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Near ω-continuous multifunctions on bitopological spaces

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Abstract:

In this paper, we introduce and study basic characterizations, several properties of upper (lower) nearly (i, j)-ω-continuous multifunctions on bitopological space.

Keywords: (i, j)-regular open set; ω-open sets; Upper nearly (i, j)-ω-continuous.

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1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This shows that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Generalized open sets play an important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the introduction of various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. A generalization of closed sets, the notion of ω-closed sets has been introduced and studied by Hdeib [8]. Several characterizations and properties of ω-closed sets has been provided in [2, 4, 5, 6, 8, 9]. In this paper, we introduce and study upper (lower) nearly (i, j)-ω-continuous multifunctions on bitopological space.

2. Preliminaries

Throughout this paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote the bitopological spaces in which no separation axioms are assumed unless explicitly stated. Bitopological spaces and its different properties have been investigated by Triparthy and Sarma ([11], [12], [14]), Sarma and Triparthy [15], Tripathy and Acharjee [13], Acharjee and Tripathy [1], Tripathy and Debnath [16], and others. For a subset A of (X, τ), $iCl(A)$ (respectively $iInt(A)$) denote the closure of A with respect to τ_i (respectively the interior of A with respect to τ_i). A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. The set A is said to be ω-closed [8] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ. The intersection of all ω-closed sets containing A is called the ω-closure[8] of A and is denoted by $\omega Cl(A)$. For each $x \in X$, the family of all ω-open sets containing x is denoted by $\omega O(X, x)$. The family of all ω-open sets of X is denoted by $\omega O(X)$. A Multifunction $F : X \to Y$ from a topological space X to a topological space Y is a point to set correspondence and is assumed that

$F(x) \neq \emptyset$ for all $x \in X$. We denote the upper and lower inverse of a subset V of Y by $F^{+}(V)$ and $F^{-}(V)$ (respectively; $F^{+}(V) = \{x \in X : F(x) \subseteq V\}$ and $F^{-}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$.

3. Upper (lower) nearly (i,j)-ω-continuous multifunctions

Definition 3.1. *[10] A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i,j)-regular open if $A = iInt(jCl(A))$. The complement of an (i,j)-regular open set is called (i,j)-regular closed set.*

Definition 3.2. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i,j)-N-closed [10] if every cover of A by (i,j)-regular open sets of X has a finite subcover.*

Definition 3.3. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

1. *upper nearly (i,j)-ω-continuous at a point $x \in X$ if for each σ_i-open set V containing $F(x)$ and having (i,j)-N-closed complement, there exists a τ_i-ω-open set U containing x such that $U \subset F^{+}(V)$.*

2. *lower nearly (i,j)-ω-continuous at a point $x \in X$ if for each σ_i-open set V of Y meeting $F(x)$ and having (i,j)-N-closed complement, there exists a τ_i-ω-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *(i,j)-upper (resp. (i,j)-lower) nearly ω-continuous on X if it has this property at every point of X.*

Example 3.4. *Consider the set $X = Y = \{a, b, c, d\}$ with topologies $\tau_1 = \sigma_1 = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\tau_2 = \sigma_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$. Define the multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ as follows: $F(a) = \{c\}$, $F(b) = \{a, b\}$, $F(c) = \{d\}$ and $F(d) = \{a, b\}$. It is easy to see that the set $\{b, c\}$ is (i,j)-regular open and the multifunction F is (i,j)-upper (resp. (i,j)-lower) nearly ω-continuous on X.*

Theorem 3.5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

1. *F is upper nearly (i,j)-ω-continuous.*

2. *$F^{+}(V)$ is τ_i-ω-open for each σ_i-open set V of Y having (i,j)-N-closed complement.*

3. *$F^-(K)$ is τ_i-ω-closed for every (i,j)-N-closed and σ_i-closed set K of Y.*

4. *$i\omega Cl(F^-(B)) \subset F^-(iCl(B))$ for every subset B of Y having (i,j)-N-closed σ_i-closure.*

5. *$F^+(iInt(B)) \subset i\omega Int(F^+(B))$ for every subset B of Y such that $Y \backslash iInt(B)$ is (i,j)-N-closed.*

