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Controllability of affine systems on free Nilpotent Lie groups $G_{m,r}$

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Abstract:

Controllability properties of affine control systems on free nilpotent Lie groups are examined and controllability of affine systems on thiskind of Lie groups are characterized by the help of their associated bilinear parts. In order to show this, an automorphism in the algebra level is found, authomosrpism orbit of the system is calculated and its properties are studied.

Keywords: Controllability; Affine algebra; Automorphism; Derivation; Free nilpotent Lie group.

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1. Introduction

The subject matter of this article is to study controllability of affine control systems on free nilpotent Lie groups using their Lie algebra properties via their bilinear parts. Control systems are important from their application point of views. In this work, we study controllability problem which is one of the most important classical problem in the area. By controllability, we mean to reach all points of the state space from an initial point by using only positive time. In [1,2], authors study controllability of affine control systems on Euclidean space of finite dimension and on Generalized Heisenberg Lie groups, respectively, and they use bilinear parts of the affine systems in order to verify controllability. In this work, we generalize this approach to affine control systems on free nilpotent Lie groups which is the larger case.

A control system Σ on a connected Lie group G is a couple together with G and a dynamic D which consists of smooth vector fields defined on G parametrized by the time t. If these vector fields are considered as a sum of invariant vector fields and derivations from the derivation algebra $Der(L(G))$, then we have affine control systems. In the next section, we will explain affine control systems on Lie groups.

This work consists of three sections. In the second section, affine control systems on Lie groups are given elaborately and in the third section, we study free nilpotent Lie algebra and Lie groups with their relations and construct an automorphism for free nilpotent Lie algebra which is necessary for a relation between the affine control system and its bilinear part. We present the automorphism orbit $Aut(G_{m,r})-$ orbit of the affine system on the free nilpotent Lie group $G_{m,r}$ and show that the automorphism orbit $Aut(G_{m,r})-$ orbit is dense. By restricting the all system to $Aut(G_{m,r})-$ orbit, we characterize controllability of the affine system on $G_{m,r}$.

2. Affine Control System on Lie Groups

First affine control systems on Euclidean spaces of finite dimension have been introduced, [3], and affine control systems on Lie groups have followed similarly. Let us consider a connected Lie group G with its Lie algebra $L(G)$ and consider their derivation algebra $Der(L(G))$ which is defined by

$$Der(L(G)) = \{D \in End(L(G)) | D[X,Y] = [D(X),Y]+[X,D(Y)], \forall X,Y \in L(G)\}, [2].$$

Derivations are special endomorphisms for which they allow to construct affine vector fields together with invariant vector fields of the Lie group G.

A general affine control system Σ on a Lie group G is defined by the following data

$$\dot{g} = (D + X)_g + \sum_{j=1}^{d} u_j (D^j + Y^j)_g$$

where $X, Y^1, \cdots, Y^d \in L(G); D, D^1, \cdots, D^d \in Der(L(G))$, u is the control which is an element of U the class of piecewise constant real-valued functions and $g \in G$, [2].

Affine Lie algebra $af(G)$ as a vector space, consists of vector fields of the form $X + D$ and Lie group $Af(G)$ of $af(G)$ is a semi-direct product of G and its set of automorphisms $Aut(G)$. $Aut(G)$ is a closed Lie subgroup of G.

For affine Lie algebra $af(G)$, Lie bracket operation is defined by

$$[(D_1, X_1), (D_2, X_2)] = ([D_1, D_2], D_1X_2 - D_2X_1 + [X_1, X_2]), [2].$$

In general affine control systems are richer class of systems than the other control systems. If we consider control systems on Lie groups and if G is an abelian Lie group, where each Lie brackets of Lie algebra is null, then affine control system on G is transformed to linear control system. Dynamic of linear control systems on Lie groups consist of vector fields which has the form $D + uX$, where D is a derivation which are the inner ones and X belongs to the Lie algebra $L(G)$ of the Lie group G and u is a piecewise constant real-valued function which is the control of the system.

