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Z_k-Magic Labeling of Star of Graphs

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Abstract:

For any non-trivial abelian group A under addition a graph G is said to be A-magic if there exists a labeling $f : E(G) \rightarrow A - \{0\}$ such that, the vertex labeling f + defined as f +(v) = Pf(uv) taken over all edges uv incident at v is a constant. An A-magic graph G is said to be Zk-magic graph if the group A is Zk, the group of integers modulo k and these graphs are referred to as k-magic graphs. In this paper we prove that the graphs such as star of cycle, flower, double wheel, shell, cylinder, gear, generalised Jahangir, lotus inside a circle, wheel, closed helm graph are Zk-magic graphs.

Keywords: A-magic labeling; Flower; Double wheel; Shell; Cylinder; Gear; Generalised Jahangir; Lotus inside a circle; Wheel; Closed helm graph.

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1. Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [1]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [10]introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [9] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [11] proved that the product and composition of A-magic graphs were also A-magic.

For any non-trivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \rightarrow A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as *k-magic graphs*. Shiu and Low [12] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Motivated by the concept of A-magic graph in [10] and the results in [9], [11] and [12] Jeyanthi and Jeya Daisy [2]-[8] proved that the open star of graphs, subdivision graphs, cycle of graphs and some standard graphs admit Z_k-magic labeling. We use the following definitions in the subsequent section.

Definition 1.1. *A star of graph G is obtained by replacing each vertex of star $K_{1,n}$ by a graph G. It is denoted by $S(n.G)$.*

Definition 1.2. *A shell graph S_n, $n \geq 4$, is obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called an apex.*

Definition 1.3. *A flower graph Fl_n, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.*

Definition 1.4. *A double wheel graph DW_n, $n \geq 3$, is obtained by joining the vertices of two cycles C_n to an extra vertex called the hub.*

Definition 1.5. *A Cartesian product of a cycle C_n, $n \geq 3$, and a path on two vertices is called a cylinder graph $C_n \square P_2$.*

Definition 1.6. *A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$ is a 3-regular graph with the vertex set $\{u_i, v_i : i = 1, 2, \ldots, n\}$ and the edge set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+m} : i = 1, 2, \ldots, n\}$, where the indices are taken over modulo n.*

Definition 1.7. *A wheel graph W_n, $n \geq 3$, is obtained by joining the vertices of a cycle C_n to an extra vertex called the centre. The vertices of degree three are called rim vertices.*

Definition 1.8. *A generalised Jahangir $J_{k,s}$, is a graph on $ks + 1$ vertices consisting of a cycle C_{ks} and one additional vertex that is adjacent to k vertices of C_{ks} at distance s to each other on C_{ks}.*

Definition 1.9. *A lotus inside a circle LC_n, $n \geq 3$, is a graph obtained from a wheel W_n by subdividing every edge forming the outer cycle and joining these new vertices to form a cycle.*

Definition 1.10. *A helm graph H_n, $n \geq 3$, is obtained by adjoining a pendant edge at each vertex of the wheel except the center.*

Definition 1.11. *A closed helm graph CH_n, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to form a cycle.*

2. Z_k-Magic Labeling of Star of Graphs

In this section we prove that the graphs such as star of cycle, flower, double wheel, shell, cylinder, gear, generalised Jahangir, lotus inside a circle, wheel, closed helm graph are Z_k-magic graphs.

Theorem 2.1. *The star of cycle $S(n.C_n)$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $V(S(n.C_n))$ be $V(S(n.C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{u_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.C_n)) = \{v_i v_{i+1}, u_i^j u_{i+1}^j : 1 \leq i \leq n-1\} \cup \{v_n v_1, u_n^j u_1^j : 1 \leq j \leq n\} \cup \{v_j u_1^j : 1 \leq j \leq n\}$.
Let $a, b \in Z_k - \{0\}$ such that $a + b \not\equiv 0(mod\ k)$. Define the edge labeling $f : E(S(n.C_n)) \to Z_k - \{0\}$ as follows:
$f(v_i v_{i+1}) = a$ for $1 \leq j \leq n-1$,
$f(v_n v_1) = a, f(v_j u_1^j) = b$ for $1 \leq j \leq n$,

$$f(u_i^j u_{i+1}^j) = \begin{cases} a, & \text{for } i \ \ is\ odd,\ 1 \leq j \leq n, \\ a+b, & \text{for } i \ \ is\ even,\ 1 \leq j \leq n. \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(n.C_n)) \to Z_k$ is $f^+(v) \equiv 2a + b(mod\ k)$ for all $v \in V(S(n.C_n))$. Hence f^+ is constant and it is equal to $2a + b(mod\ k)$. Since $S(n.C_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. $\square$

An example of Z_5-magic labeling of $S(5.C_5)$ is shown in Figure 1.

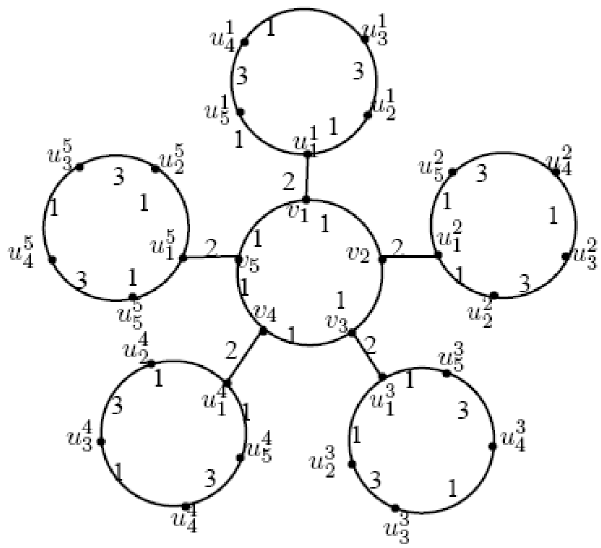

Figure 1: Z_5-magic labeling of $S(5.C_5)$

Theorem 2.2. *The star of shell graph $S(n.S_n)$ is Z_k-magic, when n is odd and $n \neq 3$.*

