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The b-radical of generalized alternative b-algebras II

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Abstract:

We prove that if (U, ω) is a finite dimensional generalized alternative b-algebra II over a field F of characteristic different from 2 and 3, then $\mathrm{rad}(U) = R(U) \cap (\mathrm{bar}(U))^3$.

Keywords: b-algebras; b-radical; Generalized alternative b-algebras II.

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1. B-algebras and generalized alternative algebras II

Baric algebras play a central role in the theory of genetic algebras. They were introduced by I. M. H. Etherington, in [2], in order to give an algebraic treatment to Genetic Populations. Several classes of b-algebras have been defined, such as: train, Bernstein, special triangular, etc.

Let U be an algebra over a field F not necessarily associative, commutative or finite dimensional. If $\omega : U \longrightarrow F$ is a nonzero homomorphism of algebras, then the ordered pair (U, ω) will be called a *b-algebra* over F and ω its weight function or simply its weight. For $x \in U$, $\omega(x)$ is called weight of x.

When B is a subalgebra of U and $B \not\subset ker\omega$, then B is called a b-subalgebra of (U, ω). In this case, (B, ω_B) is a b-algebra, where $\omega_B = \omega|_B : B \longrightarrow F$.

Let B be a b-subalgebra of (U, ω). Then the subset $\operatorname{bar}(B) = \{x \in B \mid \omega(x) = 0\}$ is a two-side ideal of B of codimension 1, called bar ideal of B. For all $b \in B$ with $\omega(b) \neq 0$, we have $B = Fb \oplus \operatorname{bar}(B)$. If B is a b-subalgebra of U and $\operatorname{bar}(B)$ is a two-side ideal of $\operatorname{bar}(U)$ (then by [2, Proposition 1.1]), it is also a two-sided ideal of U), then B is called normal b-subalgebra of (U, ω). If $I \subseteq \operatorname{bar}(B)$ is a two-side ideal of B, then I is called b-ideal of B.

Let (U, ω) be a b-algebra. A subset B is called maximal (normal) b-subalgebra of U if B is a (normal) b-subalgebra of U and there is no (normal) b-subalgebra C of U such that $B \subset C \subset U$. A subset I is called maximal b-ideal of U if I is a b-ideal of U, $I \neq \operatorname{bar}(U)$ and there is no b-ideal J of U such that $I \subset J \subset \operatorname{bar}(U)$.

A nonzero element $e \in U$ is called an *idempotent* if $e^2 = e$ and *nontrivial idempotent* if it is an idempotent different from multiplicative identity element, if the algebra has this element. If (U, ω) is a b-algebra and $e \in U$ is an idempotent, then $\omega(e) = 0$ or $\omega(e) = 1$. When $\omega(e) = 1$, then e is called *idempotent of weight 1*.

A b-algebra (U, ω) is called b-simple if for all normal b-subalgebra B of U, $\operatorname{bar}(B) = (0)$ or $\operatorname{bar}(B) = \operatorname{bar}(U)$. When (U, ω) has an idempotent of weight 1, then (U, ω) is b-simple if, and only if, its only b-ideals are (0) and $\operatorname{bar}(U)$.

Let (U, ω) be a b-algebra. We define the *b-radical* of U, denoted by $\operatorname{rad}(U)$, as: $\operatorname{rad}(U) = (0)$, if (U, ω) is b-simple, otherwise as $\operatorname{rad}(U) = \bigcap \operatorname{bar}(B)$, where B runs over the maximal normal b-subalgebra of U. Of course, $\operatorname{rad}(U)$ is a b-ideal of U.

We say that U is b-semisimple if $\mathrm{rad}(U) = (0)$.

Let U be an algebra over a field F not necessarily associative, commutative and not necessarily having a unit. Let us denote the associator by $(x, y, z) = (xy)z - x(yz)$ and the commutator by $[x, y] = xy - yx$.

By [8] we have the following definition.

Let F be a field of characteristic different from 2 and 3. An algebra U over F is called a *generalized alternative algebra II* if the following two identities are satisfied:

$$(wx, y, z) + (w, x, [y, z]) = w(x, y, z) + (w, y, z)x, \tag{1.1}$$

$$(x, y, x) = 0, \tag{1.2}$$

All associative or noncommutative Jordan algebra is a generalized alternative algebra II and all generalized alternative algebra II is a power-associative algebra [8].

An ideal $K \neq 0$ of a generalized alternative algebra II is called minimal if for any ideal J such that $0 \subseteq J \subseteq K$, then $J = 0$ or $J = K$.