If the vector fields $X, Y^1, Y^2, \cdots, Y^d$ in the affine control systems on Lie group are considered null, then the control systems can be viewed as bilinear control system and if $D, D^1, D^2, \cdots, D^d$ in the affine control systems on Lie group are considered null, then the control systems can be viewed as invariant control systems. Thus, affine control systems is more general than linear, bilinear and invariant control systems.

3. Free Nilpotent Lie Algebras and Their Lie Groups

In this section, we consider free nilpotent Lie algebras and their Lie groups, and this type of Lie groups are in the hearth of our systems as state spaces of the systems in this work. Free nilpotent Lie algebra $L(G)_{m,r}$ are acquired in the following way, [4]:

Let $m \geq 2$ and $r \geq 1$ be integers and $V_1, V_2, \cdots, V_r$ be real vector spaces. Let $V_1 := \{X_1, X_2, \cdots, X_m\}$ and

$$V_i := \{[X_k, X_l] | X_k \in V_a, X_l \in V_b, 1 \leq k, l \leq m, a + b = i\}, \qquad i = 2, ..., r.$$

We define $L(G)_{m,r}$ as

$$L(G)_{m,r} = span_{L.A.}\{X_1, X_2, ..., X_m\}.$$

Then $L(G)_{m,r}$ can be written as a direct sum of V_i, i.e. ,

$$L(G)_{m,r} = V_1 \bigoplus \cdots \bigoplus V_r.$$

Since $L(G)_{m,r}$ is a nilpotent Lie algebra; for $X, Y \in L(G)_{m,r}$,

$$X \diamond Y := \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sum_{p_i + q_i \geq 1 1 \leq i \leq n} \frac{(adX)^{p_1}(adY)^{q_1} \cdots (adX)^{p_n}(adY)^{q_{n-1}}Y}{(\sum_{j=1}^{n}(p_j + q_j))p_1!q_1! \cdots p_n!q_n!}.$$

$$\begin{aligned} = \; & X + Y + \tfrac{1}{2}[X, Y] \\ & + \tfrac{1}{12}[X, [X, Y]] - \tfrac{1}{12}[Y, [X, Y]] \\ & - \tfrac{1}{48}[Y, [X, [X, Y]]] - \tfrac{1}{48}[X, [Y, [X, Y]]] \\ & + \{\text{brackets of height } \leq 5\}. \end{aligned}$$

This operation is the well-known Campbell-Hausdorff formula and by the nilpotency of $L(G)_{m,r}$ it follows that this operation is a binary operation on $L(G)_{m,r}$.

A connected and simply connected Lie group of $L(G)_{m,r}$ can be found in the following way, [5].

Let $r \geq 1, m \geq 2$, and let $d = \dim(L(G)_{m,r})$. Then the vector fields with polynomial coefficient on $\mathbf{R}^d$, where d is the dimension of $L(G)_{m,r}$, is defined by :

$$\begin{aligned} & E_1 := \tfrac{\partial}{\partial x_1}, \\ & E_2 := \tfrac{\partial}{\partial x_2} + \textstyle\sum_{j \succ 2} P_{2,j} \tfrac{\partial}{\partial x_j}, \\ & \vdots \\ & E_m := \tfrac{\partial}{\partial x_m} + \textstyle\sum_{j \succ m} P_{m,j} \tfrac{\partial}{\partial x_j} \end{aligned}$$

We would like to explain this construction on an example.

Example 3.1. *Let $f_{3,3}$ be a free nilpotent Lie algebra with 3 generators X_1, X_2, X_3, where the rank of nilpotentcy is* 3. *We can generate other elements of the Lie algebra with the help of Hall basis, in the following way:*

$X_4 = [X_2, X_1]$, $X_5 = [X_3, X_1]$, $X_6 = [X_3, X_2]$, *each of whose height is 2;*
$X_7 = [[X_2, X_1], X_1]$, $X_8 = [[X_2, X_1], X_2]$, $X_9 = [[X_2, X_1], X_3]$,
$X_{10} = [[X_3, X_1], X_1]$, $X_{11} = [[X_3, X_1], X_2]$, $X_{12} = [[X_3, X_1], X_3]$,
$X_{13} = [[X_3, X_2], X_2]$, $X_{14} = [[X_3, X_2], X_3]$, *each of whose that is 3.*

This example for free nilpotent Lie algebras provides different generators for $f_{3,3}$ *which are isomorphic to* X_1, X_2 *and* X_3.

