Proyecciones (Antofagasta. On line) | vol. 38, n. 5 (2019) | pp. 981-998

doi:10.22199/issn.0717-6279-2019-05-0063

PROYECCIONES
Journal of Mathematics

REVISTAS CIENTÍFICAS de la Universidad Católica del Norte. revistas.ucn.cl

ISSN 0717-6279 (On line)

On approximation of signals in the generalized Zygmund class via $(E, 1)$ $(\bar{N}, p_n)$ summability means of conjugate Fourier series

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Received: September 2018 | Accepted: October 2018

Abstract:

Approximation of functions of different classes have been considered by various researchers under different summability means. In the present paper, presumably a new theorem has been established under $(E, 1)(\bar{N}\ p_n)$ -product summability mean of conjugate Fourier series of a function of $Z_r^{(\omega)}$ r $(r \geq 1)$ -class (generalized Zygmund class). Moreover, the result obtained here is a generalization of several known theorems.

Keywords: Degree of approximation; Generalized Zygmund class; Conjugate Fourier series; (E, 1) summability means; $(\bar{N}, p_n)$ summability means; $(E, 1)(\bar{N}, p_n)$ summability means

MSC (2010): 41A25; 42B05; 42B08.

Cite this article as (IEEE citation style):

T. Pradhan, S. Paikray, A. Das and H. Dutta, "On approximation of signals in the generalized Zygmund class via $(E, 1)$ $(\bar{N}, p_n)$ summability means of conjugate Fourier series", *Proyecciones (Antofagasta, On line)*, vol. 38, no. 5, pp. 981-998, Dec. 2019, doi: 10.22199/issn.0717-6279-2019-05-0063.

1. Introduction and Preliminaries

Signal analysis is concerned with the reliable estimation, detection and classification of signals (functions) which are subject to random fluctuations and it has its roots in probability theory, mathematical statistics and, more recently, approximation theory and communications theory. These approximation analysis of signals have great importance in the field of science and engineering. Moreover, it has given a new dimensions because of their vast applications in signal analysis, radar system, telecommunications and image processing system. The error estimation of functions in various function spaces such as Hölder, Lipschitz, Zygmund and Besov spaces etc. by using different summability techniques of trigonometric Fourier series has been received a growing interest of several researchers in last decades. Functions in L_r $(r \geq 1)$-spaces assumed to be most practicable in signal analysis. Particularly, L_1, L_2 and L_∞ spaces are used by engineers for designing digital filters and matrix summability or matrix transformation plays a vital role in this context. Also, matrix summability generalizes different summability methods like Cesàro summability, Nörlund summability, Riesz summability, Banach summability etc. (see [1], [3], [6], [16], and [18]. Recently, various product summability means under Lipschitz classes of functions have been considered to prove several approximation theorems. For more details, see the current works [7], [8], [9], [10] and [17]. Subsequently, the generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ is an extension of $Z_{(\alpha)}, Z_{(\alpha),r}, Z^{(\omega)}$-classes. The generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ is studied by Leindler [11], Moricz [13], Moricz and Nemeth [14]. Recently Singh et al. [18] used Hausdörff means to established some approximation results for the functions in generalized Zygmund class. Lal and Shireen [9] considered matrix-Euler summability mean of Fourier series for approximation of functions of generalized Zygmund class. Very recently, some results on statistical approximation and associated Korovkin-type theorems has been established by different researchers (see [2], [5], [8], [19], [20], [21] and [22]). To get best approximation by product summability means (Ordinary versions), in the proposed paper, we have used $(E,1)(\overline{N},p_n)$ summability mean of conjugate Fourier series to estimate the degree of approximation of a function of $Z_r^{(\omega)}$ $(r \geq 1)$ class (generalized Zygmund class).

