Proyecciones (Antofagasta. On line) | vol. 39, n. 5 (2019) | pp. 1081-1092

REVISTAS CIENTÍFICAS de la Universidad Católica del Norte. revistas.ucn.cl

doi:10.22199/issn.0717-6279-2019-05-0070

PROYECCIONES
Journal of Mathematics

ISSN 0717-6279 (On line)

Computing the Schultz polynomials and indices for ladder related graphs

Ali Ahmad[*] orcid.org/0000-0003-3434-9908

[*]Jazan University, College of Computer Sci. & Info. Technol., Jazan, Kingdom Saudi Arabia.
ahmadsms@gmail.com

Received: November 2018 | Accepted: January 2019

Abstract:

Distance is an important graph invariant that has wide applications in computing science and other fields of sciences. A topological index is a genuine number connected with compound constitution indicating for relationship of compound structure with different physical properties, synthetic reactivity or natural action. The Schultz and modified Schultz polynomials and their corresponding indices are used in synthetic graph theory as in light of vertex degrees. In this paper, the Schultz and modified

Keywords: Distance; Topological indices; Schultz indices; Schultz polynomial.

MSC (2010): 05C07, 05C12, 05C31.

Cite this article as (IEEE citation style):

A. Ahmad, "Computing the Schultz polynomials and indices for ladder related graphs", *Proyecciones (Antofagasta, On line)*, vol. 38, no. 5, pp. 1081-1092, Dec. 2019, doi: 10.22199/issn.0717-6279-2019-05-0070. [Accessed dd-mm-yyyy].

1. Introduction

Let $G(V(G), E(G))$ be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. The degree of a vertex u in an undirected simple graph, denoted as d_u is the number of edges incident with it. The separation (distance) between vertices u and v; $d(u, v)$ is the quantity of edges in a most limited way associating them. The biggest separation between any two vertices of a graph G is known as the diameter of G, denoted as $d(G)$. largest distance between any two vertices of a graph G is called the diameter of G, denoted as $d(G)$.

While working on structural determination of the paraffin boiling points, Wiener [22] characterized a descriptor that is known as Wiener index. A great deal of topological indices have been presented, Wiener index is one of the topological indices that connect with a portion of the physico-chemical properties of the compound [1, 12]. Harry Shultz [19] presented another distance based topological index known as Shultz index with some of the physicochemical properties of the compound [1, 12]. Harry Shultz [19] introduced another distance based topological index known as Shultz index

$$Sc(G) = \sum_{\{u,v\} \subseteq V(G)} (d_u + d_v)\, d(u, v)$$

In [16] showed the close relation between the Schultz index and the Wiener index. The modified Schultz index defined by Klavzar and Gutman [15] as:

$$Sc^*(G) = \sum_{\{u,v\} \subseteq V(G)} (d_u d_v)\, d(u, v)$$

The modified Schultz index is closely related to the Wiener index as shown in [11]. $Sc^*(G) = 4W(G) - n(2n - 1)$. Hosoya [14] acquainted a distance-based polynomial to generate distance distributions for graphs, called the Wiener polynomial

$$H(G, x) = \sum_{\{u,v\} \subseteq V(G)} x^{d(u,v)}$$

The first derivative of $H(G, x)$ at $x = 1$ is equal to Wiener index of G. Gutman [11] introduced new polynomials called the Schultz polynomial and the modified Schultz polynomial as

$$Sc(G, x) = \sum_{\{u,v\} \subseteq V(G)} (d_u + d_v)\, x^{d(u,v)}$$

$$Sc^{*}(G,x)=\sum_{\{u,v\}\subseteq V(G)}(d_u d_v)\,x^{d(u,v)}$$

Such that their derivative at $x = 1$ are equal to the Schultz and modified Schultz indices. He likewise acquired a few association between these polynomials and Wiener polynomial of trees. For further details we refer [2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 20].

The Schultz and modified Schultz polynomials and their corresponding indices were set up in engineered outline speculation in light of vertex degrees. In this paper, the Schultz and modified Schultz polynomials and their corresponding indices for Mongolian tent graph, diamond graph and double fan are computed.