Proof. (1)⇒(2): Let $x \in F^+(V)$ and V be a σ_i-open set of Y having (i,j)-N-closed complement. From (1), there exists a τ_i-ω-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that
$F^+(V) = \underset{x \in F^+(V)}{\cup} U_x$. Since arbitrary union of τ_i-ω-open sets is τ_i-ω-open, $F^+(V)$ is τ_i-ω-open in (X, τ_1, τ_2).
(2)⇒(3): Let K be any (i,j)-N-closed and σ_i-closed set of Y. Then by (2), $F^+(Y \backslash K) = X \backslash F^-(K)$ is an τ_i-ω-open set. Then it is obtained that $F^-(K)$ is an τ_i-ω-closed set.
(3)⇒(4): Let B be any subset of Y having (i,j)-N-closed σ_i-closure. By (3), We have $F^-(B) \subset F^-(iCl(B)) = i\omega Cl(F^-(iCl(B)))$. Hence $i\omega Cl(F^-(B)) \subset i\omega Cl(F^-(iCl(B))) = F^-(iCl(B))$.
(4)⇒(5): Let B be a subset of Y such that $Y \backslash iInt(B)$ is (i,j)-N-closed. Then by (4), we have $X \backslash i\omega Int(F^+(B)) = i\omega Cl(X \backslash F^+(B)) = i\omega Cl(F^-(Y \backslash B)) \subset F^-(iCl(Y \backslash B)) \subset X \backslash F^+(iInt(B))$. Therefore, we get $F^+(iInt(B)) \subset i\omega Int(F^+(B))$.
(5)⇒(1): Let $x \in X$ and V be any σ_i-open set of Y containing $F(x)$ and having (i,j)-N-closed complement. Then by (5). $x \in F^+(V) = F^+(iInt(V)) \subset i\omega Int(F^+(V))$. There exists a τ_i-ω-open set U containing x such that $U \subset F^+(V)$; and hence $F(U) \subset V$. This shows that F is upper nearly (i,j)-ω-continuous.

Theorem 3.6. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

1. *F is lower nearly (i,j)-ω-continuous.*

2. *$F^-(V)$ is τ_i-ω-open for each σ_i-open set V of Y having (i,j)-N-closed complement.*

3. *$F^+(K)$ is τ_i-ω-closed for every (i,j)-N-closed and σ_i-closed set K of Y.*

4. $i\omega Cl(F^+(B)) \subset F^+(iCl(B))$ *for every subset* B *of* Y *having* (i,j)*-*N*-closed* σ_i*-closure.*

5. $F^-(iInt(B)) \subset i\omega Int(F^-(B))$ *for every subset* B *of* Y *such that* $Y\backslash iInt(B)$ *is* (i,j)*-*N*-closed.*

Proof. The proof is similar to that of Theorem 3.5

Corollary 3.7. *A multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is upper nearly* (i,j)*-*ω*-continuous (resp. lower nearly* (i,j)*-*ω*-continuous) if* $F^-(K)$ *is* τ_i*-*ω*-closed (resp.* $F^+(K)$ *is* τ_i*-*ω*-closed) for every* (i,j)*-*N*-closed set* K *of* Y*.*

Proof. Let G be any σ_i-open set of Y having (i,j)-N-closed complement. Then $Y\backslash G$ is an (i,j)-N-closed. By the hypothesis, $X\backslash F^+(G) = F^-(Y\backslash G) = i\omega Int(F^-(Y\backslash G)) = i\omega Cl(X\backslash F^+(G)) = X\backslash i\omega Int(F^+(G))$ and hence, $F^+(G) = i\omega Int(F^+(G))$. It follows from Theorem 3.5 that F is upper nearly (i,j)-ω-continuous. The proof of lower nearly (i,j)-ω-continuity can be established similarly.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we define $D^+_{n(i,j)\omega}(F)$ and $D^-_{n(i,j)\omega}(F)$ as follows:
$D^+_{n(i,j)\omega}(F) = \{x \in X : F$ is not upper nearly (i,j)-ω-continuous at $x\}$.
$D^-_{n(i,j)\omega}(F) = \{x \in X : F$ is not lower nearly (i,j)-ω-continuous at $x\}$.