Let $\sum u_n$ be an infinite series with its sequence of partial sum $\{s_n\}$. Let $\{p_k\}$ for $k = 0, 1, 2, ...$ be a sequence of non-negative integers such that $p_0 > 0$ and

$$P_n = \sum_{k=0}^{n} p_k \to \infty \ as\ n \to \infty \quad (p_{-k},\ P_{-k} = 0,\ k \geq 1). \tag{1.1}$$

Let the sequence to sequence transformation,

$$\tau_n^{\overline{N}} = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k, \quad n = 0, 1, 2, ... \tag{1.2}$$

defines $(\overline{N}, p_n)$ mean of $\{s_n\}$ generated by the sequence $\{p_k\}$. The series $\sum u_n$ is known to be summable to s by $(\overline{N}, p_n)$ method, if $\lim_{n\to\infty} \tau_n^{\overline{N}} \to s$ as $n \to \infty$. Also, this $(\overline{N}, p_n)$ method is regular (see [4]).

The sequence to sequence transformation,

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{k}{v} s_k, \tag{1.3}$$

defines the $(E, 1)$ transform of the sequence $\{s_n\}$. The series $\sum u_n$ is summable to s with respect to $(E, 1)$ summability, if $E_n^1 \to s$ as $n \to \infty$. Also $(E, 1)$ method is regular (see [4]).

Now we define here a new composite transform by using the product $(E, 1)(\overline{N}, p_n)$ transform. As $(\overline{N}, p_n)$ and $(E, 1)$ summability methods are regular, the product $(E, 1)(\overline{N}, p_n)$ method is also regular.

Let

$$\tau_n^{E\overline{N}} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v s_v \right\}, \tag{1.4}$$

defines the $(E, 1)(\overline{N}, p_n)$ transform of the sequence $\{s_n\}$. We say here that $\sum u_n$ is summable to s by product $(E, 1)(\overline{N}, p_n)$ transform, if $\tau_n^{E\overline{N}} \to s$ as $n \to \infty$.

Let f is 2π periodic function belonging to $L^r[0, 2\pi]$, $r \geq 1$ with the partial sum $s_n(f)$, then

$$s_n(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.5}$$

The conjugate series of the Fourier series (1.5) is given by

$$\widetilde{s_n}(f) = \sum_{k=1}^{\infty}(a_n \cos nx - b_n \sin nx). \tag{1.6}$$

Let f be a 2π-periodic integrable function belonging to $[0, 2\pi]$ and let $\widetilde{f}$, conjugate to f be a 2π periodic function to f. We have,

$$L_r[0, 2\pi] = \{\widetilde{f} : [0, 2\pi] \to \mathbf{R}; \int_0^{2\pi} |f(x)|^r dx < \infty\}.$$

The L_r norm of a function f is defined by

$$\|f\|_r = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, \ 1 \le r < \infty \\ ess\sup_{0<x\le 2\pi} |f(x)|, \quad r = \infty. \end{cases}$$

The Zygmund modulus of continuity of f is defined by

$$\omega(f, h) = \sup_{o \le h, x \in \mathbf{R}} |f(x+t) + f(x-t)|.$$

Let $C_{2\pi}$ be the Banach space of all 2π-periodic functions (continuous) defined on $[0, 2\pi]$ under the supremum norm. For $0 < \alpha \le 1$, the function space

$$Z_{(\alpha)} = \{f \in \mathbf{C}_{2\pi} : |f(x+t) + f(x-t)| = O(|t|^\alpha)\}$$

is also a Banach space with the norm $\|\cdot\|_{(\alpha)}$, given by

$$\|f\|_{(\alpha)} = \sup_{0 \le x \le 2\pi} |f(x)| + \sup_{x, t \ne 0} \frac{|f(x+t) + f(x-t)|}{|t|^\alpha}.$$

For $f \in L_r[0, 2\pi], r \ge 1$, the integral Zygmund modulus of continuity is given by

$$\omega_r(f, h) = \sup_{0<t\le h} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+t) + f(x-t)|^r dx\right\}^{\frac{1}{r}}.$$

For $f \in \mathbf{C}_{2\pi}$ *and* $r = \infty$,

$$\omega_\infty(f,h) = \sup_{0<t\leq h} \max_x |f(x+t)+f(x-t)|.$$

Also, it is known that $\omega_r(f,h) \to 0$ as $r \to 0$.