2. Results and Discussion

The ladder graph, denoted by L_n, is the graph with vertex set

$$V(L_n)=\{u_i,v_i : 1\le i\le n\}$$

and edge set

$$E(L_n)=\{u_iu_{i+1},v_iv_{i+1} : 1\le i\le n-1\}\cup\{u_iv_i : 1\le i\le n\}.$$

L_n is isomorphic to the grid $P_2\times P_n$.

Mongolian tent, denoted by Mt_n, is the graph obtained from the ladder graph L_n by adding a new vertex z and joining each vertex v_i, $1\le i\le n$ with z. This graph is shown in Figure 1.

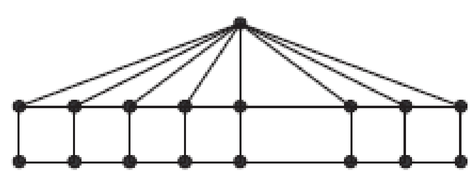

Figure 1: Mongolian tent graph Mt_n

In the next theorem, the Schultz and modified Schultz polynomials and their corresponding indices of Mongolian tent graph are studied.

Theorem 1. *Let Mt_n be a Mongolian tent graph with order $n\ge 3$. Then,*

- *The Schultz polynomial and index of Mt_n are:*
  $Sc(Mt_n, x) = (n^2+25n-24)x+(5n^2+35n-60)x^2+(72n-188)x^3+(3n^2-17n+18)x^4$
  $Sc(Mt_n) = 23n^2 + 243n - 636.$

- *The Modified Schultz polynomial and index of Mt_n are:*
  $Sc^*(Mt_n, x) = (4n^2 + 35n - 51)x + (11n^2 + 47n - 115)x^2 + (115n - 321)x^3 + (\frac{9n^2-57n+74}{2})x^4$
  $Sc^*(Mt_n) = 44n^2 + 360n - 996.$

Proof. Consider the graph of Mongolian tent Mt_n with $n \geq 3$. The order of Mt_n is equal to $2n + 1$, in which 2 vertices of Mt_n have degree 2, n vertices of Mt_n have degree 3, $n - 2$ vertices of Mt_n have degree 4 and only one vertex has degree n.

Thus, we divide the vertex set $V(Mt_n)$ in four partitions:
$V_2 = \{v \in V(Mt_n) : d_v = 2\}$
$V_3 = \{v \in V(Mt_n) : d_v = 3\}$
$V_4 = \{v \in V(Mt_n) : d_v = 4\}$
$V_n = \{v \in V(Mt_n) : d_v = n\}$

From Figure 1, the size of these four subsets are $|V_2| = 2, |V_3| = 3, |V_4| = n-2$ and $|V_n| = 1$. By using the hand shaking lemma the size of Mongolian tent graph Mt_n is equal to

$|E(Mt_n)| = \frac{1}{2}[2 \times |V_2| + 3 \times |V_3| + 4 \times |V_4| + n \times |V_n|] = \frac{4+3n+4n-8+n}{2} = 4n - 2.$

From Figure 1, it is easy to see that for every vertices $u, v \in V(Mt_n)$, $\exists\ d(u, v) \in \{1, 2, 3, 4\}$.

Now, from the structure of the Mongolain tent graph Mt_n, we compute all terms of the Schultz polynomial, modified Schultz polynomial of Mt_n, based on the number of $d(u, v)\ \forall\ u, v \in V(Mt_n)$.