We set the new generators with some formulas:
$E_1 = \frac{\partial}{\partial x_1}$,
$E_2 = \frac{\partial}{\partial x_2} + \sum\limits_{j \succ 2}^{14} P_{2,j}$,
$E_3 = \frac{\partial}{\partial x_3} + \sum\limits_{j \succ 4}^{14} P_{3,j}$,
where $P_{i,j} = \sum\limits_{j>i}^{14} \frac{(-1)^{(d(j)-d(i))}}{(I(j)-I(i))!} x^{(I(j)-I(i))}$ *and* $\succ$ *is the operation of defining order, i.e., given the element* $X_k = [[[[X_j, X_{i_1}], ..., X_{i_n}]$, *then* $k \succ j$. *With simple observation,* $4 \succ 2, 7 \succ 2, 8 \succ 2, 9 \succ 2, 5 \succ 3, 6 \succ 3, 10 \succ 3, 11 \succ 3, 12 \succ 3,\ 13 \succ 3, 14 \succ 3$. *In the formula, the function* $d(i)$ *represents the number of Lie brackets of the vector field* E_i *and* $I(i)$ *represents index, which shows the number of generators except the first one, of that. For instance,* $d(1) = d(2) = d(3) = 0$, $d(13) = 2$ *and the index of* E_{13}, $I(13) = (0, 2, 0)$.

With the above formulas, we give an illustration for $P_{i,j}$ *function. We consider* $P_{3,14}$ *and for the formula, the information what we need are* $d(3) = 0, d(14) = 2$ *and* $I(3) = (0, 0, 0), I(14) = (0, 1, 1)$. *Thus,*
$P_{3,14} = (\frac{(-1)^{(2-0)} x_2^{(1-0)}}{1!})(\frac{(-1)^{(2-0)} x_3^{(1-0)}}{1!}) = (x_2)(x_3)$.

After same calculations, we obtain
$E_1 = \frac{\partial}{\partial x_1}$,
$E_2 = \frac{\partial}{\partial x_2} + (-x_1)\frac{\partial}{\partial x_4} + (\frac{x_1^2}{2!})\frac{\partial}{\partial x_7} + (x_1 x_2)\frac{\partial}{\partial x_8} + (x_1 x_3)\frac{\partial}{\partial x_9}$,
$E_3 = \frac{\partial}{\partial x_3} + (-x_1)\frac{\partial}{\partial x_5} + (-x_2)\frac{\partial}{\partial x_6} + (\frac{x_1^2}{2!})\frac{\partial}{\partial x_{10}} + (x_1 x_2)\frac{\partial}{\partial x_{11}} + (x_1 x_3)\frac{\partial}{\partial x_{12}} + (\frac{x_2^2}{2!})\frac{\partial}{\partial x_{13}} + (x_2 x_3)\frac{\partial}{\partial x_{14}}$.

In order to characterize controllability of affine control systems on free nilpotent Lie groups, our approach is to find an automorphism in the algebra level between affine control system and its bilinear part. Then we have the following Lemma.

Lemma 3.2. *Consider the Lie algebra $L(G)_{m,r}$ of dimension d and the mapping $\varphi_\lambda : L(G)_{m,r} \to L(G)_{m,r}$ such that*

$$\varphi_\lambda\Big(\sum_{i=1}^{d} x_i E_i\Big) = \sum_{i=1}^{d} \sqrt[r]{\lambda^{a_i}} x_i E_i,$$

where $\{E_i\}_{i=1}^{d}$ is a basis for $L(G)_{m,r}$; $\lambda \in (\mathbf{R}^+ \cup \{0\})$ and a_i is the height of E_i, i.e., if $E_i \in V_k$, then a_i of E_i is k. Then, $\varphi_\lambda : L(G)_{m,r} \to L(G)_{m,r}$ is an automorphism on $L(G)_{m,r}$.