Let U be a finite dimensional generalized alternative algebra II. We define the nilradical of U, denoted by $R(U)$, as the maximal nil ideal of U. We say that U is simple when it contains no non-trivial ideals and the multiplication operation is not uniformly zero and that U is semisimple if $R(U) = 0$.

Any finite dimensional simple generalized alternative algebra II is alternative and any semisimple non-null generalized alternative algebra II is uniquely expressible as a direct sum

$$U = W_1 \oplus \cdots \oplus W_r,$$

of simple ideals W_i $(1 \leq i \leq r)$ [8].

Let U be a generalized alternative algebra II. If B and C be subalgebras of U, let us define

$$BC = \mathrm{vect}_F \{bc \mid b \in B,\ c \in C\}$$

and B^k inductively by

$$B^1 = B \textit{ and } B^{k+1} = B^k B + B^{k-1} B^2 + \cdots + B^2 B^{k-1} + B B^k.$$

We obtain a descending chain

$$B^1 \supset B^2 \supset \cdots \supset B^k \supset \cdots$$

of subalgebras of U. We call B *nilpotent* if there is some k for which $B^k = 0$.

For any subalgebra T of U one obtains a *derived series* of subalgebras
$$T^{(0)} \supset T^{(1)} \supset \cdots \supset T^{(n)} \supset \cdots$$
by defining $T^{(0)} = T$, $T^{(i+1)} = \left(T^{(i)}\right)^2$. We call T *solvable* if there is some integer n for which $T^{(n)} = 0$.

A finite dimensional generalized alternative algebra II is solvable if, and only if, it is nilpotent.

Theorem 1.1. *Let U be a generalized alternative algebra II of characteristic different from 2 with a idempotent e. If U contains no ideal $I \neq 0$, such that $I^2 = 0$, then U has a Peirce decomposition into a direct sum of subspaces*

$$U = U_{11} \oplus U_{10} \oplus U_{01} \oplus U_{00},$$

where $U_{ij} = \{x_{ij} \in U : \; ex_{ij} = ix_{ij}$ and $x_{ij}e = jx_{ij}\}$ $(i, j = 0, 1)$. The multiplication table for the Peirce decomposition is:

(i) $U_{ij}U_{kl} = 0$, $j \neq k$;

(ii) $U_{ij}U_{jl} \subseteq U_{il}$;

(iii) $U_{01}U_{01} \subseteq U_{10}$;

(iv) $U_{10}U_{10} \subseteq U_{01}$.

Proof: See [6].

Henceforth let U be a generalized alternative algebra II contains no ideal $I \neq 0$, such that $I^2 = 0$.

Proposition 1.2. *Let U be a finite dimensional generalized alternative algebra II which is not a nilalgebra. Then U has a principal idempotent.*

Proof: By [7, Proposition 3.3] U contains an idempotent e. If e is not principal, then there is an idempotent $u \neq 0$ in $U_{00}(e)$ such that $f = e + u$ is an idempotent and $U_{11}(f)$ contains properly $U_{11}(e)$. In fact, cleary f is an idempotent and for $x_{11} \in U_{11}(e)$ we have $x_{11}f = x_{11}e = x_{11}$ and similarly $fx_{11} = x_{11}$, so that $x_{11} \in U_{11}(f)$. That is, $U_{11}(e) \subseteq U_{11}(f)$. But $u \in U_{11}(f)$ and $u \notin U_{11}(e)$. Hence $\dim U_{11}(e) < \dim U_{11}(f)$ and this process of increasing dimensions must terminate yielding a principal idempotent.

Proposition 1.3. *Let U be a finite dimensional generalized alternative algebra II and e a principal idempotent of U. Then,*

$$U_{10} \oplus U_{01} \oplus U_{00} \subseteq R(U).$$

Proof: Since e is a principal idempotent of U, then the Peirce space $U_{00}(e)$ is a nilalgebra of U. This implies that in the Peirce decomposition of the quotient algebra $U/R(U)$, relative to the idempotent $\overline{e}$, the Peirce space $\Big(U/R(U)\Big)_{00}$ is a nilalgebra of $U/R(U)$. Hence $\overline{e}$ is a principal idempotent of $U/R(U)$. So $\overline{e}$ is the unity element of the algebra $U/R(U)$, by proof of [8, Theorem 2.4], from which we obtain $U/R(U) = \Big(U/R(U)\Big)_{11}$. Consequently $U_{10} \oplus U_{01} \oplus U_{00} \subseteq R(U)$.