Here, consider $d(u, v) = 1$ ($\forall\ u, v \in V(Mt_n)$), so from edge set $E(Mt_n)$, we can see that there are 4 paths with length one or 4 edges $uv \in E(Mt_n)$ for vertex $u \in V_2 \subset V(Mt_n)$ and a vertex $v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 5$ and $d_u \times d_v = 6$. For a vertex $u \in V_3 \subset V(Mt_n)$, there are $n-3$ paths with length one until to a vertex $v \in V_3$ such that $d_u + d_v = 3+3 = 6$ and $d_u \times d_v = 3 \times 3 = 9$; there are n 1-edges paths between the vertices $u \in V_3 \subset V(Mt_n)$ and $v \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 3 + 4 = 7$, $d_u \times d_v = 3 \times 4 = 12$, there are 2 1-edges paths between the vertices $u \in V_3 \subset V(Mt_n)$ and $v \in V_n \subset V(Mt_n)$ such that $d_u + d_v = 3 + n = n + 3$, $d_u \times d_v = 3 \times n = 3n$; there are $n - 3$ 1-edges paths between the vertices

$u, v \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 4 + 4 = 8$, $d_u \times d_v = 4 \times 4 = 16$; there are $n - 2$ 1-edges paths between the vertices $u \in V_4 \subset V(Mt_n)$ and $v \in V_n \subset V(Mt_n)$ such that $d_u + d_v = 4 + n = n + 4$, $d_u \times d_v = 4 \times n = 4n$. Therefore the first term of Schultz and modified Schultz polynomial of Mt_n will be $5(4)x + 6(n-3)x + 7(n)x + (n+3)2x + 8(n-3)x + (n+4)(n-2)x = (n^2 + 25n - 24)x$ and $6(4)x + 9(n-3)x + 12(n)x + (3n)2x + 16(n-3)x + (4n)(n-2)x = (4n^2 + 35n - 51)x$, respectively.

In case $d(u, v) = 2$, $u, v \in V(Mt_n)$: there are 2 2-edges paths between the vertices $u \in V_2$ and $v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 5$, $d_u \times d_v = 2 \times 3 = 6$; there are 4 2-edges paths between the vertices $u \in V_2 \subset V(Mt_n)$ and $v \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 2 + 4 = 6$, $d_u \times d_v = 2 \times 4 = 8$; there are 2 2-edges paths between the vertices $u \in V_2 \subset V(Mt_n)$ and $v \in V_n \subset V(Mt_n)$ such that $d_u + d_v = 2 + n$, $d_u \times d_v = 2 \times n = 2n$, there are $n + 1$ 2-edges paths between the vertices $u, v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 3 + 3 = 6$, $d_u \times d_v = 3 \times 3 = 9$; there are $6n - 14$ 2-edges paths between the vertices $u \in V_3 \subset V(Mt_n)$ and $v \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 3 + 4 = 7$, $d_u \times d_v = 3 \times 4 = 12$; there are $n - 2$ 2-edges paths between the vertices $u \in V_3 \subset V(Mt_n)$ and $v \in V_n \subset V(Mt_n)$ such that $d_u + d_v = 3 + n = n + 3$, $d_u \times d_v = 3 \times n = 3n$. Finally, there are $n - 4 + \frac{(n-2)(n-4)}{2}$ 2-edges paths between all vertices $u, v \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 4 + 4 = 8$, $d_u \times d_v = 4 \times 4 = 16$. Then the second term of Schultz and modified Schultz polynomial of Mt_n is equal to $\Big[10 + 24 + 2n + 4 + 6n + 6 + 7(6n-14) + (n+3)(n-2) + 8(n-4+\frac{(n-2)(n-4)}{2})\Big]x^2 = (5n^2 + 35n - 60)x^2$ and $\Big[12 + 32 + 4n + 9n + 9 + 12(6n-14) + 3n(n-2) + 16(n - 4 + \frac{(n-2)(n-4)}{2})\Big]x^2 = (11n^2 + 47n - 115)x^2$, respectively.

