



Hyers-Ulam stability of n th order linear differential equation

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Abstract:

In this paper, we investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the homogeneous linear differential equation of n th order with initial and boundary conditions by using Taylor's Series formula.

Keywords: Hyers-Ulam stability; Hyers-Ulam-Rassias stability; Initial and boundary conditions; Taylor's series method.

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1. Introduction

The study of stability problem for various functional equations originated from a famous talk of S. M. Ulam [1]. In 1940, Ulam [1] posed a problem concerning the stability of functional equations: "Give Conditions in order for a linear function near an approximately linear function to exist." Since then, this question has attracted the attention of many researchers. Note that first solution to this question was given by Hyers [2] in 1941. He made a significant breakthrough, when he gave an affirmative answer to the Ulam's problem for additive functions defined on Banach Spaces. Thereafter, the result by Hyers [2] was generalized by Rassias [3], Aoki [4] and Bourgin [5]. After then, many Mathematicians have extended the Ulam's problem to other functional equations and generalized the Hyers results in various directions (see [6, 7, 8]).

Definition of Hyers-Ulam stability have applicable significance since it means that if one is studying the Hyers-Ulam stable system then one does not have to reach the exact solution. (Which usually is quite difficult or time consuming). This is quite useful in many applications, for example Fluid Dynamics, Numerical Analysis, Optimization, Biology, and Economics etc., where finding the exact solution is quite difficult. It also helps if the stochastic effects are small, to use deterministic model to approximate a stochastic one. It is very important to note that there are many other applications for Hyers-Ulam stability in other areas like, nonlinear analysis problems including partial differential equation and integral equations.

The theory of stability is an important branch of the qualitative theory of differential equations. The generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\phi(f, x, x', x'', \dots x^{(n)}) = 0$ has the Hyers - Ulam stability if for a given $\epsilon > 0$ and a function x such that $|\phi(f, x, x', x'', \dots x^{(n)})| \leq \epsilon$, there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \rightarrow 0} K(\epsilon) = 0$. Obloza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equation [9]. Thereafter, Alsina and Ger [10] published their papers, which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The result obtained by Alsina and Ger was generalized by Takahasi et. al., [11] to the case of the complex Banach Space valued differential equation $y'(t) = \lambda y(t)$, (see also [12, 13]).

Now a days, the Hyers-Ulam stability of ordinary differential equations has been investigated (see [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26,

29, 30, 31, 32, 33]) and the investigation is going on. In this paper, our main aim is to investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the n th order homogeneous linear differential equation

$$x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \dots + a_2(t)x''(t) + a_1(t)x'(t) + (a_0(t) - p(t))x(t) = 0 \quad (1.1)$$

with initial conditions

$$(1.2) \quad x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0,$$

and with boundary conditions

$$(1.3) \quad x(a) = x(b) = 0,$$

for all $t \in I$, $x(t) \in C^n(I)$, $p(t) \in C^0(I)$ and $a_i(t) \in C(I)$, $i = 0, 1, 2, \dots, n-1$. Where $I = [a, b] \subseteq \mathbf{R}$, $p > 0$ and $p(t)$ is a bounded for all sufficiently large t in \mathbf{R} .

2. Preliminaries

Firstly, we give the definitions of Hyers-Ulam stability and Hyers-Ulam-Rassias stability for the differential equation (1.1) with initial and boundary conditions (1.2) and (1.3) respectively.

Definition 2.1. We say that the differential equation (1.1) has the Hyers-Ulam stability with initial conditions (1.2), if there exists a constant $K > 0$ such that for every $\epsilon > 0$, $x \in C^n(I)$, if

$$\left| x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \dots + a_2(t)x''(t) + a_1(t)x'(t) + (a_0(t) - p(t))x(t) \right| \leq \epsilon,$$

with $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$, then there exists some $y(t) \in C^n(I)$ satisfying the differential equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_2(t)y''(t) + a_1(t)y'(t) + (a_0(t) - p(t))y(t) = 0,$$

with $y(a) = y'(a) = y''(a) = \dots = y^{(n-1)}(a) = 0$, such that $|x(t) - y(t)| \leq K\epsilon$.