Theorem 1.4. *Let U be a finite dimensional generalized alternative algebra II and J a ideal of U. Then, $R(J) = J \cap R(U)$.*

Proof: Let us consider the canonical homomorphism $\varphi : U \to U/R(U)$, an ideal J of U and K a nilideal of J. Then $\varphi(J)$ is an ideal of $U/R(U)$ semisimple and $\varphi(K)$ is a nilideal of $\varphi(J)$. This implies $\varphi(K) = 0$ which results in $K \subseteq R(U)$. Hence $R(J) \subseteq R(U)$. So $R(J) \subseteq J \cap R(U)$. Consequently, $R(J) = J \cap R(U)$.

More details on the definitions and properties mentioned above will be found in [5]-[8].

2. The b-radical

In this section we characterize the b-radical of a finite dimensional generalized alternative b-algebra II. The characterization of b-radical is fundamental for the demonstration of Wedderburn decomposition as can be seen in [3], [4]. Let us observe that, if (U, ω) is a generalized alternative b-algebra II, then U has an idempotent of weight 1, by [1, Corollary 3.1].

Lemma 2.1. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. Then every principal idempotent e of U has weight* 1.

Proof: Let us consider $U = U_{11} \oplus U_{10} \oplus U_{01} \oplus U_{00}$ the Peirce decompositions of U, relative to e. Since $R(U) \subseteq \mathrm{bar}(U)$, then $U_{10} \oplus U_{01} \oplus U_{00} \subseteq \mathrm{bar}(U)$, by Proposition 1.3. Hence, if $\omega(e) = 0$, then $U_{11} \subseteq \mathrm{bar}(U)$ which yields $U = \mathrm{bar}(U)$. This implies that ω is the zero homomorphism, which is absurd.

Proposition 2.2. *Let (U,ω) be a finite dimensional generalized alternative b-algebra II. Then $R(U)=R\Big(\mathrm{bar}(U)\Big)$.*

Proof: Let us again observe that, $R(U)$ is a nil ideal of U with $R(U)\subseteq \mathrm{bar}(U)$. Thus $R(U)\subseteq R\Big(\mathrm{bar}(U)\Big)$. To prove the other inclusion, we will show that $R\Big(\mathrm{bar}(U)\Big)$ is an ideal of U. In fact, if $R(U)=\mathrm{bar}(U)$ then clearly $R\Big(\mathrm{bar}(U)\Big)\subseteq R(U)$. Now, if $R(U)\neq \mathrm{bar}(U)$, then $\mathrm{bar}(U)$ has a principal idempotent f, by Proposition 1.2. Let us consider a nonzero orthogonal idempotent e to f, by Lemma 2.1. Certainly, e has weight 1 since f is principal in $\mathrm{bar}(U)$. Let us take $U=Fe\oplus \mathrm{bar}(U)$ and the Peirce decompositions of U and of $\mathrm{bar}(U)$, relative to f,

$$U=U_{11}(f)\oplus U_{10}(f)\oplus U_{01}(f)\oplus U_{00}(f)$$

and

$$\mathrm{bar}(U)=\mathrm{bar}(U)_{11}(f)\oplus \mathrm{bar}(U)_{10}(f)\oplus \mathrm{bar}(U)_{01}(f)\oplus \mathrm{bar}(U)_{00}(f).$$

Then:

1. $U_{11}(f)=\mathrm{bar}(U)_{11}(f)$.

Clearly $\mathrm{bar}(U)_{11}(f)\subseteq U_{11}(f)$. Otherwise for all $x_{11}\in U_{11}(f)$, $fx_{11}=x_{11}$ which implies $U_{11}(f)\subseteq \mathrm{bar}(U)_{11}(f)$.

2. $U_{10}(f)=\mathrm{bar}(U)_{10}(f)$ and $U_{01}(f)=\mathrm{bar}(U)_{01}(f)$.

Clearly $\mathrm{bar}(U)_{10}(f)\subseteq U_{10}(f)$ and for all $x_{10}\in U_{10}(f)$, we have $fx_{10}=x_{10}$ which implies $U_{10}(f)\subseteq \mathrm{bar}(U)_{10}(f)$. Similarly, we show $U_{01}(f)=\mathrm{bar}(U)_{01}(f)$.

3. $U_{00}(f)=Fe\oplus \mathrm{bar}(U)_{00}(f)$.