In case $d(u, v) = 3$, $u, v \in V(Mt_n)$: there are 2 3-edges paths between the vertices $u \in V_2$ and $v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 5$, $d_u \times d_v = 2 \times 3 = 6$; there are $2n$ 3-edges paths between the vertices $u \in V_2 \subset V(Mt_n)$ and $v \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 2 + 4 = 6$, $d_u \times d_v = 2 \times 4 = 8$; there are $3n - 5$ 3-edges paths between the vertices $u, v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 3 + 3 = 6$, $d_u \times d_v = 3 \times 3 = 9$. Finally, there are $6n - 24$ 3-edges paths between the vertices $u \in V_3 \subset V(Mt_n)$ and $u \in V_4 \subset V(Mt_n)$ such that $d_u + d_v = 3 + 4 = 7$, $d_u \times d_v = 3 \times 4 = 12$. Then the third term of Schultz and modified Schultz polynomial of Mt_n is equal to $\Big[10 + 12n + 6(3n-5) + 7(6n-24)\Big]x^3 = (72n - 188)x^3$ and $\Big[12 + 16n + 9(3n-5) + 12(6n-24)\Big]x^3 = (115n - 321)x^3$, respectively.

In case $d(u, v) = 4$, $u, v \in V(Mt_n)$: there are 1 4-edges paths between all vertices $u, v \in V_2 \subset V(Mt_n)$ such that $d_u + d_v = 2 + 2 = 4$, $d_u \times d_v =$

$2 \times 2 = 4$; there are $2n-8$ 4-edges paths between the vertices $u \in V_2 \subset V(Mt_n)$ and $v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 2+3 = 5$, $d_u \times d_v = 2 \times 3 = 6$. Finally, there are $\frac{n^2-9n+18}{2}$ 4-edges paths between all vertices $u, v \in V_3 \subset V(Mt_n)$ such that $d_u + d_v = 3+3 = 6$, $d_u \times d_v = 3 \times 3 = 9$. Then the fourth term of Schultz and modified Schultz polynomial of Mt_n is equal to $\Big[4 + 5(2n-8) + 6(\frac{n^2-9n+18}{2})\Big]x^4 = (3n^2 - 17n + 18)x^4$ and $\Big[4 + 6(2n-8) + 9(\frac{n^2-9n+18}{2})\Big]x^4 = (\frac{9n^2-57n+74}{2})x^4$, respectively.

Hence, Schultz and modified Schultz polynomials of Mt_n are:

$Sc(Mt_n, x) = \sum\limits_{u,v \in V(Mt_n)} (d_u + d_v)x^{d(u,v)}$

$= (n^2+25n-24)x+(5n^2+35n-60)x^2+(72n-188)x^3+(3n^2-17n+18)x^4$

$Sc^*(Mt_n, x) = \sum\limits_{u,v \in V(Mt_n)} (d_u \times d_v)x^{d(u,v)}$

$= (4n^2+35n-51)x+(11n^2+47n-115)x^2+(115n-321)x^3+(\frac{9n^2-57n+74}{2})x^4$

By definitions of the Schultz and modified Schultz indices, we have

$Sc(Mt_n) = \frac{\partial Sc(Mt_n,x)}{\partial x}|_{x=1}$

$= \frac{\partial}{\partial x}\Bigg((n^2+25n-24)x+(5n^2+35n-60)x^2+(72n-188)x^3+(3n^2-17n+18)x^4\Bigg)|_{x=1}$

$= 23n^2 + 243n - 636.$

$Sc^*(Mt_n) = \frac{\partial Sc^*(Mt_n,x)}{\partial x}|_{x=1}$

$= \frac{\partial}{\partial x}\Bigg((4n^2 + 35n - 51)x + (11n^2 + 47n - 115)x^2 + (115n - 321)x^3 + (\frac{9n^2-57n+74}{2})x^4\Bigg)|_{x=1}$

$= 44n^2 + 360n - 996$. Which completes the proof.

□ *Diamond graph*, denoted by D_n, is the graph obtained from the Mongolian tent graph Mt_n by adding a new vertex z_1 and joining each vertex x_i, $1 \le i \le n$ with z_1. This graph is shown in Figure 2. The Schultz and modified Schultz polynomials and their corresponding indices of Diamond are determined in the following theorem.