We call such K as a Hyers-Ulam stability constant for the differential equation (1.1) with (1.2).

Definition 2.2. We say that the differential equation (1.1) has the Hyers-Ulam stability with boundary conditions (1.3), if there exists a constant $K > 0$ such that for every $\epsilon > 0$, $x \in C^n(I)$, if

$$\left| x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) \right| \leq \epsilon,$$

with $x(a) = x(b) = 0$, then there exists some $y(t) \in C^n(I)$ satisfying the differentail equation

$$y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \dots + a_2(t) y''(t) + a_1(t) y'(t) + (a_0(t) - p(t)) y(t) = 0,$$

with $y(a) = y(b) = 0$, such that $|x(t) - y(t)| \leq K \epsilon$.

We call such K as a Hyers-Ulam stability constant for the differential equation (1.1) with (1.3).

Definition 2.3. We say that the differential equation (1.1) has the Hyers-Ulam-Rassias stability with initial conditions (1.2), if there exists $\phi \in C(I, R_+)$ such that for every $\epsilon > 0$, $x \in C^n(I)$, if

$$\left| x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) \right| \leq \epsilon \phi(t),$$

with $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(t) = 0$, then there exists some $y(t) \in C^n(I)$ satisfying the differentail equation

$$y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \dots + a_2(t) y''(t) + a_1(t) y'(t) + (a_0(t) - p(t)) y(t) = 0,$$

with $y(a) = y'(a) = y''(a) = \dots = y^{(n-1)}(t) = 0$, such that $|x(t) - y(t)| \leq \phi(t) K(\epsilon)$.

We call such K as a Hyers-Ulam-Rassias stability constant for the differential equation (1.1) with (1.2).

Definition 2.4. We say that the differential equation (1.1) has the Hyers-Ulam-Rassias stability with boundary conditions (1.3), if there exists $\theta_\phi \in C(I, R_+)$ such that for every $\epsilon > 0$, $x \in C^n(I)$, if

$$\left| x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) \right| \leq \epsilon \phi(t),$$

with $x(a) = x(b) = 0$, then there exists some $y(t) \in C^n(I)$ satisfying the differentail equation

$$y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \dots + a_2(t) y''(t) + a_1(t) y'(t) + (a_0(t) - p(t)) y(t) = 0,$$

with $y(a) = y(b) = 0$, such that $|x(t) - y(t)| \leq \phi(t) K(\epsilon)$.

We call such K as a Hyers-Ulam-Rassias stability constant for the differential equation (1.1) with (1.3).

3. Hyers-Ulam Stability

In this section, we investigate the Hyers-Ulam stability of the differential equation (1.1) with (1.2) and (1.3). Now, we prove the Hyers-Ulam stability of the differential equation (1.1) with (1.2).

Theorem 3.1. *If $\sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| < \frac{n!}{(b-a)^n}$, then the differential equation (1.1) has the Hyers-Ulam stability with initial conditions (1.2).*

Proof. For every $\epsilon > 0$ and $x \in C^n(I)$ satisfies $|x^{(n-1)}(t)| < \dots < |x''(t)| < |x'(t)| < |x(t)|$ if $|x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t)| \leq \epsilon$,

with $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$. Then by Taylor's formula, we have

$$(3.1) \quad x(t) = x(a) + x'(a)(t-a) + \frac{x''(a)}{2!}(t-a)^2 + \dots + \frac{x^{(n)}(\xi)}{n!}(t-a)^n$$