Clearly $Fe\oplus \mathrm{bar}(U)_{00}(f)\subseteq U_{00}(f)$. Now, for all $x_{00}\in U_{00}(f)$, we have $x_{00}=\alpha e+x$, where $x\in \mathrm{bar}(U)$. Since $fx_{00}=x_{00}f=0$, then $fx=f(\alpha e+x)=0$ and $xf=(\alpha e+x)f=0$. Thus $U_{00}(f)\subseteq Fe\oplus \mathrm{bar}(U)_{00}(f)$.

4. $\mathrm{bar}(U)_{10}(f)\oplus \mathrm{bar}(U)_{01}(f)\oplus \mathrm{bar}(U)_{00}(f)\subseteq R\Big(\mathrm{bar}(U)\Big)$.

This follows from the fact that f is a principal idempotent of $\mathrm{bar}(U)$.

Hence, for every element $x\in U$ and $y\in R\Big(\mathrm{bar}(U)\Big)$, let us write $x=\alpha e+x_{11}+x_{10}+x_{01}+x_{00}$, where $\alpha\in F$ and $x_{ij}\in \mathrm{bar}(U)_{ij}(f)$ $(i,j=0,1)$, and $y=y_{11}+y_{10}+y_{01}+y_{00}$, with $y_{ij}\in \mathrm{bar}(U)_{ij}(f)$ $(i,j=0,1)$. Then $xy=\alpha ey+x_{11}y+x_{10}y+x_{01}y+x_{00}y$. Since $x_{11}y+x_{10}y+x_{01}y+x_{00}y\in R\Big(\mathrm{bar}(U)\Big)$, $ey_{11}=0$, $ey_{10}=0$, $ey_{01}\in R\Big(\mathrm{bar}(U)\Big)$ and $ey_{00}\in R\Big(\mathrm{bar}(U)\Big)$, then $ey \in R\Big(\mathrm{bar}(U)\Big)$ which yields $UR\Big(\mathrm{bar}(U)\Big) \subseteq R\Big(\mathrm{bar}(U)\Big)$. Similarly, we have $R\Big(\mathrm{bar}(U)\Big)U \subseteq R\Big(\mathrm{bar}(U)\Big)$. Thus $R\Big(\mathrm{bar}(U)\Big)$ is an ideal of U and therefore $R\Big(\mathrm{bar}(U)\Big) \subseteq R(U)$. Consequently, $R(U) = R\Big(\mathrm{bar}(U)\Big)$.

Proposition 2.3. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. If $R(U) = 0$, then we have the following conditions:*

(i) There is a principal idempotent $f \in \mathrm{bar}(U)$ and an orthogonal idempotent e to f of weight 1 such that $U = Fe \oplus U_{11}(f)$;

(ii) There is a primitive idempotent e such that Fe is a simple ideal of U;

(iii) $\mathrm{rad}(U) = 0$.

Proof: *(i).* From the demonstration of the Proposition 2.2, there is a principal idempotent $f \in \mathrm{bar}(U)$ and an orthogonal idempotent e to f of weight 1 such that $U = Fe \oplus \mathrm{bar}(U)_{11}(f) \oplus \mathrm{bar}(U)_{10}(f) \oplus \mathrm{bar}(U)_{01}(f) \oplus \mathrm{bar}(U)_{00}(f)$ and $\mathrm{bar}(U)_{10}(f) \oplus \mathrm{bar}(U)_{01}(f) \oplus \mathrm{bar}(U)_{00}(f) \subset R\Big(\mathrm{bar}(U)\Big)$. This implies $U = Fe \oplus U_{11}(f)$.
(ii) From Proposition 2.2 again, we have $U_{00}(f) = Fe \oplus \mathrm{bar}(U)_{00}(f) = Fe$. Thus Fe is a simple ideal of U.
(iii) If $R(U) = 0$, then U is a semisimple algebra and uniquely expressible as a direct sum $U = W_1 \oplus \cdots \oplus W_r$ of nonzero simple ideals W_i of U. Let us consider elements e_i in W_i for $(1 \leq i \leq r)$ such that $e = \sum_{i=1}^{r} e_i$. Let us observe that e_i $(1 \leq i \leq r)$ are nonzero idempotents, of U, the unity elements of W_i, respectively and $e_i e_j = 0$ for all $i, j = 1, \ldots, r$. It is easy to see that there is an only k $(1 \leq k \leq r)$ such that $\omega(e_k) = 1$ and $\{e_1, \ldots, e_r\} \setminus \{e_k\} \subseteq \mathrm{bar}(U)$. Without loss of generality, we can suppose that $\omega(e_1) = 1$. Hence $W_1 = Fe_1 \oplus \mathrm{bar}W_1$. Since $\mathrm{bar}W_1$ is an ideal of W_1, then $\mathrm{bar}W_1 = 0$. So $U = Fe_1 \oplus W_2 \oplus \cdots \oplus W_r$. Next, let us observe that $W_k \subseteq \mathrm{bar}(U)$ for every k $(2 \leq k \leq r)$. This implies that $\mathrm{bar}(U) = W_2 \oplus \cdots \oplus W_r$. Since W_k is a b-simple ideal of U, then U is b-semisimple and therefore $\mathrm{rad}(U) = 0$, by [5, Proposition 4.2].