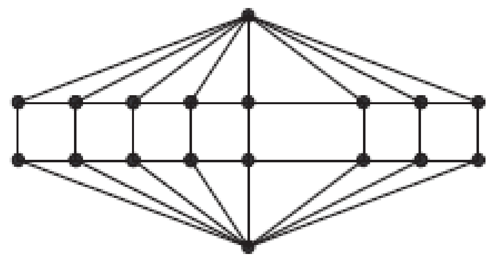

Figure 2: Diamond graph D_n

Theorem 2. *Let D_n be a Diamond graph with order $n \geq 3$. Then,*

- *The Schultz polynomial and index of D_n are:*
  $Sc(D_n, x) = (2n^2 + 32n - 28)x + (10n^2 + 28n - 42)x^2 + (18n^2 - 24n - 24)x^3$
  $Sc(D_n) = 76n^2 + 16n - 184.$
- *The Modified Schultz polynomial and index of D_n are:*
  $Sc^*(D_n, x) = (8n^2 + 44n - 62)x + (24n^2 + 28n - 86)x^2 + (n^3 + 32n^2 - 64n - 28)x^3$
  $Sc^*(D_n) = 3n^3 + 152n^2 - 92n - 318.$

Proof. Consider the graph of Diamond D_n with $n \geq 3$. The order of D_n is equal to $2n+2$, in which 4 vertices of D_n have degree 3, $2n-4$ vertices of D_n have degree 4 and two vertices have degree n.

Thus, we divide the vertex set $V(D_n)$ in three partitions:
$V_3 = \{v \in V(D_n) : d_v = 3\}$
$V_4 = \{v \in V(D_n) : d_v = 4\}$
$V_n = \{v \in V(D_n) : d_v = n\}$

From Figure 2, the size of these three subsets are $|V_3| = 4, |V_4| = 2n-4$ and $|V_n| = 2$. By using the hand shaking lemma the size of Diamond graph D_n is equal to

$|E(D_n)| = \frac{1}{2}[3 \times |V_3| + 4 \times |V_4| + n \times |V_n|] = \frac{12+8n-8+2n}{2} = 5n-2.$

From Figure 2, we can see that there are distance between vertices of graph D_n are up to three and the diameter equal to 3.

Now, from the structure of the Diamond graph D_n, we compute all terms of the Schultz polynomial, modified Schultz polynomial of D_n, based on the number of $d(u,v)\ \forall\ u, v \in V(D_n)$.

Table 2.1: All cases of $d(u,v)$-edge-paths $d(u,v) = 1, 2, 3$ of the Diamond graph D_n.

distance $d(u,v) = i$	degrees of d_u & d_v	Number of i-edges paths	Term of Schultz polynomial	Term of Modified Schultz polynomial
1	$(3,3)$	2	12	18
	$(3,4)$	4	28	48
	$(3,n)$	4	$4n+12$	$12n$
	$(4,4)$	$3n-8$	$24n-64$	$48n-128$
	$(4,n)$	$2n-4$	$2n^2+4n-16$	$8n^2-16n$
2	$(3,3)$	2	12	18
	$(3,4)$	$4n+2$	$28n+14$	$48n+24$
	$(3,n)$	4	$4n+12$	$12n$
	$(4,4)$	n^2-n-8	$8n^2-8n-64$	$16n^2-16n-128$
	$(4,n)$	$2n-4$	$2n^2+4n-16$	$8n^2-16n$
3	$(3,3)$	4	24	36
	$(3,4)$	$8n-16$	$56n-112$	$96n-192$
	$(4,4)$	$2n^2-10n+8$	$16n^2-80n+64$	$32n^2-160n+128$
	(n,n)	n	$2n^2$	n^3

By using the Table 1, we obtain the followings:

- coefficient first term of the Schultz polynomial: $2n^2 + 32n - 28$
- coefficient second term of the Schultz polynomial: $10n^2 + 28n - 42$
- coefficient third term of the Schultz polynomial: $18n^2 - 24n - 24$
- coefficient first term of the modified Schultz polynomial: $8n^2+44n-62$
- coefficient second term of the modified Schultz polynomial: $24n^2 + 28n - 86$
- coefficient third term of the modified Schultz polynomial: $n^3+32n^2-64n-28$

Hence, Schultz and modified Schultz polynomials of D_n are:

