}{v_k}\right| = \left|\frac{L - \lambda}{L}\right|$. If $\lambda \in \sigma_r\left(\Delta_v, \ell_p\right)$, from Theorem 18, we get $\left|\frac{L - \lambda}{L}\right| < 1$. Thus from Lemma 3 , the infinite product $\prod\limits_{k=0}^{\infty} \frac{v_k - \lambda}{v_k}$ is divergent. This means that $\lim\limits_{n\to\infty} |x_n|^q \neq 0$. Hence $\lambda \in \sigma_r\left(\Delta_v, \ell_p\right)$ and for all $k \in \mathbf{N}$, $\lambda \neq v_k$ implies $x \notin \ell_q$. In this case, $\lambda I - \Delta_v^*$ is not surjective and from Lemma 1, $\lambda I - \Delta_v$ does not have bounded inverse. Now, we assume that $\lambda \in \sigma_r\left(\Delta_v, \ell_p\right)$ and for some $k_0 \in \mathbf{N}$, $\lambda = v_k$. In this case, since the products are zero on the right hand of (2.5), $x_n = \frac{y_{n-1}}{v_{n-1}}$. Let us take elements v_k such that $|v_k - L| < |L|$, for $k \in \mathbf{N}$. If $L > 0$, then all v_k are positive in circle, if $L < 0$, then all v_k are negative in circle. Hence there are two cases for elements of the set $\{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$.

1. case: If $L > 0$, then there exists $M > 0$ such that $M < v_k < L$. Hence $M < |v_k| < L$ and $\frac{1}{L} < \frac{1}{|v_k|} < \frac{1}{M}$. Thus we get

$$\sum_{n=2}^{\infty} |x_n|^q < \frac{1}{M^q} \sum_{n=1}^{\infty} |y_n|^q \leq \frac{1}{M^q} \|y\|_{\ell_q}^q.$$

2. case: If $L < 0$, then $2L < v_k < L$. Thus $|L| < |v_k| < 2|L|$ and $\frac{1}{2|L|} < \frac{1}{|v_k|} < \frac{1}{|L|}$. Therefore we have

$$\sum_{n=2}^{\infty} |x_n|^q < \frac{1}{|L|^q} \sum_{n=1}^{\infty} |y_n|^q \leq \frac{1}{L^q} \|y\|_{\ell_q}^q.$$

That is, the operator $(\lambda I - \Delta_v)^*$ is surjective if and only if $\{\lambda \in v_k : k \in \mathbf{N}, |v_k - L| < |L|\}$. Hence from Lemma 1, $\lambda I - \Delta_v$ has a bounded inverse.

Corollary 5.

$$III_2\sigma(\Delta_v, \ell_p) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\}$$

$$\setminus \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$$

.

Proof. It is clear from Theorem 18 and Theorem 20, since $III_2\sigma(\Delta_v, c) = \sigma_r(\Delta_v, c) \setminus III_1\sigma(\Delta_v, c)$.

Theorem 21. $III_3\sigma(\Delta_v, \ell_p) = E$.

Proof. Let we find $\ker(\lambda I - \Delta_v^*)$. If $(\lambda I - \Delta_v^*)x = 0$, then we have

$$
\begin{aligned}
(\lambda - v_0)x_0 + v_0 x_1 &= 0 \\
(\lambda - v_1)x_1 + v_1 x_1 &= 0 \\
&\vdots \\
(\lambda - v_n)x_n + v_n x_{n+1} &= 0 \\
&\vdots
\end{aligned}
$$

Thus we have
$$x_n = x_0 \prod_{k=0}^{n-1} \frac{v_k - \lambda}{v_k}, n \geq 1.$$
If $\lambda \in E$, then $\lambda = v_k$ and $|v_k - L| > |L|$ and so
$$\ker(\lambda I - \Delta_v^*) = \left\{ (x_0, 0, 0, \ldots), \left(x_0, \frac{v_0 - \lambda}{v_0}x_0, 0, 0, \ldots\right), \right.$$
$$\left. \left(x_0, \frac{v_0 - \lambda}{v_0}x_0, \frac{v_0 - \lambda}{v_0}\frac{v_1 - \lambda}{v_1}x_0, 0, 0, \ldots\right), \ldots\right\}$$
$$\neq \{(0, 0, 0, \ldots)\}.$$

From here, if $\lambda \in E$, then $\lambda I - \Delta_v^*$ is not injective. Hence from Lemma 2, if $\lambda \in E$, then $\lambda I - \Delta_v$ does not have dense range. Therefore we obtain that $III_3\sigma(\Delta_v, \ell_p) = E$.