Proof. Let the vertex set and the edge set of $S(n.S_n)$ be $V(S(n.S_n)) = \{v_i, u_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.S_n)) = \{v_1 v_{i+2} : 1 \leq i \leq n-3\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_j u_1^j : 1 \leq i, j \leq n\} \cup \{u_1^j u_{i+2}^j : 1 \leq i \leq n-3, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{u_n^j u_1^j : 1 \leq j \leq n\}$.
Let a be an integer and $k > (n-2)2a$.
Define the edge labeling $f : E(S(n.S_n)) \to Z_k - \{0\}$ as follows:
$f(v_1 v_{i+2}) = f(u_1^j u_{i+2}^j) = 2a$ for $1 \leq i \leq n-3, 1 \leq j \leq n$,
$f(v_1 u_1^1) \;\; = k - (n-2)2a$, $f(u_1^1 u_2^1) = f(u_1^1 u_n^1) = a$,
$f(u_i^1 u_{i+1}^1) = k - a$ for $2 \leq i \leq n-1$,
$f(v_2 u_1^2) \;\; = f(v_n u_1^n) = k - 2a$,
$f(u_1^2 u_2^2) \;\; = f(u_1^2 u_n^2) = k - (n-4)a$,
$$f(u_i^2 u_{i+1}^2) = \begin{cases} (n-4)a, & \text{for } i \;\; \textit{is even}, \\ k-(n-2)a, & \text{for } i \;\; \textit{is odd}, \; i \neq 1, n. \end{cases}$$
$f(u_1^n u_2^n) \;\; = f(u_1^n u_n^n) = k - (n-4)a$,

$$f(u_i^n u_{i+1}^n) = \begin{cases} (n-4)a, & \text{for } i \ \ is\ even, \\ k-(n-2)a, & \text{for } i \ \ is\ odd,\ i \neq 1, n, \end{cases}$$
$$f(v_j u_1^j) \ \ = k-4a \text{ for } 3 \leq j \leq n-1,$$
$$f(u_1^j u_2^j) \ \ = f(u_1^j u_n^j) = k-(n-5)a \text{ for } 3 \leq j \leq n-1,$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} (n-5)a, & \text{for } i \ \ is\ even,\ 3 \leq j \leq n-1, \\ k-(n-3)a, & \text{for } i \ \ is\ odd,\ i \neq 1, n,\ 3 \leq j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(n.S_n)) \rightarrow Z_k$ is $f^+(v) \equiv \ 0(mod\, k)$ for all $v \in V(S(n.S_n))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $S(n.S_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. □ An example of Z_{13}-magic labeling of $S(7.S_7)$ is shown in Figure 2.

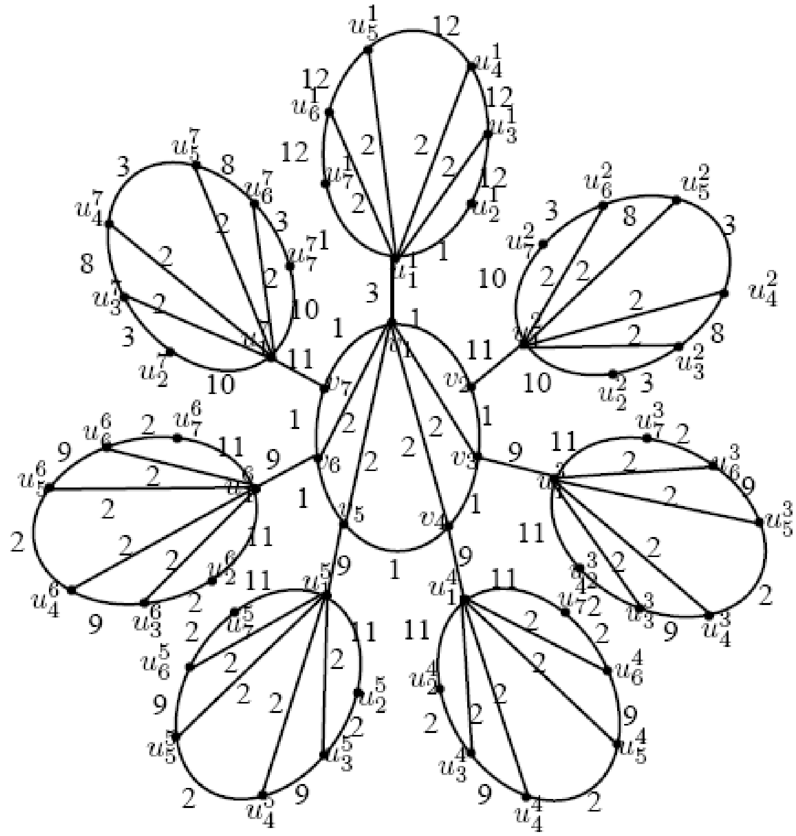

Figure 2: Z_{13}-magic labeling of $S(7.S_7)$

Theorem 2.3. *The star of flower $S(n.Fl_n)$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.Fl_n)$ be $V(S(n.Fl_n)) = \{v, v_i, u_i : 1 \leq i \leq n\} \cup \{w_i, u_i^j, v_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.Fl_n)) = \{vv_i, u_iv_i, u_iv : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{w_jv_i^j : 1 \leq i, j \leq n\} \cup \{u_i^jv_i^j : 1 \leq i, j \leq n\} \cup \{u_i^jw_j : 1 \leq i, j \leq n\} \cup \{v_i^jv_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{v_n^jv_1^j : 1 \leq j \leq n\} \cup \{v_jv_1^j : 1 \leq j \leq n\}$.
Let a be an integer and $k > 4a$.
Define the edge labeling $f : E(S(n.Fl_n)) \rightarrow Z_k - \{0\}$ as follows:
$f(vv_i) = a$ for $1 \leq i \leq n$,
$f(v_iu_i) = a$ for $1 \leq i \leq n$, $f(vu_i) = k - a$ for $1 \leq i \leq n$,
$f(v_iv_{i+1}) = a$ for $1 \leq i \leq n-1$, $f(v_nv_1) = a$,
$f(w_jv_i^j) = a$ for $1 \leq i, j \leq n$,
$f(u_i^jv_i^j) = a$ for $1 \leq i, j \leq n$,
$f(w_ju_i^j) = k - a$ for $1 \leq i, j \leq n$,

$$f(v_i^jv_{i+1}^j) = \begin{cases} a, & \text{for } i \ \ is\ odd,\ 1 \leq j \leq n, \\ k - 3a, & \text{for } i \ \ is\ even,\ 1 \leq j \leq n, \end{cases}$$

$f(v_jv_1^j) = k - 4a$ for $1 \leq j \leq n$.
Then the induced vertex labeling $f^+ : V(S(n.Fl_n)) \rightarrow Z_k$ is $f^+(v) \equiv 0(mod\, k)$ for all $v \in V(S(n.Fl_n))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $S(n.Fl_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. □

An example of Z_{10}-magic labeling of $S(3.Fl_3)$ is shown in Figure 3.