$Sc(D_n, x) = \sum\limits_{u,v \in V(D_n)} (d_u + d_v)x^{d(u,v)}$

$= (2n^2 + 32n - 28)x + (10n^2 + 28n - 42)x^2 + (18n^2 - 24n - 24)x^3$

$Sc^*(D_n, x) = \sum\limits_{u,v \in V(D_n)} (d_u \times d_v)x^{d(u,v)}$

$= (8n^2 + 44n - 62)x + (24n^2 + 28n - 86)x^2 + (n^3 + 32n^2 - 64n - 28)x^3$

By definitions of the Schultz and modified Schultz indices, we have
$Sc(D_n) = \frac{\partial Sc(D_n,x)}{\partial x}|_{x=1}$
$= \frac{\partial}{\partial x}\Big((2n^2 + 32n - 28)x + (10n^2 + 28n - 42)x^2 + (18n^2 - 24n - 24)x^3\Big)|_{x=1}$
$= 76n^2 + 16n - 184.$
$Sc^*(D_n) = \frac{\partial Sc^*(D_n,x)}{\partial x}|_{x=1}$
$= \frac{\partial}{\partial x}\Big((8n^2+44n-62)x+(24n^2+28n-86)x^2+(n^3+32n^2-64n-28)x^3\Big)|_{x=1}$
$= 3n^3 + 152n^2 - 92n - 318$. Which completes the proof. □

Fan graph, denoted by f_n, is the graph obtained from the path with n vertices P_n, where $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \le i \le n-1\}$ by adding a new vertex z and joining each $v_i, 1 \le i \le n$ with z.

Double fan graph, denoted by df_n, is the graph obtained from the fan f_n by adding a new vertex z_1 joining each $v_i, 1 \le i \le n$ with z_1.

Theorem 3. *Let df_n be a double fan graph with order $n \ge 5$. Then,*

- *The Schultz polynomial and index of df_n are:*
  $Sc(df_n, x) = (2n^2 + 16n - 14)x + (18n^2 - 76n - 218)x^2$
  $Sc(df_n) = 38n^2 - 136n - 450.$
- *The Modified Schultz polynomial and index of df_n are:*
  $Sc^*(df_n, x) = (8n^2 + 12n - 24)x + (n^3 + 32n^2 - 160n - 422)x^2$
  $Sc^*(df_n) = 2n^3 + 72n^2 - 308n - 868.$

Proof. Consider the graph of double fan df_n with $n \ge 5$. The order of df_n is equal to $n+2$, in which 2 vertices of df_n have degree 3, $n-2$ vertices of df_n have degree 4 and two vertices have degree n.

Thus, we divide the vertex set $V(df_n)$ in three partitions:
$V_3 = \{v \in V(df_n) : d_v = 3\}$
$V_4 = \{v \in V(df_n) : d_v = 4\}$
$V_n = \{v \in V(df_n) : d_v = n\}$

The size of these three subsets are $|V_3| = 2, |V_4| = n-2$ and $|V_n| = 2$. By using the hand shaking lemma the size of double fan df_n is equal to
$|E(df_n)| = \frac{1}{2}[3 \times |V_3| + 4 \times |V_4| + n \times |V_n|] = \frac{6+4n-8+2n}{2} = 3n - 2.$

The diameter of double fan df_n equal to 2.

Now, from the structure of the double fan df_n, we compute all terms of the Schultz polynomial, modified Schultz polynomial of df_n, based on the number of $d(u,v)$ $\forall\ u, v \in V(df_n)$.