Now define,

$$Z_{(\alpha),r} = \left\{ f \in L_r[0,2\pi] : \left(\int_0^{2\pi} |f(x+t)+f(x-t)|^r dx \right)^{\frac{1}{r}} = O(|t|^\alpha) \right\}$$

and also the space $Z_{(\alpha),r}, r \geq 1, 0 < \alpha \leq 1$ is a Banach space with the norm $\|\cdot\|_{(\alpha),r}$ of the form

$$\|f\|_{(\alpha),r} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(\cdot+t)+f(\cdot-t)\|_r}{|t|^\alpha}.$$

The class of function $Z^{(\omega)}$ is defined as

$$Z^{(\omega)} = \{f \in \mathbf{C}_{2\pi} : |f(x+t)+f(x-t)| = O(\omega(t))\},$$

where ω is a Zygmund modulus of continuity, that is, ω is positive, non-decreasing continuous function with the sub linearity property. That is,

(i) $\omega(0) = 0$ and

(ii) $\omega(t_1+t_2) \leq \omega(t_1) + \omega(t_2)$.

Let $\omega : [0,2\pi] \to \mathbf{R}$ be an arbitrary function with $\omega(t) > 0$ for $0 \leq t < 2\pi$ and let $\lim_{t\to 0^+} \omega(t) = \omega(0) = 0$, define

$$Z_r^{(\omega)} = \left\{ \widetilde{f} \in L_r : 1 \leq r \leq \infty, \sup_{t\neq 0} \frac{\|f(\cdot+t)+f(\cdot-t)\|_r}{\omega(t)} < \infty \right\},$$

where

$$\|\widetilde{f}\|_r^{(\omega)} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(\cdot+t)+f(\cdot-t)\|_r}{\omega(t)}, r \geq 1.$$

Clearly $\|\cdot\|_r^{(\omega)}$ is a norm on $Z_r^{(\omega)}$. As we know L_r $(r \geq 1)$ is complete, the space $Z_r^{(\omega)}$ is also complete. Hence we can say $Z_r^{(\omega)}$ is a Banach space under the norm $\|\cdot\|_r^{(\omega)}$.

We use the following notations through out this paper:

$$\psi(x,t) = f(x+t) + f(x-t);$$
$$\widetilde{K}_n^{E\overline{N}}(t) = \frac{1}{\pi 2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\}.$$

2. Main Theorems

Theorem 2.1. *Let $\widetilde{f}$ be conjugate to 2π periodic function f, Lebesgue integrable in $[0,2\pi]$ and belonging to generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$. Then the degree of approximation of signal (function) $\widetilde{f}$, using product $(E,1)(\overline{N},p_n)$ mean of conjugate Fourier series (1.6) is given by*

$$E_n(\widetilde{f}) = \inf_{\widetilde{\tau}_n^{E\overline{N}}} \|\widetilde{\tau}_n^{E\overline{N}} - \widetilde{f}\|_r^v = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t\, v(t)} \right), \tag{2.1}$$

where $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that $\frac{\omega(t)}{v(t)}$ is positive and increasing.

Theorem 2.2. *Let $\widetilde{f}$ be conjugate to 2π periodic function f, Lebesgue integrable in $[0,2\pi]$ and belonging to generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$. Then the degree of approximation of signal (function) $\widetilde{f}$, using product $(E,1)(\overline{N},p_n)$ mean of conjugate Fourier series (1.6) is given by*

$$E_n(\widetilde{f}) = \inf_{\widetilde{\tau}_n^{E\overline{N}}} \|\widetilde{\tau}_n^{E\overline{N}} - \widetilde{f}\|_r^v = O\left(\frac{\omega(\frac{1}{n+1})}{(n+1)^2 v(\frac{1}{n+1})} (\pi(n+1)-1) \right), \tag{2.2}$$

where $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that $\frac{\omega(t)}{tv(t)}$ is positive and decreasing.