Theorem 3.8. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties hold:*

$$\begin{aligned} D^+_{n(i,j)\omega} &= \bigcup_{G\in\sigma NC}\{F^+(G)\backslash i\omega Int(F^+(G))\} \\ &= \bigcup_{B\in iNC}\{F^+(iInt(B))\backslash i\omega Int(F^+(B))\} \\ &= \bigcup_{B\in NC}\{i\omega Cl(F^-(B))\backslash F^-(iCl(B))\} \\ &= \bigcup_{H\in\mathcal{F}}\{i\omega Cl(F^-(H))\backslash F^-(H)\}. \end{aligned}$$

Where
σNC *is the family of all* σ_i*-open sets of* Y *having* (i,j)*-*N*-closed complement,*
iNC *is the family of all subsets* B *of* Y *such that* $Y\backslash iInt(B)$ *is* (i,j)*-*N*-closed,*
NC *is the family of all subsets* B *of* Y *having the* (i,j)*-*N*-closed* σ_i*-closure,*
$\mathcal{F}$ *is the family of all* σ_i*-closed and* (i,j)*-*N*-closed sets of* (Y, σ_1, σ_2)*.*

Proof. We shall establish only the first equality and the last equality since the proofs of other are similar to the first. Let $x \in D^+_{n(i,j)\omega}(F)$. Then there exists an σ_i-open set V of Y containing $F(x)$ and having (i,j)-N-closed complement such that $x \in i\omega Int(F^+(V))$. Therefore, we have $x \in F^+(V)\backslash i\omega Int(F^+(V)) \subset \underset{G\in\sigma NC}{\cup}\{F^+(G)\backslash i\omega Int(F^+(G))\}$. Conversely, let $x \in \underset{G\in\sigma NC}{\cup}\{F^+(G)\backslash i\omega Int(F^+(G))\}$. Then there exists a σ_i-open set V of Y having (i,j)-N-closed complement such that $x \in F^+(V)\backslash i\omega Int(F^+(V))$. Hence $x \in D^+_{n(i,j)}(F)$. We prove the last equality. $\underset{H\in\mathcal{F}}{\cup}\{i\omega Cl(F^-(H))\backslash F^-(H)\} \subset \underset{B\in NC}{\cup}\{i\omega Cl(F^-(B))\backslash F^-(iCl(B))\} = D^+_{n(i,j)\omega}(F)$. Conversely, we have $D^+_{n(i,j)\omega}(F) =$
$\underset{B\in NC}{\cup}\{i\omega Cl(F^-(B))\backslash F^-(iCl(B))\} \subset \underset{H\in\mathcal{F}}{\cup}\{i\omega Cl(F^-(H))\backslash F^-(H)\}$

Theorem 3.9. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties hold:*

$$\begin{aligned} D^-_{n(i,j)\omega} &= \underset{G\in\sigma NC}{\cup}\{F^-(G)\backslash i\omega Int(F^-(G))\} \\ &= \underset{B\in iNC}{\cup}\{F^-(iInt(B))\backslash i\omega Int(F^-(B))\} \\ &= \underset{B\in NC}{\cup}\{i\omega Cl(F^+(B))\backslash F^+(iCl(B))\} \\ &= \underset{H\in\mathcal{F}}{\cup}\{i\omega Cl(F^+(H))\backslash F^+(H)\}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 3.8

Definition 3.10. *Let (X,τ) be a topological space and A a subset of X. The ω-frontier of A, ω-$Fr(A)$, is defined by ω-$Fr(A) = \omega Cl(A) \cap \omega Cl(X\backslash A) = \omega Cl(A)\backslash \omega Int(A)$.*

Theorem 3.11. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, $D^+_{n(i,j)\omega}(F)$ (resp. $D^-_{n(i,j)\omega}(F)$) is identical with the union of ω-frontiers of the (i,j)-upper (resp. (i,j)-lower) inverse images of σ_i-open sets containing (resp. meeting) $F(x)$ and having (i,j)-N-closed complement.*

Proof. We shall established the first case since the proof of the second can be established similar.
Let $x \in D^+_{n(i,j)\omega}(F)$. Then, there exists a σ_i-open set V of Y containing $F(x)$ and having (i,j)-N-closed complement such that $U \cap (X\backslash F^+(V)) \neq \emptyset$ for every (i,j)-open set U containing x. Then we have $x \in i\omega Cl(X\backslash F^+(V))$.

On the other hand, since $x \in F^+(V) \subset i\omega Cl(F^+(V))$ and hence $x \in i\omega\text{-}Fr(F^+(V))$. Conversely, suppose that F is upper nearly (i,j)-ω-continuous at $x \in X$. Then for any σ_i-open set V of Y containing $F(x)$ and having (i,j)-N-closed complement, there exists $U \in \tau_i\text{-}\omega O(X)$ containing x such that $F(U) \subset V$; hence $x \in U \subset F^+(V)$. Therefore, $x \in U \subset i\omega Int(F^+(V))$. This contradicts to the fact that $x \in i\omega\text{-}Fr(F^+(V))$.

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