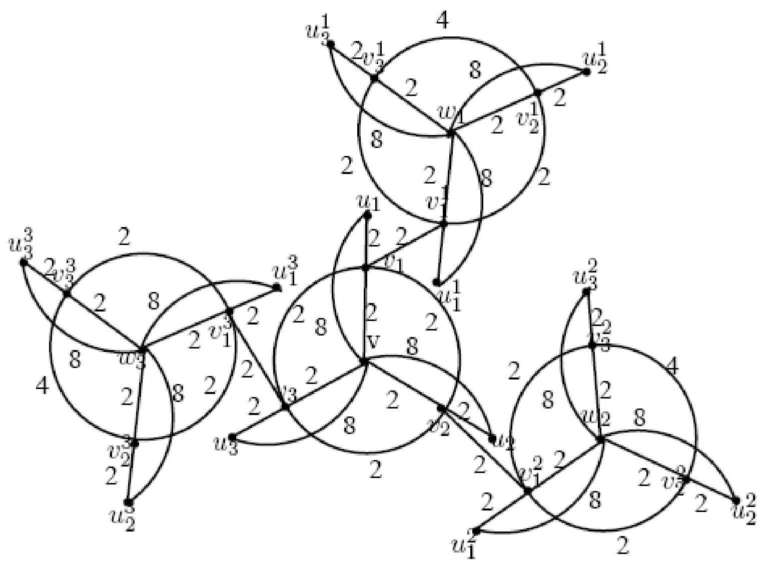

Figure 3: Z_{10}-magic labeling of $S(3.Fl_3)$

Theorem 2.4. *The star of double wheel graph $S(n.DW_n)$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.DW_n)$ be $V(S(n.DW_n)) = \{v, v_i, u_i : 1 \leq i \leq n\} \cup \{w_i, u_i^j, v_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.DW_n)) = \{vv_i, vu_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1, u_nu_1\} \cup \{w_jv_i^j, w_ju_i^j : 1 \leq i, j \leq n\} \cup \{v_i^jv_{i+1}^j, u_i^ju_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{v_n^jv_1^j, u_n^ju_1^j : 1 \leq j \leq n\} \cup \{u_ju_1^j : 1 \leq j \leq n\}$.
Let a be an integer and $k > 4a$.
Define the edge labeling $f : E(S(n.DW_n)) \to Z_k - \{0\}$ as follows:
$f(vv_i) = f(w_jv_i^j) = 2a$ for $1 \leq i, j \leq n$,
$f(vu_i) = f(w_ju_i^j) = k - 2a$ for $1 \leq i, j \leq n$,
$f(v_iv_{i+1}) = f(u_iu_{i+1}) = k - a$ for $1 \leq i \leq n-1$,
$f(v_nv_1) = f(u_nu_1) = k - a$,
$f(u_ju_1^j) = 4a$ for $1 \leq j \leq n$,
$f(v_i^jv_{i+1}^j) = k - a$ for $1 \leq i \leq n-1, 1 \leq j \leq n$,

$f(v_n^j v_1^j) \quad = k - a$ for $1 \leq j \leq n$,

$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a, & \text{for } i \;\; \textit{is odd}, \;\; 1 \leq j \leq n, \\ 3a, & \text{for } i \;\; \textit{is even}, \;\; 1 \leq j \leq n. \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(n.DW_n)) \rightarrow Z_k$ is $f^+(v) \equiv \; 0(mod\, k)$ for all $v \in V(S(n.DW_n))$. Hence f^+ is constant and it is equal to $0(mod\, k)$. Since $S(n.DW_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. $\Box$

An example of Z_5-magic labeling of $S(3.DW_3)$ is shown in Figure 4.

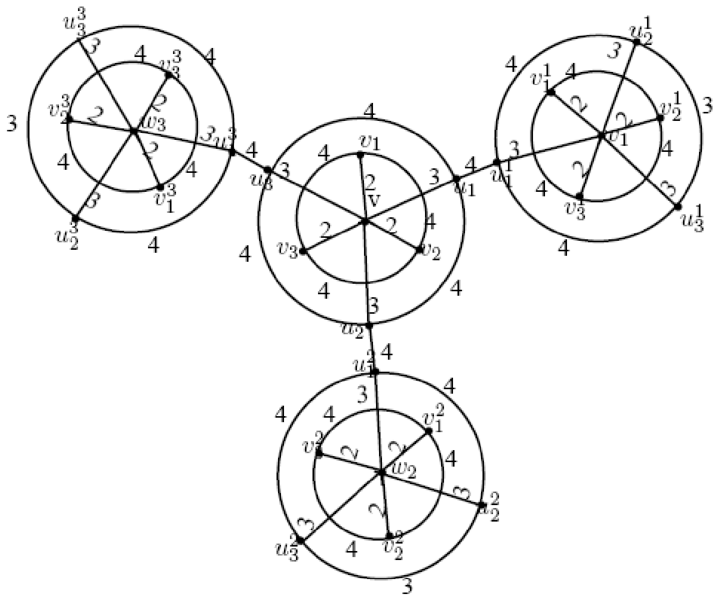

Figure 4: Z_5-magic labeling of $S(3.DW_3)$

Theorem 2.5. *The star of cylinder graph $C_n \Box P_2$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.(C_n \Box P_2))$ be $V(S(n.(C_n \Box P_2))) = \{v_i, u_i : \; 1 \leq i \leq n\} \cup \{v_i^j, u_i^j : \; 1 \leq i, j \leq n\}$ and $E(S(n.(C_n \Box P_2))) = \{v_i v_{i+1}, u_i u_{i+1} : \; 1 \leq i \leq n-1\} \cup \{v_n v_1, u_n u_1\} \cup \{v_i u_i : \; 1 \leq i \leq n\} \cup \{u_j u_1^j : \; 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j, u_i^j u_{i+1}^j : \; 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{v_n^j v_1^j, u_n^j u_1^j : \; 1 \leq j \leq n\} \cup \{v_i^j u_i^j : \; 1 \leq i, j \leq n\} \cup \{u_j u_1^j : 1 \leq j \leq n\}$.