To prove the theorems we need the following lemmas.

Lemma 2.3. $|\widetilde{K_n}(t)| = O(n)$, *for* $0 \leq t \leq \frac{1}{n+1}$.

Proof. For $0 \leq t \leq \frac{1}{n+1}$, we have $\sin\ nt \leq n \sin\ t$.

$$\begin{aligned}
|\widetilde{K_n}(t)| &= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}} \right\} \right| \\
&= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\frac{\cos\frac{t}{2} - \cos vt.\cos\frac{t}{2} + \sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}} \right) \right\} \right| \\
&= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\frac{\cos\frac{t}{2}(2\sin^2 v\frac{t}{2})}{\sin\frac{t}{2}} + \sin vt \right) \right\} \right| \\
&\leq \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(O(2\sin v\frac{t}{2}\sin v\frac{t}{2}) \right) + v\sin t \right\} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(O(v) + O(v) \right) \right\} \right| \\
&= \frac{1}{\pi 2^{n}} \left| \sum_{k=0}^{n} \binom{n}{k} \frac{O(k)}{P_k} \sum_{v=0}^{k} p_v \right| \\
&= O(n).
\end{aligned}$$

Lemma 2.4. *$|\widetilde{K_n}(t)| = O(\frac{1}{t})$, for $\frac{1}{n+1} < t \leq \pi$.*

Proof. For $\frac{1}{n+1} < t \leq \pi$ and by using Jordans lemma, $sin\frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$.

$$\begin{aligned}
|\widetilde{K_n}(t)| &= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}} \right\} \right| \\
&= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\frac{\cos\frac{t}{2} - \cos vt.\cos\frac{t}{2} + \sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}} \right) \right\} \right| \\
&\leq \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\cos\frac{t}{2}(2\sin^2 v\frac{t}{2}) + \sin vt \right) \right\} \right|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_v\right\}\right| \\ &= O\left(\frac{1}{t}\right). \end{aligned}$$

Lemma 2.5. *Let $f \in Z_r^{(\omega)}$, then for $0 < t \leq \pi$,*

(i) $\|\psi(\cdot, t)\|_r = O(\omega(t))$ *and*

(ii) $\|\psi(\cdot + y, t) + \psi(\cdot - y, t)\|_r = \begin{cases} O\left(w\left(t\right)\right) \\ O\left(w\left(y\right)\right). \end{cases}$

If $\omega(t)$ and $v(t)$ defined as in Theorem 1, then

$\|\psi(\cdot+y, t)+\psi(\cdot-y, t)\|_r = O\left(v(y)\frac{\omega(t)}{v(t)}\right)$, *where* $\psi(x, t) = f(x+t)+f(x-t)$.

Proof. This Lemma can be proved easily by following [15].

3. Proof of Main Results

3.1. Proof of Theorem 3

Let $\widetilde{s}_k(f; x)$ denotes the k^{th} partial sum of the series (1.6) and following [23], we have

$$\widetilde{s}_k(f; x) - \widetilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(x; t)\frac{\cos\frac{t}{2} - \cos(v + \frac{1}{2})t}{\sin(\frac{t}{2})}dt. \tag{3.1}$$

Therefore using (1.2), the $(\overline{N}, p_n)$ transform of $\widetilde{s}_k(f; x)$ is given by

$$\frac{1}{P_n}\sum_{k=0}^{n}p_k(\widetilde{s}_k(\widetilde{f}; x) - \widetilde{f}(x)) = \frac{1}{2\pi}\int_0^{\pi}\phi(x; t)\frac{1}{P_n}\sum_{k=0}^{n}p_k\frac{\cos\frac{t}{2} - \cos(v + \frac{1}{2})t}{\sin(\frac{t}{2})}dt.$$