Since we have $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$, then (4.1) becomes

$$x(t) = \frac{x^{(n)}(\xi)}{n!}(t-a)^n.$$

So,

$$\begin{aligned} |x(t)| &= \left| \frac{x^{(n)}(\xi)}{n!}(t-a)^n \right| \\ \max |x(t)| &\leq \frac{(b-a)^n}{n!} \max |x^{(n)}(t)| \\ \max |x(t)| &\leq \frac{(b-a)^n}{n!} \left\{ \max \left| x^{(n)}(t) + \sum_{i=1}^{n-1} a_i(t) x^{(i)}(t) + (a_0(t) - p(t)) x(t) \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^{n-1} a_i(t) x^{(i)}(t) - (a_0(t) - p(t)) x(t) \right| \right\} \\ \max |x(t)| &\leq \frac{(b-a)^n}{n!} \left\{ \epsilon + \left\{ \sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| \right\} \max |x(t)| \right\} \\ \max |x(t)| &\leq \frac{(b-a)^n}{n!} \epsilon + \frac{(b-a)^n}{n!} \lambda \max |x(t)| \end{aligned}$$

where $\lambda = \sum_{i=1}^{n-1} \max |a_i(t)| + \max |a_0(t) - p(t)|$. Now, let us choose

$\delta = \frac{(b-a)^n}{n!} \lambda$. Hence we get,

$$\max |x(t)| \leq \frac{(b-a)^n}{n! (1-\delta)} \epsilon,$$

choose K as $\frac{(b-a)^n}{n! (1-\delta)}$, hence we have $\max |x(t)| \leq K \epsilon$. Obviously, we have $y_0(t) = 0$ is a solution of

$$x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) = 0$$

with initial conditions $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$, such that $|x(t) - y_0(t)| \leq K\epsilon$.

Hence, by the virtue of Definition 2.1, the differential equation (1.1) has the Hyers-Ulam stability with initial conditions (1.2).

Now, we investigate the Hyers-Ulam stability of the differential equation (1.1) with boundary conditions (1.3).

Theorem 3.2. If $\sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| < \frac{n 2^n n!}{(b-a)^n}$, then the differential equation (1.1) has the Hyers-Ulam stability with boundary conditions (1.3).

Proof. For every $\epsilon > 0$ and $x \in C^n(I)$ satisfying

$$\left| x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) \right| \leq \epsilon,$$

with $x(a) = x(b) = 0$. Let $M = \max \left\{ |x^{(i)}(t)| : t \in [a, b] \right\}$, $i = 0, 1, 2, \dots, n-1$. Since $x(a) = x(b) = 0$, there exists $t_0 \in (a, b)$ such that $|x(t_0)| = M$. Then by Taylor's formula, we have

$$(3.2) \quad x(a) = x(t_0) + x'(t_0)(t_0 - a) + \frac{x''(t_0)}{2!}(t_0 - a)^2 + \dots + \frac{x^{(n)}(\xi)}{n!}(t_0 - a)^n$$

$$(3.3) \quad x(b) = x(t_0) + x'(t_0)(b - t_0) + \frac{x''(t_0)}{2!}(b - t_0)^2 + \dots + \frac{x^{(n)}(\zeta)}{n!}(b - t_0)^n$$

Since we have $x(a) = 0$, (4.2) becomes

$$\frac{x^{(n)}(\xi)}{n!}(t_0 - a)^n + \dots + \frac{x''(t_0)}{2!}(t_0 - a)^2 + x'(t_0)(t_0 - a) + x(t_0) = 0,$$

thus $|x^{(n)}(\xi)| \geq \frac{n! M}{(t_0 - a)^n}$. Now, let $t_0 \in \left(a, \frac{a+b}{2}\right]$, we have

$$\frac{n! M}{(t_0 - a)^n} \geq \frac{n! M}{(b-a)^n} = \frac{n! 2^n M}{(b-a)^n} = \frac{n! 2^n}{(b-a)^n} \max |x(t)|.$$