Proof. Let $X, Y \in L(G)_{m,r}$.

$$\varphi_\lambda(X \diamond Y) = \varphi_\lambda\Big(\Big(\sum_{i=1}^{d} x_i E_i\Big) \diamond \Big(\sum_{i=1}^{d} y_i E_i\Big)\Big)$$

$$= \varphi_\lambda\Big(\sum_{i=1}^{m}(x_i + y_i)E_i + \frac{1}{2}\sum_{i=1,j=1 j>i}^{m}(x_j y_i - x_i y_j)[E_i, E_j] + \frac{1}{12}\sum_{i=1,j=1,k=1}$$

$$j > i \leq k^m (x_k - y_k)(x_j y_i - x_i y_j)[E_k, [E_i, E_j]] + \cdots\Big)$$

$$= \sqrt[r]{\lambda}\Big(\sum_{i=1}^{m}(x_i + y_i)E_i + \sqrt[r]{\lambda^2}\frac{1}{2}\sum_{i=1,j=1 j>i}^{m}(x_j y_i - x_i y_j)[E_i, E_j]$$

$$+ \sqrt[r]{\lambda^3}\frac{1}{12}\sum_{i=1,j=1,k=1 j>i\leq k}^{m}(x_k - y_k)(x_j y_i - x_i y_j)[E_k, [E_i, E_j]] + \cdots\Big)$$

$$= \Big(\sum_{i=1}^{m}(x_i + y_i)\sqrt[r]{\lambda}E_i + \frac{1}{2}\sum_{i=1,j=1 j>i}^{m}(x_j y_i - x_i y_j)[\sqrt[r]{\lambda}E_i, \sqrt[r]{\lambda}E_j]$$

$$+\frac{1}{12}\sum_{i=1,j=1,k=1 j>i\leq k}^{m}(x_k - y_k)(x_j y_i - x_i y_j)[\sqrt[r]{\lambda}E_k, [\sqrt[r]{\lambda}E_i, \sqrt[r]{\lambda}E_j]] + \cdots\Big)$$

$$= \varphi_\lambda(X) \diamond \varphi_\lambda(Y).$$

So, φ_λ is a homomorphism with the help of Campbell-Hausdorff formula. For injection of φ_λ,

$$\varphi_\lambda(X) = \varphi_\lambda(Y)$$

$$= \varphi_\lambda\Big(\sum_{i=1}^{d} x_i E_i\Big) = \varphi_\lambda\Big(\sum_{i=1}^{d} y_i E_i\Big) = \sum_{i=1}^{d} \sqrt[r]{\lambda^{a_i}} x_i E_i = \sum_{i=1}^{d} \sqrt[r]{\lambda^{a_i}} y_i E_i$$

from equality of vector fields, each x_i equals y_i for all $i = 1, \cdots, m$. Then, $X = Y$.

It is obvious that φ_λ is onto. This proves that φ_λ is an automorphism.

Lemma 3.3. *Let $G_{m,r}$ be a free nilpotent Lie group. Then there exists a dense $Aut(G_{m,r})$ – orbit.*

Proof. The set

$$\mathcal{O} := exp(L(G)_{m,r} - \underbrace{[...[L(G)_{m,r}, L(G)_{m,r}]...]}_{r-1}) = G_{m,r} - \underbrace{[...[G_{m,r}, G_{m,r}]...]}_{r-1}$$

is an $Aut(G_{m,r})$ – orbit of $G_{m,r}$. In fact, the exponential map is a global diffeomorphism for simply connected nilpotent Lie groups. If the binary operation $x \circ y$ on $G_{m,r}$ can be seen as left translation of y, then Jacobian basis $Z_1, Z_2, \cdots, Z_H$ of $L(G)_{m,r}$ can be obtained from Jacobian matrix of left translation of y at origin, where $H := \dim(L(G)_{m,r})$. Since $L(G)_{m,r}$ is nilpotent of step r,

$$L(G)_{m,r} = V_1 \bigoplus \cdots \bigoplus V_r,$$

Lie brackets of each elements of V_r and the other elements of $L(G)_{m,r}$ which has height $a_i \leq r$ vanishes. Therefore, each Z_i which belongs to V_r has $\frac{\partial}{\partial x_i}$ at origin. When we eliminate the elements of V_r from $L(G)_{m,r}$, we delete finitely many lines from $G_{m,r}$ because of simply connectedness of $G_{m,r}$. So this process does not change the dimension of the state space. Moreover, $Aut(G_{m,r})$ – orbit of $G_{m,r}$ is open.