Proposition 2.4. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. Then* $\mathrm{rad}(U) \subseteq R(U)$.

Proof: Let $e \in U$ be an idempotent of weight 1 and $B = Fe \oplus R(U)$. Then B is a normal b-subalgebra of (U, ω). Since the quotient algebra

$U/\mathrm{bar}(B) = U/R(U)$ is semisimple, then we have $\mathrm{rad}\Big(U/\mathrm{bar}(B)\Big) = 0$, by Proposition 2.3. Hence $\mathrm{rad}(B) \subseteq \mathrm{rad}(U) \subseteq \mathrm{bar}(B) = R(U)$, by [5, Proposition 3.4]. So $\mathrm{rad}(U) \subseteq R(U)$.

Proposition 2.5. *Let (U, ω) be a generalized alternative b-algebra II and J a b-ideal of U, then J^3 is a b-ideal of U. Moreover, if J is a minimal b-ideal of U, then $J^3 = 0$ or $J^3 = J$.*

Proof: By, [8, Lemma 2.3], J^3 is b-ideal of U. Since J is minimal, then $J^3 = 0$ or $J^3 = J$.

Corollary 2.6. *Let (U, ω) be a generalized alternative b-algebra II. Then $(\mathrm{bar}(U))^3$ is a b-ideal of U.*

Proposition 2.7. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. Then*

$$\mathrm{rad}(U) \subseteq \Big(\mathrm{bar}(U)\Big)^3.$$

Proof: Let $e \in U$ be an idempotent of weight 1, $U = Fe \oplus \mathrm{bar}(U)$ and $U = U_{11} \oplus U_{10} \oplus U_{01} \oplus U_{00}$ the Peirce decomposition of U, relative to e. According with the properties of this decomposition, there are vectorial subspaces $Y_{ij} \subseteq \mathrm{bar}(U) \cap U_{ij}$ $(i, j = 0, 1)$ of U such that

$$\mathrm{bar}(U) = \Big(\mathrm{bar}(U)\Big)^3 \oplus Y_{11} \oplus Y_{10} \oplus Y_{01} \oplus Y_{00}.$$

For each subspace Y_{ij} $(i, j = 0, 1)$, let us take a basis $Z_{ij} = \{z_{1ij}, \ldots, z_{n_{ij}ij}\}$ and let us define the subspaces

$$\mathrm{J}(\mathrm{m}_{ij}, ij) = \Big(\mathrm{bar}(U)\Big)^3 \oplus < (Z_{11} \cup Z_{10} \cup Z_{\frac{1}{2}\frac{1}{2}} \cup Z_{01} \cup Z_{00}) \setminus \{z_{m_{ij}ij}\} >,$$

for $(i, j = 0, 1)$ *and* $1 \leq m_{ij} \leq n_{ij}$. Let us prove that $J(m_{ij}, ij)$ is a maximal b-ideal of U. In fact, let us be $x \in U$ and $y \in J(m_{ij}, ij)$. There are scalars $\alpha, \alpha_{k_{pq}} \in F$ $(1 \leq k_{pq} \leq n_{pq}$ *and* $k_{pq} \neq m_{ij};\ p, q = 0, 1)$ and elements $a \in \mathrm{bar}(U)$ and $b \in \Big(\mathrm{bar}(U)\Big)^3$ such that $x = \alpha e + a$ *and* $y = b + \sum_{p,q=0,1} \Big(\sum_{k_{pq}=1(\neq m_{ij})}^{n_{pq}} \alpha_{k_{pq}} z_{k_{pq}pq}\Big)$. It follows that

$$xy = \alpha(eb) + ab + \sum_{p,q=0,1} \Big(\sum_{k_{pq}=1(\neq m_{ij})}^{n_{pq}} \alpha_{k_{pq}} a z_{k_{pq}pq}\Big) + \sum_{p,q=0,1} \Big(\sum_{k_{pq}=1(\neq m_{ij})}^{n_{pq}} p\alpha\alpha_{k_{pq}} z_{k_{pq}pq}\Big).$$

This implies that $xy \in J(m_{ij}, ij)$. Simirlarly, we have that $yx \in J(m_{ij}, ij)$.