Since we have $x(b) = 0$, (4.3) becomes

$$\frac{x^{(n)}(\zeta)}{n!}(b - t_0)^n + \dots + \frac{x''(t_0)}{2!}(b - t_0)^2 + x'(t_0)(b - t_0) + x(t_0) = 0.$$

Thus we have $|x^{(n)}(\zeta)| \geq \frac{n! M}{(t_0 - b)^n}$. Now, let $t_0 \in \left[\frac{a+b}{2}, b\right)$, we have

$$\frac{n! M}{(b - t_0)^n} \geq \frac{n! M}{(b-a)^n} = \frac{n! 2^n M}{(b-a)^n} = \frac{n! 2^n}{(b-a)^n} \max |x(t)|$$

Hence, we obtain $\max |x(t)| \leq \frac{(b-a)^n}{n! 2^n} \max |x^{(n)}(t)|$. Thus we have

$$\begin{aligned} \max |x(t)| &\leq \frac{(b-a)^n}{n! 2^n} \left\{ \max \left| x^{(n)}(t) + \sum_{i=1}^{n-1} a_i(t)x^{(i)}(t) + (a_0(t) - p(t)) x(t) \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^{n-1} a_i(t)x^{(i)}(t) - (a_0(t) - p(t)) x(t) \right| \right\} \\ \max |x(t)| &\leq \frac{(b-a)^n}{n! 2^n} \left\{ \epsilon + \left\{ \sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| \right\} \max |x(t)| \right\} \\ \max |x(t)| &\leq \frac{(b-a)^n}{n! 2^n} \epsilon + \frac{(b-a)^n}{n! 2^n} \lambda \max |x(t)| \end{aligned}$$

where $\lambda = \sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))|$. Now, let us choose $\delta =$

$\frac{(b-a)^n}{n! 2^n} \lambda$. Hence we arrive that,

$$\max |x(t)| \leq \frac{(b-a)^n}{n! 2^n (1-\delta)} \epsilon.$$

Choose

$$K = \frac{(b-a)^n}{n! 2^n (1-\delta)}.$$

Hence we have $\max |x(t)| \leq K \epsilon$. Obviously, we have $y_0(t) = 0$ is a solution of

$$x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \dots + a_2(t)x''(t) + a_1(t)x'(t) + (a_0(t) - p(t))x(t) = 0$$

with boundary conditions $x(a) = x(b) = 0$, such that

$$|x(t) - y_0(t)| \leq K\epsilon.$$

Hence, by the virtue of Definition 2.2, the differential equation (1.1) has the Hyers-Ulam stability with boundary conditions (1.3).

4. Hyers-Ulam-Rassias stability

The following Theorems are shows the Hyers-Ulam-Rassias stability of the differential equation (1.1) with (1.2) and (1.3).

Theorem 4.1. *If $\sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| < \frac{n!}{(b-a)^n}$, then the differential equation (1.1) has the Hyers-Ulam-Rassias stability with the initial conditions (1.2).*

Proof. For every $\epsilon > 0$ and $x \in C^n(I)$ satisfies

$$\left| x^{(n-1)}(t) \right| < \dots < |x''(t)| < |x'(t)| < |x(t)|, \text{ there exists a } \phi : I \rightarrow [0, \infty) \text{ if}$$

$$\left| x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \dots + a_2(t)x''(t) + a_1(t)x'(t) + (a_0(t) - p(t))x(t) \right| \leq \phi(t)\epsilon,$$

with $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$. Then by Taylor's formula, we have

$$(4.1) \quad x(t) = x(a) + x'(a)(t-a) + \frac{x''(a)}{2!}(t-a)^2 + \dots + \frac{x^{(n)}(\xi)}{n!}(t-a)^n$$