Table 2.2: All cases of $d(u,v)$-edge-paths $d(u,v) = 1, 2$ of the double fan df_n.

distance $d(u,v) = i$	degrees of d_u & d_v	Number of i-edges paths	Term of Schultz polynomial	Term of Modified Schultz polynomial
1	$(3,4)$	2	14	24
	$(3,n)$	4	$4n+12$	$12n$
	$(4,4)$	$n-3$	$8n-24$	$16n-48$
	$(4,n)$	$2n-4$	$2n^2+4n-16$	$8n^2-16n$
2	$(3,3)$	2	12	18
	$(3,4)$	$4n-10$	$28n-70$	$48n-120$
	$(4,4)$	$2n^2-13n+20$	$16n^2-104n-160$	$32n^2-208n-320$
	(n,n)	n	$2n^2$	n^3

By using the Table 2, we obtain the followings:

- coefficient first term of the Schultz polynomial: $2n^2 + 16n - 14$
- coefficient second term of the Schultz polynomial: $18n^2 - 76n - 218$
- coefficient first term of the modified Schultz polynomial: $8n^2+12n-24$
- coefficient second term of the modified Schultz polynomial:$n^3+32n^2-160n - 422$

Hence, Schultz and modified Schultz polynomials of df_n are:
$Sc(df_n, x) = \sum\limits_{u,v \in V(df_n)} (d_u + d_v)x^{d(u,v)}$
$= (2n^2 + 16n - 14)x + (18n^2 - 76n - 218)x^2$
$Sc^*(df_n, x) = \sum\limits_{u,v \in V(df_n)} (d_u \times d_v)x^{d(u,v)}$
$= (8n^2 + 12n - 24)x + (n^3 + 32n^2 - 160n - 422)x^2$
By definitions of the Schultz and modified Schultz indices, we have
$Sc(df_n) = \frac{\partial Sc(df_n,x)}{\partial x}|_{x=1}$
$= \frac{\partial}{\partial x}\bigg((2n^2 + 16n - 14)x + (18n^2 - 76n - 218)x^2\bigg)|_{x=1}$
$= 38n^2 - 136n - 450.$
$Sc^*(df_n) = \frac{\partial Sc^*(df_n,x)}{\partial x}|_{x=1}$
$= \frac{\partial}{\partial x}\bigg((8n^2 + 12n - 24)x + (n^3 + 32n^2 - 160n - 422)x^2\bigg)|_{x=1}$
$= 2n^3 + 72n^2 - 308n - 868$. Which completes the proof. $\square$

Closing Remarks

In this paper, we proved Schultz and modified Schultz polynomials for Mongolian tent graph (Mt_n), Diamond graph (D_n), double fan (df_n). In future, we are interested to plan some beginning designs/systems and after that review their topological indices which will be very helper to comprehend their hidden topology.

References

[1] A. Dobrynin, R. Entringer and I Gutman, "Wiener index of trees: theory and applications", *Acta applicandae mathematicae*, vol. 66, no. 3 pp. 211-249, May 2001, doi: 10.1023/A:1010767517079.

[2] M. Eliasi and B. Taeri, "Schultz polynomials of composite graphs", *Applicable analysis and discrete mathematics*, vol. 2, no. 2, pp. 285-296, Apr. 2008. [On line]. Available: https://bit.ly/34yFuvy

[3] M. Farahani and M. Vlad, "On the Schultz, modified Schultz and Hosoya polynomials and derived indices of capradesigned planar benzenoid", *Studia universitatis Babeş-Bolyai, chemia,* vol. 57, no. 4, pp. 55-63, 2012. [On line]. Available: https://bit.ly/2YWgSvK

[4] M. Farahani, "Hosoya, Schultz, modified Schultz polynomials and their topological indices of benzene molecules, first members of polycyclic aromatic hydrocarbons (PAHs)", *International journal of theoretical chemistry,* vol. 1, no. 2, pp. 9-16, Oct. 2013. [On line]. Available: https://bit.ly/2S4WwPC

[5] M. Farahani, "On the Schultz and modified Schultz polynomials of some harary graphs", *International journal of applications of discrete mathematics,* vol. 1, no. 1, pp. 1-8, Sep. 2013. [On line]. Available: https://bit.ly/2r4PgIl

[6] M. Farahani, "On the Schultz polynomial and Hosoya polynomial of circumcoronene series of benzenoid", *Journal of applied mathematics & informatics,* vol. 31, no. 5-6, pp. 595-608, 2013, doi: 10.14317/jami.2013.595.