Let a be an integer and $k > 4a$.
Define the edge labeling $f : E(S(n.(C_n \Box P_2))) \to Z_k - \{0\}$ as follows:

$$f(v_i v_{i+1}) = f(v_i^j v_{i+1}^j) = a \text{ for } 1 \le i \le n-1, 1 \le j \le n,$$
$$f(u_i u_{i+1}) = k - a \text{ for } 1 \le i \le n-1,\ f(u_n u_1) = k - a,$$
$$f(v_i u_i) \quad = f(v_i^j u_i^j) = k - 2a \text{ for } 1 \le i \le n, 1 \le j \le n,$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} k-a, & \text{for } i \ \ \textit{is odd},\ 1 \le j \le n, \\ 3a, & \text{for } i \ \ \textit{is even},\ 1 \le j \le n, \end{cases}$$
$$f(u_j u_1^j) \quad = 4a \text{ for } 1 \le j \le n.$$

Then the induced vertex labeling $f^+ : V(S(n.(C_n \Box P_2))) \to Z_k$ is $f^+(v) \equiv \ 0(mod\ k)$ for all $v \in V(S(n.(C_n \Box P_2)))$. Hence f^+ is constant and it is equal to $\ 0(mod\ k)$. Since $S(n.(C_n \Box P_2))$ admits Z_k-magic labeling, it is a Z_k-magic graph. $\Box$

An example of Z_9-magic labeling of $S(5.(C_5 \Box P_2))$ is shown in Figure 5.

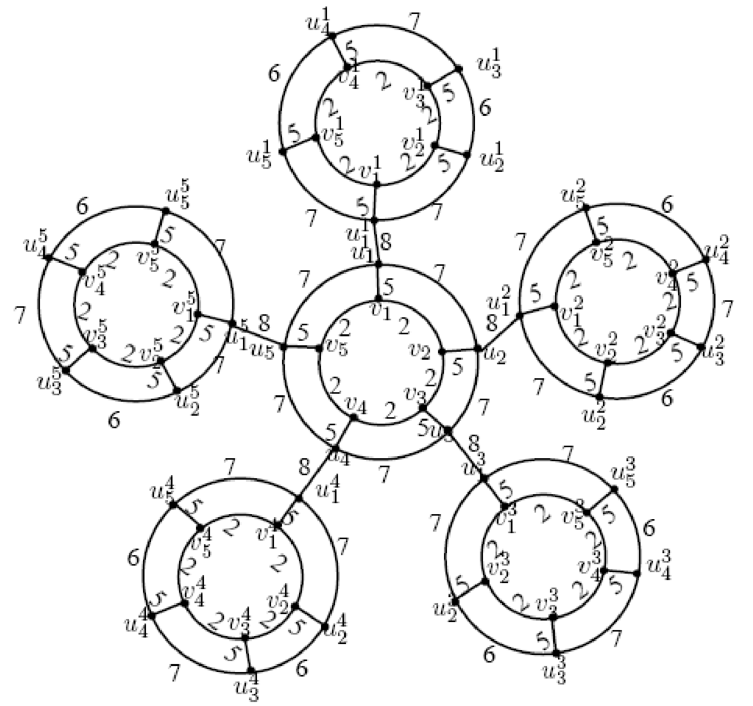

Figure 5: Z_9-magic labeling of $S(5.C_5 \times P_2)$

Theorem 2.6. *The star of generalised Jahangir graph $S(n.J_{n,s})$ is Z_k-magic, when n is odd, $n \ge 3$ and $s \ge 2$.*

Proof. Let the vertex set and the edge set of $S(n.J_{n,s})$ be $V(S(n.J_{n,s})) = \{v, v_i : 1 \le i \le n\} \cup \{v_{i,j} : 1 \le i \le n, 1 \le j \le s-1\} \cup \{w_l : 1 \le l \le n\} \cup \{v_i^l : 1 \le i, l \le n\} \cup \{v_{i,j}^l : 1 \le i, l \le n, 1 \le j \le s-1\}$ and $E(S(n.J_{n,s})) = \{vv_i : 1 \le i \le n\} \cup \{v_j v_{1,j} : 1 \le j \le n\} \cup \{v_{i,j} v_{i,j+1} : 1 \le i \le n, 1 \le j \le s-1\} \cup \{v_{i,s-1} v_{i+1} : 1 \le i \le n-1\} \cup \{v_{n,s-1} v_1\} \cup \{v_i v_1^i : 1 \le i \le n\} \cup \{w_l v_i^l, u_n^j u_1^j : 1 \le i, l \le n\} \cup \{v_i^l v_{1,j}^l : 1 \le i, l \le n, 1 \le j \le s-1\} \cup \{v_{i,j}^l v_{i+1,j}^l : 1 \le i \le n-1, 1 \le j \le s-1, 1 \le l \le n\} \cup \{v_{i,s-1}^l v_{i+1} : 1 \le i \le n-1, 1 \le l \le n\} \cup \{v_{n,s-1}^n v_1^l : 1 \le l \le n\}$.

Let a be an integer and $k > (n-1)a$.
Define the edge labeling $f : E(S(n.J_{n,s})) \to Z_k - \{0\}$ as follows:
$f(vv_1) = k - (n-1)a$,
$f(vv_i) = a$ for $2 \le i \le n$,

For $1 \le j \le n$,
$$f(v_j v_{1,j}) = \begin{cases} k-a, & \text{for } j \ \textit{is odd}, \\ k-2a, & \text{for } j \ \textit{is even}, \end{cases}$$

For i is odd.
$$f(v_{i,j} v_{i,j+1}) = \begin{cases} a, & \text{for } j \ \textit{is odd}, \\ k-a, & \text{for } j \ \textit{is even}, \end{cases}$$

For i is even.
$$f(v_{i,j} v_{i,j+1}) = \begin{cases} 2a, & \text{for } j \ \textit{is odd}, \\ k-2a, & \text{for } j \ \textit{is even}, \end{cases}$$