Since we have $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$, then (4.1) becomes

$$x(t) = \frac{x^{(n)}(\xi)}{n!}(t-a)^n.$$

So,

$$\begin{aligned}
 |x(t)| &= \left| \frac{x^{(n)}(\xi)}{n!} (t-a)^n \right| \\
 \max |x(t)| &\leq \frac{(b-a)^n}{n!} \max |x^{(n)}(t)| \\
 \max |x(t)| &\leq \frac{(b-a)^n}{n!} \left\{ \max \left| x^{(n)}(t) + \sum_{i=1}^{n-1} a_i(t) x^{(i)}(t) + (a_0(t) - p(t)) x(t) \right. \right. \\
 &\quad \left. \left. - \sum_{i=1}^{n-1} a_i(t) x^{(i)}(t) - (a_0(t) - p(t)) x(t) \right| \right\} \\
 \max |x(t)| &\leq \frac{(b-a)^n}{n!} \left\{ \phi(t) \epsilon + \left\{ \sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| \right\} \max |x(t)| \right\} \\
 \max |x(t)| &\leq \frac{(b-a)^n}{n!} \phi(t) \epsilon + \frac{(b-a)^n}{n!} \lambda \max |x(t)|
 \end{aligned}$$

where $\lambda = \sum_{i=1}^{n-1} \max |a_i(t)| + \max |a_0(t) - p(t)|$. Now, let us choose $\delta = \frac{(b-a)^n}{n!} \lambda$. Hence we get, $\max |x(t)| \leq \frac{(b-a)^n}{n! (1-\delta)} \phi(t) \epsilon$, choose K as $\frac{(b-a)^n}{n! (1-\delta)}$, hence we have $\max |x(t)| \leq K \phi(t) \epsilon$. Obviously, we have $y_0(t) = 0$ is a solution of

$$x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) = 0$$

with initial conditions $x(a) = x'(a) = x''(a) = \dots = x^{(n-1)}(a) = 0$, such that

$$|x(t) - y_0(t)| \leq K \phi(t) \epsilon.$$

Hence, by the virtue of Definition 2.3, the differential equation (1.1) has the Hyers-Ulam-Rassias stability with initial conditions (1.2).

Now, we prove the Hyers-Ulam-Rassias stability of the linear differential equation (1.1) with boundary conditions (1.3).

Theorem 4.2. If $\sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| < \frac{n 2^n n!}{(b-a)^n}$, then the differential equation (1.1) has the Hyers-Ulam-Rassias stability with the boundary conditions (1.3).

Proof. For every $\epsilon > 0$ and $x \in C^n(I)$ if there exists $\phi : I \rightarrow [0, \infty)$ satisfies the inequality

$$\left| x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) \right| \leq \phi(t) \epsilon,$$

with $x(a) = x(b) = 0$. Let $M = \max \left\{ |x^{(i)}(t)| : t \in [a, b], i = 0, 1, 2, \dots, n-1 \right\}$.

Since $x(a) = x(b) = 0$, there exists $t_0 \in (a, b)$ such that $|x(t_0)| = M$. Then by Taylor's formula, we have

$$(4.2) \quad x(a) = x(t_0) + x'(t_0)(t_0 - a) + \frac{x''(t_0)}{2!}(t_0 - a)^2 + \dots + \frac{x^{(n)}(\xi)}{n!}(t_0 - a)^n$$

$$(4.3) \quad x(b) = x(t_0) + x'(t_0)(b - t_0) + \frac{x''(t_0)}{2!}(b - t_0)^2 + \dots + \frac{x^{(n)}(\zeta)}{n!}(b - t_0)^n$$

Since we have $x(a) = 0$, (4.2) becomes

$$\frac{x^{(n)}(\xi)}{n!}(t_0 - a)^n + \dots + \frac{x''(t_0)}{2!}(t_0 - a)^2 + x'(t_0)(t_0 - a) + x(t_0) = 0,$$

thus $|x^{(n)}(\xi)| \geq \frac{n n! M}{(t_0 - a)^n}$. Now, let $t_0 \in \left(a, \frac{a+b}{2}\right]$, we have

$$\frac{n n! M}{(t_0 - a)^n} \geq \frac{n n! M}{\frac{(b-a)^n}{2^n}} = \frac{n n! 2^n M}{(b-a)^n} = \frac{n n! 2^n}{(b-a)^n} \max |x(t)|.$$