Corollary 6. $I_3\sigma(\Delta_v, \ell_p) = II_3\sigma(\Delta_v, \ell_p) = \emptyset$.

Proof. It is clear from Theorem 21, since from Table 1, $\sigma_p\left(\Delta_v, \ell_p\right) = I_3\sigma\left(\Delta_v, \ell_p\right) \cup II_3\sigma\left(\Delta_v, \ell_p\right) \cup III_3\sigma\left(\Delta_v, \ell_p\right) = E$ and $I_3\sigma\left(\Delta_v, \ell_p\right) \cap II_3\sigma\left(\Delta_v, \ell_p\right) \cap III_3\sigma\left(\Delta_v, \ell_p\right) = \emptyset$.

Theorem 22. (a) $\sigma_{ap}\left(\Delta_v, \ell_p\right) = (D \cup E) \setminus \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$,
(b) $\sigma_\delta\left(\Delta_v, \ell_p\right) = D \cup E$,
(c) $\sigma_{co}\left(\Delta_v, \ell_p\right) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\} \cup E$.

Proof. (a) It is clear from Theorem 15 and Theorem 20. (b) It is clear from Theorem 15 and Conclusion 6, since from Table 1, $\sigma_\delta\left(\Delta_v, \ell_p\right) = \sigma\left(\Delta_v, \ell_p\right) \setminus I_3\sigma\left(\Delta_v, \ell_p\right)$. (c) Since from Table 1, $\sigma_{co}\left(\Delta_v, \ell_p\right) = III_1\sigma\left(\Delta_v, \ell_p\right) \cup III_2\sigma\left(\Delta_v, \ell_p\right) \cup III_3\sigma\left(\Delta_v, \ell_p\right) = \sigma_r\left(\Delta_v, \ell_p\right) \cup III_3\sigma\left(\Delta_v, \ell_p\right)$, the proof is finished from Theorem 11 and Theorem 21.

Corollary 7. (a) $\sigma_{ap}\left(\Delta_v^*, \ell_q\right) = D \cup E$,
(b) $\sigma_\delta\left(\Delta_v^*, \ell_q\right) = (D \cup E) \setminus \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$.

Proof. It is clear from Theorem 22 and Proposition 1 (c) and (d).

2.2.2. Partition of the spectrum of the modified operator Δ_v on c

Akhmedov and El-Shabrawy [2] have examined the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the modified operator Δ_v on the sequence space c. Herein we mention the main results.

Theorem 23 ([2], Theorem 3.10). $\sigma_p\left(\Delta_v^*, c^*\right) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\} \cup E \cup \{0\}$.

Theorem 24 ([2], Theorem 3.11). (a) $\sigma\left(\Delta_v, c\right) = D \cup E$,
(b) $\sigma_p\left(\Delta_v, c\right) = E$,
(c) $\sigma_r\left(\Delta_v, c\right) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\} \cup \{0\}$,
(d) $\sigma_c\left(\Delta_v, c\right) = \{\lambda \in \mathbf{C} : |\lambda - L| = |L|\} \setminus \{0\}$.

Theorem 25. $III_1\sigma\left(\Delta_v, \ell_p\right) = \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$.

Proof. It can be shown as in Theorem 6 that $0 \notin III_1\sigma\left(\Delta_v, c\right)$. Now, we assume that $\lambda \neq 0$. Let us investigate whether the operator $\left(\lambda I - \Delta_v\right)^* = \lambda I - \Delta_v^*$ is surjective or not. Does there exist $x \in \ell_1$ for all $y \in \ell_1$ such that $\left(\lambda I - \Delta_v^*\right) x = y$. If $\left(\lambda I - \Delta_v^*\right) x = y$, then from (2.4) we get

$$\mathrm{x}_n = x_1 \prod_{k=0}^{n-2} \frac{v_k - \lambda}{v_k} + y_1 \prod_{k=1}^{n-2} \frac{v_k - \lambda}{v_k} + \sum_{k=2}^{n-2} \frac{y_k}{v_{k-1}} \prod_{i=k}^{n-2} \frac{v_i - \lambda}{v_i} + \frac{1}{v_{n-2}} y_{n-1}, \, n \geq 4.$$

Now we show that $x \in \ell_1$. That is, is the series $\sum_{n=0}^{\infty} |x_n|$ convergent?
Since

$$\lim_{n \to \infty} x_n = x_1 \prod_{k=0}^{\infty} \frac{v_k - \lambda}{v_k} + y_1 \prod_{k=1}^{\infty} \frac{v_k - \lambda}{v_k} + \sum_{k=1}^{\infty} \frac{y_k}{v_{k-1}} \prod_{i=k}^{\infty} \frac{v_i - \lambda}{v_i} + \lim_{n \to \infty} \frac{y_{n-1}}{v_{n-2}},$$

the result is obtained as in Theorem 20.