$f(v_1 v_1^1) = k - (n-3)a$,
$f(v_i v_1^i) = \frac{(n-1)a}{2}$ for $2 \le i \le n$,
$f(w_l v_1^l) = k - (n-1)a$ for $1 \le l \le n$,
$f(w_l v_i^l) = a$ for $2 \le i \le n, 1 \le l \le n$,
$$f(v_i^1 v_{i,1}^1) = \begin{cases} (n-2)a, & \text{for } i \ \textit{is odd}, \\ k-(n-2)a, & \text{for } i \ \textit{is even}. \end{cases}$$

For i is odd.
$$f(v_{i,j}^1 v_{i,j+1}^1) = \begin{cases} k-(n-2)a, & \text{for } j \text{ is odd}, \\ (n-2)a, & \text{for } j \text{ is even}. \end{cases}$$

For i is even.
$$f(v_{i,j}^1 v_{i,j+1}^1) = \begin{cases} (n-1)a, & \text{for } j \text{ is odd}, \\ k-(n-1)a, & \text{for } j \text{ is even}, \end{cases}$$

$$f(v_i^l v_{i,1}^l) = \begin{cases} \frac{(n-3)a}{2}, & \text{for } i \text{ is odd, } 2 \leq l \leq n, \\ k - \frac{(n-1)a}{2}, & \text{for } i \text{ is even, } 2 \leq l \leq n. \end{cases}$$

For i is odd.

$$f(v_{i,j}^l v_{i,j+1}^l) = \begin{cases} k - \frac{(n-3)a}{2}, & \text{for } j \text{ is odd, } 2 \leq l \leq n, \\ \frac{(n-3)a}{2}, & \text{for } j \text{ is even, } 2 \leq l \leq n. \end{cases}$$

For i is even.

$$f(v_{i,j}^l v_{i,j+1}^l) = \begin{cases} \frac{(n-1)a}{2}, & \text{for } j \text{ is odd, } 2 \leq l \leq n, \\ k - \frac{(n-1)a}{2}, & \text{for } j \text{ is even, } 2 \leq l \leq n. \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(n.J_{n,s})) \rightarrow Z_k$ is $f^+(v) \equiv 0 (mod\, k)$ for all $v \in V(S(n.J_{n,s}))$. Hence f^+ is constant and it is equal to $0(mod\, k)$. Since $S(n.J_{n,s})$ admits Z_k-magic labeling, it is a Z_k-magic graph. □

An example of Z_{10}-magic labeling of $S(5.J_{5,2})$ is shown in Figure 6.

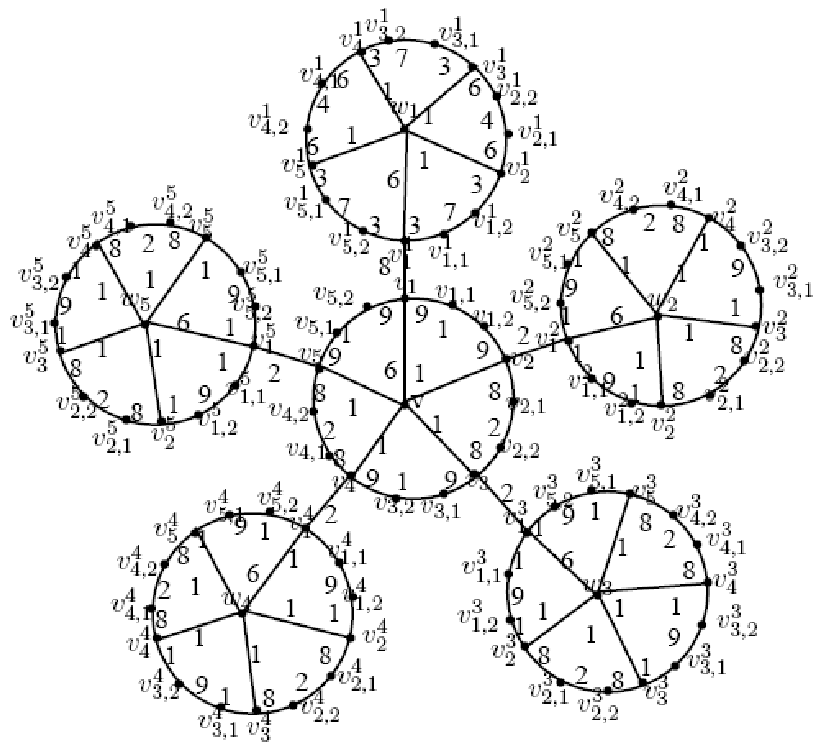

Figure 6: Z_{10}-magic labeling of $S(5.J_{5,2})$

Theorem 2.7. *The star of wheel graph $S(n.W_n)$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.W_n)$ be $V(S(n.W_n)) = \{v, u_i, v_j, u_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.W_n)) = \{vu_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{w_ju_1^j : 1 \leq j \leq n\} \cup \{u_i^ju_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{u_n^ju_1^j : 1 \leq j \leq n\} \cup \{u_ju_1^j : 1 \leq j \leq n\}$.

Let a be an integer and $k > (n-1)a$.

Define the edge labeling $f : E(S(n.W_n)) \to Z_k - \{0\}$ as follows:

$f(vu_1) = k - (n-1)a$,
$f(w_ju_1^j) = k - (n-1)a$ for $1 \leq j \leq n$,
$f(vu_i) = a$ for $2 \leq i \leq n$,
$f(w_ju_i^j) = a$ for $2 \leq i \leq n, 1 \leq j \leq n$,

$$f(u_iu_{i+1}) = \begin{cases} \frac{(n+1)a}{2}, & \text{for } i \ \textit{is odd}, \\ k - \frac{(n-1)a}{2}, & \text{for } i \ \textit{is even}, \end{cases}$$

$$f(u_i^ju_{i+1}^j) = \begin{cases} \frac{(n+1)a}{2}, & \text{for } i \text{ is odd}, 1 \leq j \leq n, \\ k - \frac{(n+3)a}{2}, & \text{for } i \text{ is even}, 1 \leq j \leq n, \end{cases}$$

$f(u_ju_1^j) = k - 2a$.

Then the induced vertex labeling $f^+ : V(S(n.W_n)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(S(n.W_n))$. Hence f^+ is constant and it is equal to $0(mod\ k)$. Since $S(n.W_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. □

An example of Z_8-magic labeling of $S(7.W_7)$ is shown in Figure 7.