Since we have $x(b) = 0$, (4.3) becomes

$$\frac{x^{(n)}(\zeta)}{n!}(b - t_0)^n + \dots + \frac{x''(t_0)}{2!}(b - t_0)^2 + x'(t_0)(b - t_0) + x(t_0) = 0.$$

Thus we have $|x^{(n)}(\zeta)| \geq \frac{n n! M}{(t_0 - b)^n}$. Now, let $t_0 \in \left[\frac{a+b}{2}, b\right)$, we have

$$\frac{n n! M}{(b - t_0)^n} \geq \frac{n n! M}{\frac{(b-a)^n}{2^n}} = \frac{n n! 2^n M}{(b-a)^n} = \frac{n n! 2^n}{(b-a)^n} \max |x(t)|$$

Hence, we obtain $\max |x(t)| \leq \frac{(b-a)^n}{n n! 2^n} \max |x^{(n)}(t)|$. Thus we have

$$\begin{aligned} \max |x(t)| &\leq \frac{(b-a)^n}{n n! 2^n} \left\{ \max \left| x^{(n)}(t) + \sum_{i=1}^{n-1} a_i(t)x^{(i)}(t) + (a_0(t) - p(t)) x(t) \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^{n-1} a_i(t)x^{(i)}(t) - (a_0(t) - p(t)) x(t) \right| \right\} \\ \max |x(t)| &\leq \frac{(b-a)^n}{n n! 2^n} \left\{ \phi(t) \epsilon + \left\{ \sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))| \right\} \max |x(t)| \right\} \\ \max |x(t)| &\leq \frac{(b-a)^n}{n n! 2^n} \phi(t) \epsilon + \frac{(b-a)^n}{n n! 2^n} \lambda \max |x(t)| \end{aligned}$$

where $\lambda = \sum_{i=1}^{n-1} \max |a_i(t)| + \max |(a_0(t) - p(t))|$. Now, let us choose $\delta = \frac{(b-a)^n}{n! 2^n} \lambda$. Hence we arrive that,

$$\max |x(t)| \leq \frac{(b-a)^n}{n! 2^n (1-\delta)} \phi(t) \epsilon,$$

choose K as $\frac{(b-a)^n}{n! 2^n (1-\delta)}$. Hence we have $\max |x(t)| \leq K \phi(t) \epsilon$. Obviously, we have $y_0(t) = 0$ is a solution of

$$x^{(n)}(t) + a_{n-1}(t) x^{(n-1)}(t) + \dots + a_2(t) x''(t) + a_1(t) x'(t) + (a_0(t) - p(t)) x(t) = 0$$

with boundary conditions $x(a) = x(b) = 0$, such that

$$|x(t) - y_0(t)| \leq K \phi(t) \epsilon.$$

Hence, by the virtue of Definition 2.4, the differential equation (1.1) has the Hyers-Ulam-Rassias stability with boundary conditions (1.3).

5. Conclusion

The definition has studied in this work has applicable significance since it means that if we studying the Hyers-Ulam stability of a system, then we does not have to reach the exact solution (which usually is quite difficult or time consuming), all what is required is to get a function, that is a close exact solution. Therefore Hyers-Ulam stability guarantees that there is a closed exact solution of the system under study. That is, we proved the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the homogeneous linear differential equation of n th order with initial and boundary conditions by using Taylor's Series formula. It is very useful to readers to get the approximate solution of these kind of differential equations. Researchers are still on going on the Hyers-Ulam stability of first, second order and higher order homogeneous and non-homogeneous differential equations, in partial differential equations and integral equations.

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