(3.2)

Now denoting the $(E,1)(\overline{N},p_n)$ transform of $\widetilde{s}_k(f;x)$ by $\widetilde{\tau}_n^{E\overline{N}}$, we write

$$\widetilde{\tau}_n^{E\overline{N}} - \widetilde{f}(x) = \frac{1}{\pi 2^{n+1}} \sum_{k=0}^{n} \int_0^{pi} \frac{\psi(x;t)}{\sin(t/2)} \frac{1}{P_k} \sum_{k=0}^{v} p_v(\cos\frac{t}{2} - \cos(v+\frac{1}{2})t)dt.$$

(3.3)

Let

$$\widetilde{\mathcal{L}}_n(x) = \widetilde{\tau}_n^{E\overline{N}} - \widetilde{f}(x) = \int_0^{\pi} \psi(x;t)\widetilde{K}_n^{E\overline{N}}(t)dt, \tag{3.4}$$

then

$$\widetilde{\mathcal{L}}_n(x+y) + \widetilde{\mathcal{L}}_n(x-y) = \int_0^{\pi} \left[\psi(x+y;t) + \psi(x-y;t)\right] \widetilde{K}_n^{E\overline{N}}(t)dt.$$

(3.5)

Using generalized Minkowski's inequality, we have

$$\begin{aligned}
&\|\widetilde{\mathcal{L}}_n(x+y) + \widetilde{\mathcal{L}}_n(x-y)\|_p \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} |\widetilde{\mathcal{L}}_n(x+y) + \widetilde{\mathcal{L}}_n(x-y)|^p dx\right\}^{1/p} \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\int_0^{\pi}[\psi(x+y;t) + \psi(x-y;t)]\widetilde{K}_n^{E\overline{N}}(t)dt\right|^p dx\right\}^{1/p} \\
&\le \int_0^{\pi} \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|[\psi(x+y;t) + \psi(x-y;t)]\widetilde{K}_n^{E\overline{N}}(t)dx\right|^p\right\}^{1/p} dt \\
&= \int_0^{\pi} (|\widetilde{K}_n^{E\overline{N}}(t)|)^{1/p} \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|[\psi(x+y;t) + \psi(x-y;t)]\widetilde{K}_n^{E\overline{N}}(t)\right|^p dx\right\}^{1/p} dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi} \|\psi(\cdot + y; t) + \psi(\cdot - y; t)\|_p |\widetilde{K}_n^{E\overline{N}}(t)| dt \\
&= \int_0^{\frac{1}{n+1}} \|\psi(\cdot + y; t) + \psi(\cdot - y; t)\|_p |\widetilde{K}_n^{E\overline{N}}(t)| dt \\
&+ \int_{\frac{1}{n+1}}^{\pi} \|\psi(\cdot + y; t) + \psi(\cdot - y; t)\|_p |\widetilde{K}_n^{E\overline{N}}(t)| dt \\
&= I_1 + I_2 \quad (say).
\end{aligned}
\tag{3.6}
$$

Now, using Lemma 2.3, Lemma 2.5 and monotonicity of $(\omega(t)/v(t))$ with respect to t, we have

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{n+1}} \|\psi(\cdot + y; t) + \psi(\cdot - y; t)\|_p |\widetilde{K}_n^{E\overline{N}}(t)| dt \\
&= \int_0^{\frac{1}{n+1}} O\left(v(y) \frac{\omega(t)}{v(t)} \right) O(n) dt \\
&\leq O\left(n\, v(y) \int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)} dt \right).
\end{aligned}
\tag{3.7}
$$

Next, by using 2nd mean value theorem of integral, we have

$$
\begin{aligned}
I_1 &\leq O\left(nv(y) \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \int_0^{\frac{1}{n+1}} dt \right) \\
&= O\left(\frac{n}{n+1}\ v(y) \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \right) \\
&= O\left(v(y)\ \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \right) \quad \left(\frac{n}{n+1} = O(1) \right).
\end{aligned}
\tag{3.8}
$$