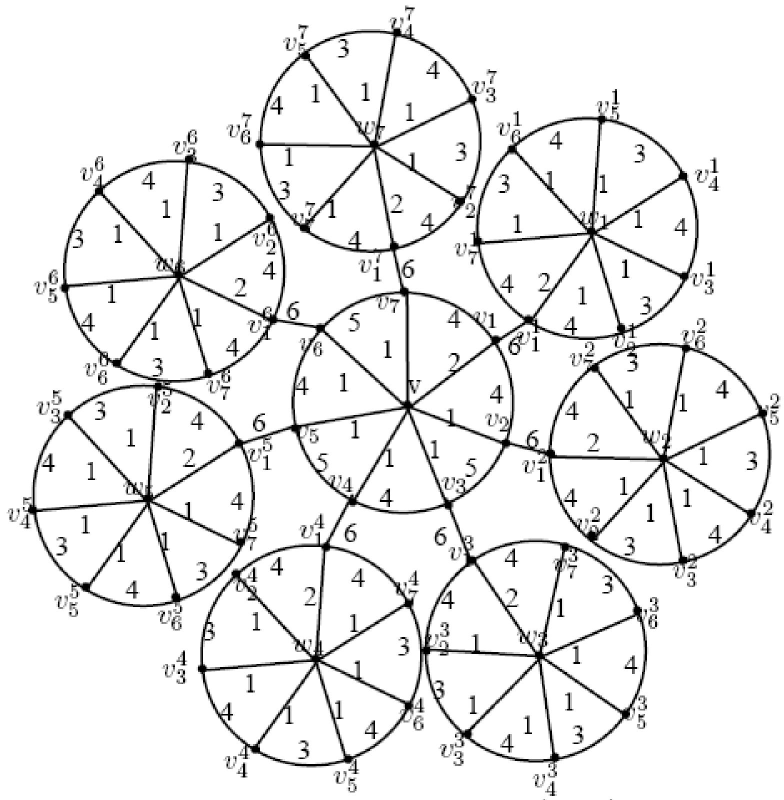

Figure 7: Z_8-magic labeling of $S(7.W_7)$

Theorem 2.8. *The star of generalised Peterson graph $S(n.P_{n,m})$ is Z_k-magic, when n is odd, $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.P_{n,m})$ be $V(S(n.P_{n,m})) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{u_i^j, v_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.P_{n,m})) = \{v_i v_{i+m}, v_i u_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_j u_1^j : 1 \leq j \leq n\} \cup \{v_i^j v_{i+m}^j : 1 \leq i, j \leq n\} \cup \{v_i^j u_i^j : 1 \leq i, j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{u_n^j u_1^j : 1 \leq j \leq n\}$.
Let a be an integer and $k > 4a$.
Define the edge labeling $f : E(S(n.P_{n,m})) \to Z_k - \{0\}$ as follows:
$f(v_i v_{i+m}) = a$ for $1 \leq i \leq n$,
$f(v_i^j v_{i+m}^j) = a$ for $1 \leq i, j \leq n$,

$$
\begin{array}{ll}
f(v_iu_i) & = k-2a \text{ for } 1 \le i \le n, \\
f(v_i^j u_i^j) & = k-2a \text{ for } 1 \le i, j \le n, \\
f(u_iu_{i+1}) & = k-a \quad \text{for } 1 \le i \le n-1, \\
f(u_nu_1) & = k-a, \\
f(u_i^j u_{i+1}^j) & = \begin{cases} k-a, & \text{for } i \ \ is\ odd,\ 1 \le j \le n, \\ 3a, & \text{for } i \ \ is\ even,\ 1 \le j \le n, \end{cases} \\
f(u_j u_1^j) & = 4a \qquad\qquad \text{for } 1 \le j \le n.
\end{array}
$$

Then the induced vertex labeling $f^+ : V(S(n.P_{n,m})) \rightarrow Z_k$ is $f^+(v) \equiv\ 0(mod\, k)$ for all $v \in V(S(n.P_{n,m}))$. Hence f^+ is constant and it is equal to $0(mod\, k)$. Since $S(n.P_{n,m})$ admits Z_k-magic labeling, it is a Z_k-magic graph. $\square$

An example of Z_9-magic labeling of $S(5.P_{5,2})$ is shown in Figure 8.