Thus $J(m_{ij}, ij)$ is a b-ideal of U. Moreover, as all $J(m_{ij}, ij)$ are of codimension 1, then it is also maximal. Consequently,

$$\mathrm{rad}(U) \subseteq \bigcap_{z_{m_{ij}ij}} J(m_{ij}, ij) = \Big(\mathrm{bar}(U)\Big)^3,$$

where

$$z_{m_{ij}ij} \in Z_{11} \cup Z_{10} \cup Z_{01} \cup Z_{00} \ (1 \leq m_{ij} \leq n_{ij};\ i, j = 0, 1).$$

Proposition 2.8. *Let (U, ω) be a finite dimensional generalized alternative b-algebra I. Then*

$$R(U) \bigcap \Big(\mathrm{bar}(U)\Big)^3 \subseteq \mathrm{rad}(U).$$

Proof: Let us take the quotient b-algebra $U/\mathrm{rad}(U)$. By [5, Corollary 3.1], we have $\mathrm{rad}\Big(U/\mathrm{rad}(U)\Big) = 0$ which implies $U/\mathrm{rad}(U)$ b-semisimple, by [5, Theorem 4.2]. Hence $\mathrm{bar}\Big(U/\mathrm{rad}(U)\Big)$ is a direct sum of minimal ideals

$$I_1 \oplus \ldots \oplus I_s \oplus J_{s+1} \oplus \ldots \oplus J_r,$$

of $U/\mathrm{rad}(U)$, where $I_i^3 = I_i$ $(1 \leq i \leq s)$ and $J_j^3 = 0$ $(s+1 \leq j \leq r)$. Let us take the ideal of $U/\mathrm{rad}(U)$

$$\Big(\mathrm{bar}\Big(U/\mathrm{rad}(U)\Big)\Big)^3 = I_1 \oplus \ldots \oplus I_s.$$

Since

$$\begin{aligned} & R\Big(\Big(\mathrm{bar}\Big(U/\mathrm{rad}(U)\Big)\Big)^3\Big) \\ = \ & R\Big(U/\mathrm{rad}(U)\Big) \bigcap \Big(\mathrm{bar}\Big(U/\mathrm{rad}(U)\Big)\Big)^3, \end{aligned}$$

by Theorem 1.4, and $R\Big(U/\mathrm{rad}(U)\Big) \bigcap I_i$ $(1 \leq i \leq s)$ are ideals of $U/\mathrm{rad}(U)$, then

$$R\Big(U/\mathrm{rad}(U)\Big) \bigcap I_i = 0.$$

It follows that

$$R\Big(\Big(\mathrm{bar}\Big(U/\mathrm{rad}(U)\Big)\Big)^3\Big) = 0.$$

Hence

$$0 = R\Big(\big(\mathrm{bar}\big(U/\mathrm{rad}(U)\big)\big)^3\Big) \cong R\Big(\big(\mathrm{bar}(U)\big)^3\Big)\Big/\mathrm{rad}(U)$$

which implies

$$R\Big(\big(\mathrm{bar}(U)\big)^3\Big) \subseteq \mathrm{rad}(U).$$

Consequently,

$$R(U)\bigcap\big(\mathrm{bar}(U)\big)^3 \subseteq \mathrm{rad}(U).$$

From the propositions 2.4, 2.7 and 2.8, we can conclude the main result of this paper.

Theorem 2.9. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. Then*

$$\mathrm{rad}(U) = R(U)\bigcap\Big(\big(\mathrm{bar}(U)\big)^3\Big).$$

Corollary 2.10. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. Then* $\mathrm{bar}(U)$ *is nilpotent if, and only if,* $\mathrm{rad}(U) = \big(\mathrm{bar}(U)\big)^3$.

Corollary 2.11. *Let (U, ω) be a finite dimensional generalized alternative b-algebra II. If* $\mathrm{bar}(U) = \big(\mathrm{bar}(U)\big)^3$, *then* $\mathrm{rad}(U) = R(U)$.

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