Next, using Lemma 2.4 and Lemma 2.5, we get

$$
I_2 = \int_{\frac{1}{n+1}}^{\pi} \|\psi(\cdot + y; t) + \psi(\cdot - y; t)\|_p |\widetilde{K}_n^{E\overline{N}}(t)| dt
$$

$$\leq \left(\int_{\frac{1}{n+1}}^{\pi} v(y)\ \frac{\omega(t)}{v(t)}\ \frac{1}{t}dt\right)$$

$$= O\left(v(y)\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)}dt\right). \tag{3.9}$$

From (3.6), (3.8) and (3.9), we have

$$\|\widetilde{\mathcal{L}}_n(\cdot+y)+\widetilde{\mathcal{L}}_n(\cdot-y)(\cdot)\|_r = O\left(v(y)\ \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right)+O\left(v(y)\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)}dt\right). \tag{3.10}$$

Therefore, we have

$$\sup_{y\neq 0}\frac{\|\widetilde{\mathcal{L}}_n(\cdot+y)+\widetilde{\mathcal{L}}_n(\cdot-y)\|_r}{v(y)} = O\left(\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right)+O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)}dt\right). \tag{3.11}$$

Clearly,

$$\psi(x;t) = |f(x+t)+f(x-t)|.$$

Now applying Minkowski's inequality, we have

$$\begin{aligned}\|\psi(\cdot,t)\|_r &= \|f(x+t)+f(x-t)\|_r \\ &= O(\omega(t)).\end{aligned} \tag{3.12}$$

Now, using Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}\|\widetilde{\mathcal{L}}_n(\cdot)\|_r &\leq \left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\|\psi(\cdot,t)\|_r|\widetilde{K_n^{E\overline{N}}}(t)|dt \\ &= O\left(n\int_0^{\frac{1}{n+1}}\omega(t)dt\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}dt\right)\end{aligned}$$

$$
\begin{aligned}
&= O\left(n\omega(\frac{1}{n+1})\int_0^{\frac{1}{n+1}}\omega(t)dt\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t}dt\right)\\
&= O\left(\frac{n}{n+1}\ w(\frac{1}{n+1})\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}dt\right)\\
&= O\left(\omega(\frac{1}{n+1})\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}dt\right). \qquad (3.13)
\end{aligned}
$$

From (3.11) and (3.13), we have

$$
\begin{aligned}
\|\widetilde{\mathcal{L}}_n(\cdot)\|_r^v &= \|\widetilde{\mathcal{L}}_n(\cdot)\|_r+\sup_{y\neq 0}\frac{\|\widetilde{\mathcal{L}}_n(\cdot+y)+\widetilde{\mathcal{L}}_n(\cdot-y)\|_r}{v(y)}\\
&= O\left(\omega(\frac{1}{n+1})\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}dt\right)+O\left(\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{tv(t)}dt\right)\\
&= \sum_{i=1}^{4}J_i.
\end{aligned}
$$

(3.14)

Now, we write J_1 in terms of J_3 and further J_2, J_3 in terms of J_4.

In view of monotonicity of $v(t)$ for $0<t\leq\pi$, we have

$$\omega(t)=\frac{\omega(t)}{v(t)}.v(t)\leq v(\pi)\frac{\omega(t)}{v(t)}.v(t)=O\left(\frac{\omega(t)}{v(t)}\right).$$

Therefore, we can write

$$J_1=O(J_3) \qquad (3.15)$$

Again by using monotonicity of $v(t)$,

$$
\begin{aligned}
J_2=\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}dt &= \int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{tv(t)}v(t)dt\\
&\leq v(\pi)\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{tv(t)}dt\\
&= O(J_4). \qquad (3.16)
\end{aligned}
$$

Using the fact that $\frac{\omega(t)}{v(t)}$ is positive and non decreasing, we have

$$J_4 = \int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)} dt = \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})} \int_{\frac{1}{n+1}}^{\pi} \frac{dt}{t} \geq \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}. \tag{3.17}$$