[7] M. Farahani, "Schultz indices and Schultz polynomials of Harary graph", *Pacific journal of applied mathematics*, vol. 6, no. 3, pp. 77-84, 2014.

[8] M. Farahani and W. Gao, "The Schultz index and Schultz polynomial of the Jahangir Graphs J5,m", *Applied mathematics*, vol. 6, pp. 2319-2325, Dic. 2015, doi: 10.4236/am.2015.614204.

[9] M. Farahani, M. Kanna and W. Gao, "The Schultz, modified Schultz indices and their polynomials of the Jahangir graphs Jn,m for integer numbers n = 3, m ≥ 3", *Asian journal of applied sciences*, vol. 3, no. 6, pp. 823-827, Dec. 2015. [On line]. Available: https://bit.ly/2PxBF5w

[10] M. Farahani and M. Jamil, "The Schultz and modified Schultz polynomials of certain subdivision and line subdivision graphs", *Journal of chemical and pharmaceutical research*, vol. 8, no. 3, pp. 51—57, 2016. [On line]. Available: https://bit.ly/2M55j08

[11] I. Gutman, "Selected properties of the Schultz molecular topological index", *Journal of chemical information and modeling*, vol. 34, no. 5, pp. 1087-1089, Sep. 1994, doi: 10.1021/ci00021a009.

[12] I. Gutman and O. Polansky, *Mathematical concepts in organic chemistry,* Berlin: Springer, 1986, doi: 10.1007/978-3-642-70982-1.

[13] F. Hassani, A. Iranmanesh and S. Mirzaie, "Schultz and modified Schultz polynomials of C100 Fullerene", *MATCH communications in mathematical and in computer chemistry*, vol. 69, no. 1 pp. 87-92, 2013. [On line]. Available: https://bit.ly/2PWgSrA

[14] H. Hosoya, "On some counting polynomials in chemistry", *Discrete applied mathematics*, vol. 19, no. 1, pp. 239-257, Mar. 1988, doi: 10.1016/0166-218X(88)90017-0.

[15] S. Klavžar and I. Gutman, "Wiener number of vertex-weighted graphs and a chemical application", *Discrete applied mathematics*, vol. 80, no. 1, pp. 73-81, Dec. 1997, doi: 10.1016/S0166-218X(97)00070-X.

[16] D. Klein, Z. Mihalić, D. Plavšić, N. Trinjastić, "Molecular topological index: a relation with the Wiener index", *Journal of chemical information and modeling,* vol. 32, no. 4, pp. 304-305, Jul. 1992, doi: 10.1021/ci00008a008.

[17] M. Nadeem, S. Zafar and Z. Zahid, "On certain topological indices of the line graph of subdivision graphs", *Applied mathematics and computation,* vol. 271, pp. 790-794, Nov. 2015, doi: 10.1016/j.amc.2015.09.061.

[18] M. Nadeem, S. Zafar and Z. Zahid,"On topological properties of the line graphs of subdivision graphs of certain nanostructures", *Applied mathematics and computation*, vol. 273, pp. 125—130, Jan. 2016, doi: 10.1016/j.amc.2015.10.010.

[19] H. Schultz, "Topological organic chemistry 1. Graph theory and topological indices of alkanes", *Journal of chemical information and modeling*, vol. 29, no. 3, pp. 227-228, Aug. 1989, doi: 10.1021/ci00063a012.

[20] M. Siddiqui, M. Imran and A. Ahmad, "On Zagreb indices, Zagreb polynomials of some nanostar dendrimers", *Applied mathematics and computation*, vol. 280, pp. 132-139, Apr. 2016, doi: 10.1016/j.amc.2016.01.041.

[21] G. Su and L. Xu, "Topological indices of the line graph of subdivision graphs and their Schur-bounds", *Applied mathematics and computation,* vol. 253, pp. 395-401, Feb. 2015, doi: 10.1016/j.amc.2014.10.053.

[22] H. Wiener, "Structural determination of the paraffin boiling points", *Journal of the American chemical society*, vol. 69, no. 1, pp. 17-20, Jan. 1947, doi: 10.1021/ja01193a005.