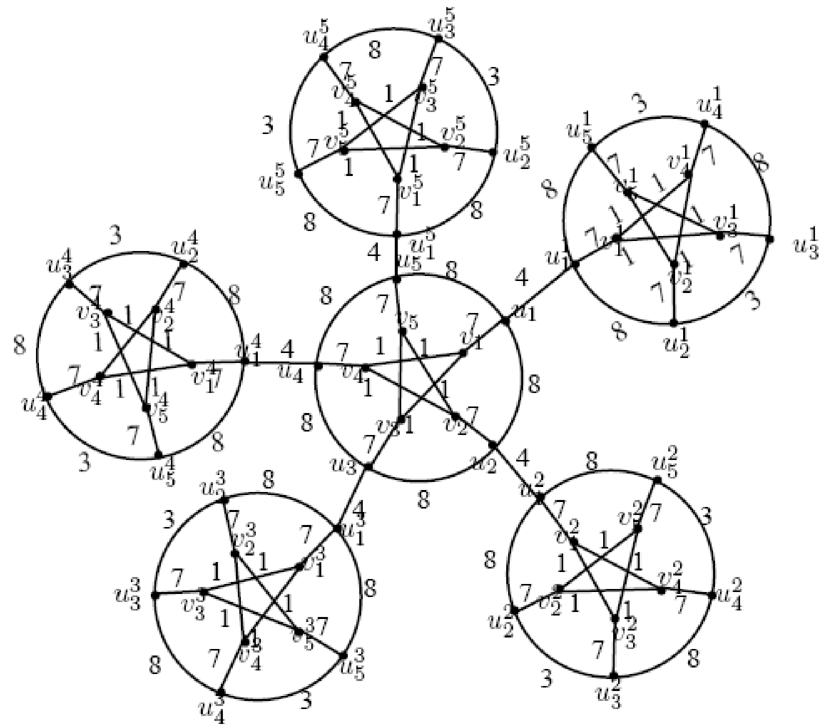

Figure 8: Z_9-magic labeling of $S(5.P(5,2))$

Theorem 2.9. *The star of lotus inside a circle $S(n.LC_n)$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.LC_n)$ be $V(S(n.LC_n)) = \{u, v_i, u_i : 1 \leq i \leq n\} \cup \{w_j, u_i^j, v_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.LC_n)) = \{uv_i, v_iw_i : 1 \leq i \leq n\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nu_1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{u_jv_1^j : 1 \leq j \leq n\} \cup \{w_jv_i^j : 1 \leq i, j \leq n\} \cup \{v_i^ju_i^j : 1 \leq i, j \leq n\} \cup \{v_i^ju_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{v_n^ju_1^j : 1 \leq j \leq n\}$.
Let a be an integer and $k > (n-1)a$.
Define the edge labeling $f : E(S(n.LC_n)) \to Z_k - \{0\}$ as follows:
$f(uv_1) = f(w_jv_1^j) = k - (n-1)a$ for $1 \leq j \leq n$,
$f(uv_i) = f(w_jv_i^j) = a$ for $2 \leq i \leq n, 1 \leq j \leq n$,
$f(v_1u_1) = f(v_1^ju_1^j) = (n-2)a$ for $1 \leq j \leq n$,
$f(v_iu_i) = f(v_i^ju_i^j) = k - 2a$ for $2 \leq i \leq n,\ 1 \leq j \leq n$,
$f(v_iu_{i+1}) = f(v_i^ju_{i+1}^j) = a$ for $1 \leq i \leq n-1,\ 1 \leq j \leq n$,
$f(v_nu_1) = f(v_n^ju_1^j) = a$ for $1 \leq j \leq n$,

$$f(u_iu_{i+1}) = \begin{cases} k-a, & \text{for } i \ \ \textit{is odd}, \\ (n-1)a, & \text{for } i \ \ \textit{is even}, \end{cases}$$

$$f(u_i^ju_{i+1}^j) = \begin{cases} k-a, & \text{for } i \ \ \textit{is odd},\ 1 \leq j \leq n, \\ 2a, & \text{for } i \ \ \textit{is even},\ 1 \leq j \leq n. \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(n.LC_n)) \to Z_k$ is $f^+(v) \equiv 0(mod\, k)$ for all $v \in V(S(n.LC_n))$. Hence f^+ is constant and it is equal to $0(mod\ k)$.
Since $S(n.LC_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. □

An example of Z_6-magic labeling of $S(5.LC_5)$ is shown in Figure 9.

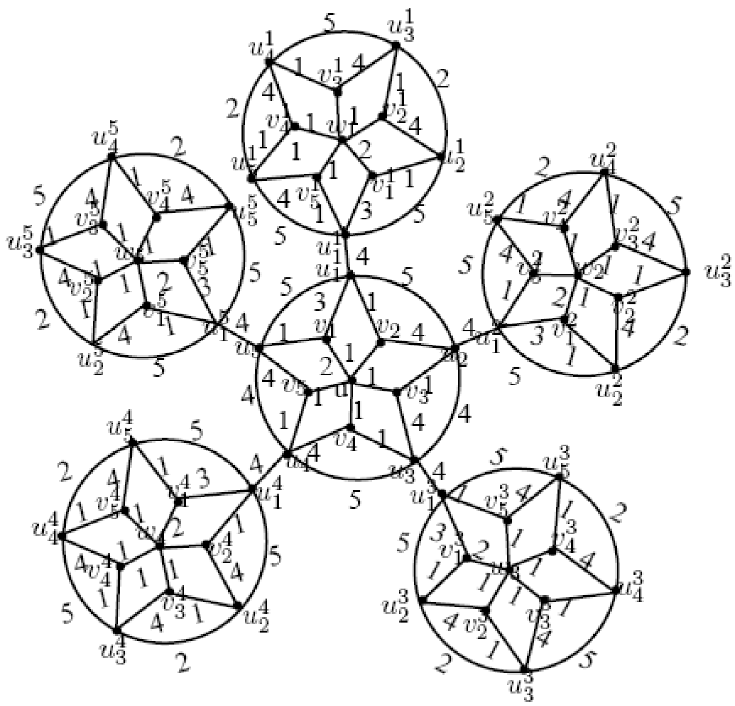

Figure 9: Z_6-magic labeling of $S(5.LC_5)$

Theorem 2.10. *The star of closed helm $S(n.CH_n)$ is Z_k-magic, when n is odd and $n \geq 3$.*

Proof. Let the vertex set and the edge set of $S(n.CH_n)$ be $V(S(n.CH_n)) = \{v, v_i, u_i, w_j, u_i^j, v_i^j : 1 \leq i, j \leq n\}$ and $E(S(n.CH_n)) = \{uv_i, v_iu_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1, u_nu_1\} \cup \{u_ju_1^j : 1 \leq j \leq n\} \cup \{v_i^jv_{i+1}^j, u_i^ju_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq n\} \cup \{v_n^jv_1^j, u_n^ju_1^j : 1 \leq j \leq n\} \cup \{w_jv_i^j, v_i^ju_i^j : 1 \leq i, j \leq n\}$.
Let a be an integer and $k > (n-1)a$.
Define the edge labeling $f : E(S(n.CH_n)) \to Z_k - \{0\}$ as follows:
$f(vv_1) = k - (n-1)a$,
$f(w_jv_1^j) = k - (n-1)a$ for $1 \leq j \leq n$,
$f(vv_i) = a$ for $2 \leq i \leq n$,
$f(w_jv_i^j) = a$ for $2 \leq i \leq n, 1 \leq j \leq n$,

$f(v_1u_1) = k-(n-1)a,$
$f(v_1^ju_1^j) = k-(n-1)a$ for $1 \leq j \leq n,$
$f(v_iu_i) \;\; = k-a$ for $2 \leq i \leq n,$
$f(v_i^ju_i^j) = k-a$ for $2 \leq i \leq n,\ 1 \leq j \leq n,$

$$f(v_iv_{i+1}) = \begin{cases} (n-1)a, & \text{for } i \;\; is\ odd, \\ k-(n-1)a, & \text{for } i \;\; is\ even, \end{cases}$$

$$f(v_i^jv_{i+1}^j) = \begin{cases} (n-1)a, & \text{for } i \;\; is\ odd,\ 1 \leq j \leq n, \\ k-(n-1)a, & \text{for } i \;\; is\ even,\ 1 \leq j \leq n, \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} (n-1)a, & \text{for } i \;\; is\ odd, \\ a, & \text{for } i \;\; is\ even, \end{cases}$$

$$f(u_i^ju_{i+1}^j) = \begin{cases} (n-1)a, & \text{for } i \;\; is\ odd,\ 1 \leq j \leq n, \\ k-(n-2)a, & \text{for } i \;\; is\ even,\ 1 \leq j \leq n, \end{cases}$$