Corollary 8.

$$III_2\sigma\left(\Delta_v, c\right) = \left(\{\lambda \in \mathbf{C} : |\lambda - L| < |L|\} \cup \{0\}\right)$$

$$\setminus \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$$

.

Proof.   It is clear from Theorem 24 and Theorem 25, since
$III_2\sigma\left(\Delta_v, c\right) = \sigma_r\left(\Delta_v, c\right) \setminus III_1\sigma\left(\Delta_v, c\right)$.

Theorem 26. $III_3\sigma\left(\Delta_v, \ell_p\right) = E$.

Proof.   Let we find $\ker\left(\lambda I - \Delta_v^*\right)$. If $\left(\lambda I - \Delta_v^*\right)x = 0$, then we get

$$\begin{aligned}
\lambda x_0 &= 0 \\
(\lambda - v_0) x_1 + v_0 x_2 &= 0 \\
(\lambda - v_1) x_2 + v_1 x_3 &= 0 \\
&\vdots \\
(\lambda - v_{n-1}) x_n + v_{n-1} x_{n+1} &= 0 \\
&\vdots
\end{aligned}$$

Hence we have
$\mathrm{x}_n = x_1 \prod_{k=0}^{n-2} \frac{v_k - \lambda}{v_k}, n \geq 2$.
If $\lambda \in E$, then $\lambda = v_k$ and $|v_k - L| > |L|$. And so, we get

$$\ker\left(\lambda I - \Delta_v^*\right) = \left\{ \begin{array}{l} \left(\frac{1}{\lambda}, x_1, 0, 0, \ldots\right), \left(\frac{1}{\lambda}, x_1, \frac{v_0 - \lambda}{v_0} x_1, 0, 0, \ldots\right) \\ , \left(\frac{1}{\lambda}, x_1, \frac{v_0 - \lambda}{v_0} x_1, \frac{v_0 - \lambda}{v_0} \frac{v_1 - \lambda}{v_1} x_1, 0, 0, \ldots\right), \ldots \end{array} \right\}$$

$\neq \{(0, 0, 0, \ldots)\}$.
This means that if $\lambda \in E$, then $\lambda I - \Delta_v^*$ is not injective. Hence from
Lemma 2, if $\lambda \in E$, then $\lambda I - \Delta_v$ does not have dense range. Therefore
$III_3\sigma\left(\Delta_v, \ell_p\right) = E$.

Corollary 9. $I_3\sigma\left(\Delta_v, c\right) = II_3\sigma\left(\Delta_v, c\right) = \emptyset$.

Proof. It is clear from Theorem 26, since from Table 1 $\sigma_p(\Delta_v, c) = I_3\sigma(\Delta_v, c) \cup II_3\sigma(\Delta_v, c) \cup III_3\sigma(\Delta_v, c) = E$ and $I_3\sigma(\Delta_v, c) \cap II_3\sigma(\Delta_v, c) \cap III_3\sigma(\Delta_v, c) = \emptyset$.

Theorem 27. (a) $\sigma_{ap}(\Delta_v, c) = (D \cup E) \setminus \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$,
(b) $\sigma_\delta(\Delta_v, c) = D \cup E$,
(c) $\sigma_{co}(\Delta_v, c) = \{\lambda \in \mathbf{C} : |\lambda - L| < |L|\} \cup \{0\} \cup E$.

Proof. (a) It is clear from Theorem 24 (a) and Theorem 25. (b) It is clear from Theorem 24 (a) and Conclusion 9, since from Table 1, $\sigma_\delta(\Delta_v, c) = \sigma(\Delta_v, c) \setminus I_3\sigma(\Delta_v, c)$. (c) It is clear from Theorem 24 (c) and Theorem 26, since from Table 1, $\sigma_{co}(\Delta_v, c) = III_1\sigma(\Delta_v, c) \cup III_2\sigma(\Delta_v, c) \cup III_3\sigma(\Delta_v, c) = \sigma_r(\Delta_v, c) \cup III_3\sigma(\Delta_v, c)$.

Corollary 10. (a) $\sigma_{ap}(\Delta_v^*, \ell_1) = D \cup E$,
(b) $\sigma_\delta(\Delta_v^*, \ell_1) = (D \cup E) \setminus \{v_k : k \in \mathbf{N}, \ |v_k - L| < |L|\}$.

Proof. It is clear from Theorem 27 and Proposition 1 (c) and (d).

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