Therefore, we can write

$$J_3 = O(J_4). \tag{3.18}$$

Now combining (3.14) and (3.18), we have

$$\|\widetilde{\mathcal{L}}_n(\cdot)\|_r^v = O(J_4) = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)}\right). \tag{3.19}$$

Hence,

$$E_n(\widetilde{f}) = \inf_n \|\widetilde{\mathcal{L}}_n(\cdot)\|_r^v = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)}\right). \tag{3.20}$$

This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2

Following the proof of Theorem 1, we have

$$E_n(\widetilde{f}) = \inf_n \|\widetilde{\mathcal{L}}_n(\cdot)\|_r^v = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{tv(t)} dt\right). \tag{3.21}$$

In Theorem 2, we assumed $\frac{\omega(t)}{tv(t)}$ is positive and decreasing in t. Thus, we have

$$\begin{aligned} E_n(\widetilde{f}) = \inf_n \|\widetilde{\mathcal{L}}_n(\cdot)\|_r^v &= O\left(\frac{\omega(\frac{1}{n+1})}{(n+1)v(\frac{1}{n+1})} \int_{\frac{1}{n+1}}^{\pi} dt\right) \\ &= O\left(\frac{\omega(\frac{1}{n+1})}{(n+1)v(\frac{1}{n+1})} [t]_{\frac{1}{n+1}}^{\pi}\right) \\ &= O\left(\frac{\omega(\frac{1}{n+1})}{(n+1)^2 v(\frac{1}{n+1})} (\pi(n+1)-1)\right). \end{aligned} \tag{3.22}$$

This completes the proof of Theorem 2.

4. Derivations

The following corollaries can be obtained as direct consequences of the main result established in this paper. This indicates the applicability of our findings.

Corollary 4.1. *If we replace $(E,1)(\overline{N},p_n)$ mean by $(E,1)(C,1)$ mean in Theorem 2.1, then the degree of approximation of a function $\widetilde{f} \in Z_r^{(\omega)}$ by $(E,1)(C,1)$ mean*

$$\tau_n^{EC} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{v=0}^{k} s_v \quad [15]$$

of conjugate Fourier series (1.6) is given by

$$E_n(\widetilde{f}) = O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{tv(t)}dt\right). \tag{4.1}$$

Corollary 4.2. *If we replace $(E,1)(\overline{N},p_n)$ mean by $(E,q)(N,p_n,q_n)$ mean in Theorem 2.1, then the degree of approximation of a function $\widetilde{f} \in Z_r^{(\omega)}$ by $(E,q)(N,p_n,q_n)$ mean*

$$\tau_n^{EN} = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{R_k}\sum_{v=0}^{k}p_{k-v}q_v s_v\right\} \quad [12]$$

of conjugate Fourier series (1.6) is given by

$$E_n(\widetilde{f}) = O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{tv(t)}dt\right). \tag{4.2}$$

Corollary 4.3. *If we replace $(E,1)(\overline{N},p_n)$ mean by Euler-Hausdörff mean in Theorem 2.1, then the degree of approximation of a function $\widetilde{f} \in Z_r^{(\omega)}$ by Euler-Hausdörff mean*

$$\tau_n^{EH} = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}h_{k,v}s_v \quad [10]$$

of conjugate Fourier series (1.6) is given by

$$E_n(\widetilde{f}) = O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^2v(t)}dt\right). \tag{4.3}$$

Conclusion

In the present study various types of results concerning the degrees of approximation of periodic functions belonging to the different Lipschitz classes and Zygmund classes of functions are reviewed. The established theorems in this paper is an attempt to study the approximation of signals in the generalized Zygmund class via $(E, 1)(\overline{N}, p_n)$ summability means of conjugate Fourier series, which generalizes several known theorems. Further the result can be extended for other functions belonging to Weighted Zygmund class.

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