$f(u_ju_1^j) \;\; = k-(n-1)a.$

Then the induced vertex labeling $f^+ : V(S(n.CH_n)) \rightarrow Z_k$ is $f^+(v) \equiv \; 0(mod\, k)$ for all $v \in V(S(n.CH_n))$. Hence f^+ is constant and it is equal to $0(mod\, k)$. Since $S(n.CH_n)$ admits Z_k-magic labeling, it is a Z_k-magic graph. □

An example of Z_{10}-magic labeling of $S(5.CH_5)$ is shown in Figure 10.

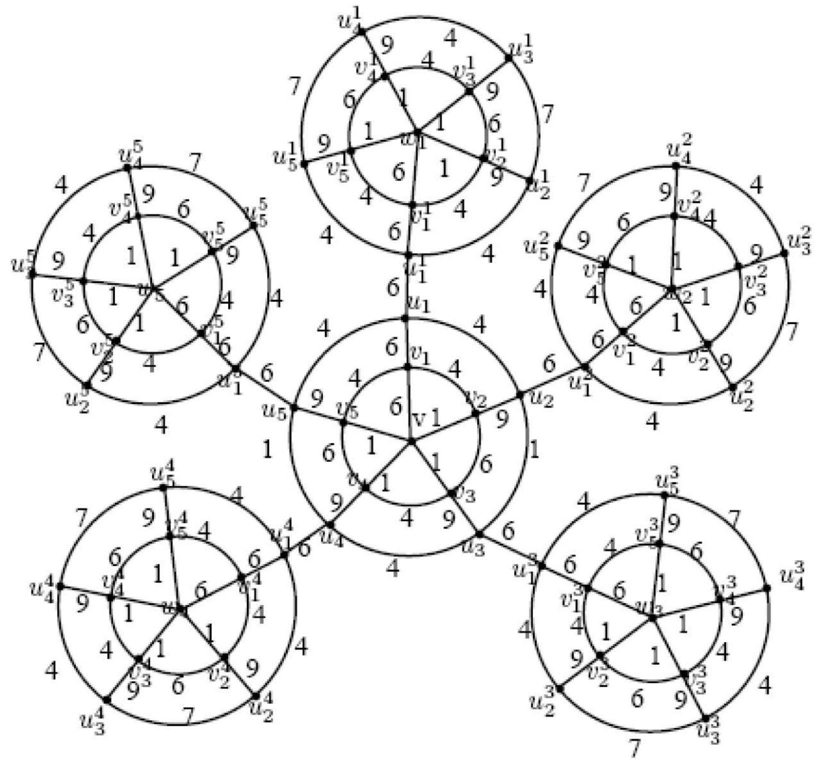

Figure 10: Z_{10}-magic labeling of $S(5.CH_5)$

References

[1] J. A. Gallian, "A Dynamic Survey of Graph Labeling", 21th ed. *The electronics journal of combinatorics*, vol. # DS6, p. 502, 2007. [On line]. Available: https://bit.ly/2tE6Eow

[2] P. Jeyanthi and K. J. Daisy, "Zk-Magic labeling of subdivision graphs", *Discrete mathematics, algorithms and applications*, vol. 8, no. 3, Art. ID 1650046, Jun. 2016, doi: 10.1142/ S1793830916500464.

[3] P. Jeyanthi and K. Jeya Daisy, "Zk-magic labeling of open star of graphs", *Bulletin of the international mathematical virtual institute*, vol. 7, no. 2, pp. 243-255, 2017. [On line]. Available: https://bit.ly/2ukchs4

[4] P. Jeyanthi and K. Jeya Daisy, "Certain classes of Zk-magic graphs", *Journal of graph labeling*, vol. 4, no. 1, pp. 38-47, 2018. [On line]. Available: https://bit.ly/2tFW6oL

[5] P. Jeyanthi and K. Jeya Daisy, "Zk-magic labeling of some families of graphs", *Journal of algorithms and computation,* vol. 50, no. 2, pp. 1-12, Dec. 2018. [On line]. Available: https://bit.ly/36bXWel

[6] P. Jeyanthi and K. Jeya Daisy, "Zk-magic labeling of cycle of graphs", *International journal of mathematical combinatorics*, vol. 1, pp. 88- 102, Mar. 2019. [On line]. Available: https://bit.ly/2TLobpi

[7] P. Jeyanthi and K. Jeya Daisy, "Some results on Zk-magic labeling", *Palestine journal of mathematics*, vol. 8, no. 2, pp. 400-412, 2019. [On line]. Available: https://bit.ly/2Rd7bqv

[8] P. Jeyanthi, K. J. Daisy, and A. Semanicová-Fenovníková, "Zk-magic labeling of path union of graphs", *Cubo (Temuco)*, vol. 21, no. 2, pp. 15–35, Aug. 2019, doi: 10.4067/s0719-06462019000200015.

[9] R. M. Low and S. M Lee, "On the products of group-magic graphs", *Australasian journal of combinatorics*, vol. 34, pp. 41-48, Feb. 2006. [On line]. Available: https://bit.ly/2Gamutw

[10] J. Sedláček, "On magic graphs", *Mathematica slovaca*, vol. 26, no. 4, pp. 329-335, 1976. [On line]. Available: https://bit.ly/38E4Yuf

[11] W. C. Shiu, P.C.B. Lam, and P. K. Sun, "Construction of group-magic graphs and some A-magic graphs with A of even order", *Congressus numerantium*, vol. 167, pp. 97-107, 2004. [On line]. Available: https://bit.ly/3at77KO

[12] W. C. Shiu and R. M. Low, "Zk-magic labeling of fans and wheels with magic-value zero", *Australasian journal of combinatorics*, vol. 45, pp. 309-316, Oct. 2009. [On line]. Available: https://bit.ly/2RERuY5