For the density, any $x \in \underbrace{[...[G_{m,r}, G_{m,r}]...]}_{r-1}$, every ball $B(x, \delta)$

$$B(x, \delta) \cap G_{m,r} - \underbrace{[...[G_{m,r}, G_{m,r}]...]}_{r-1} \neq \emptyset$$

Theorem 3.4. *An affine control system $\Sigma = (G_{m,r}, \mathcal{D})$ on free nilpotent Lie group $G_{m,r}$ is controllable if Σ does not have any equilibrium point and the associated bilinear control system $\Sigma_b = (G_{m,r}, \mathcal{D}_b)$, where*

$$\mathcal{D}_b = \{D + \sum_{j=1}^{n} u_j D^j | D, D^j \in Der(L(G)_{m,r}) \text{ and } u \in \mathbf{R}^n\}$$

is controllable on the automorphism orbit $Aut(G_{m,r})$ – orbit of $G_{m,r}$.

Proof. Any equilibrium point in a control system is a problem for establishing controllability, because of their unreachability. Therefore, to not have any equilibrium point is necessary for controllability. Define the automorphism $\xi_\lambda : Der(L(G)_{m,r}) \times L(G)_{m,r} \to Der(L(G)_{m,r}) \times L(G)_{m,r}$ such that $\xi(D + X) = D + \varphi_\lambda(X)$, where $Der(L(G)_{m,r})$ denotes the derivation algebra of $L(G)_{m,r}$, φ_λ is the automorphism given in Lemma 3.2.
$\varphi_\lambda = (\sqrt[r]{\lambda}Id, \sqrt[r]{\lambda}Id, \cdots, \sqrt[r]{\lambda}Id, \sqrt[r-1]{\lambda}Id, \cdots, \sqrt[r-1]{\lambda}Id, \cdots, \lambda Id, \cdots, \lambda Id)$ and, for all $X \in L(G)_{m,r}$, $\varphi_\lambda \to 0$ as $\lambda \to 0$. Therefore, $\xi(\Sigma) \to \Sigma_b$ as $\lambda \to 0$.

By the Lemma 3.3., we have the existence of a dense $Aut(G_{m,r})$ – orbit for our control system. Via φ_λ, affine control system approaches continuously to bilinear control system as $\lambda \to 0$. Hence, Σ_b is controllable on every point.

For λ sufficiently small, $\xi_\lambda(\Sigma)$ is controllable on $S(1,1) - \underbrace{[...[L(G)_{m,r}, L(G)_{m,r}]...]}_{r-1}$, where $S(1,1)$ is the unit sphere $S(1,1)$ centered at 1 which is the boundary of the unit ball $B(1,1)$ centered at 1, because, complete controllability is preserved under small perturbations, [6]. Then $\xi_\lambda(\Sigma)$ is controllable on $B(1,1) - \underbrace{[...[L(G)_{m,r}, L(G)_{m,r}]...]}_{r-1}$. Indeed, finite systems normally controllable on $S(1,1)$ are open, [6]. Therefore, Σ is controllable on $B(1_{\varphi_\lambda^{-1}}, 1) - \underbrace{[...[L(G)_{m,r}, L(G)_{m,r}]...]}_{r-1}$, where
$1_{\varphi_\lambda^{-1}} = (1, (\frac{e}{\sqrt[r]{\lambda}}, \cdots, \frac{e}{\sqrt[r]{\lambda}}, \frac{e}{\sqrt[r-1]{\lambda}}), \cdots, \frac{e}{\sqrt[r-1]{\lambda}}, \cdots, \frac{e}{\lambda}, \cdots, \frac{e}{\lambda})$. Then the positive orbit of the affine system through the identity element is open and its interior is non-empty and Σ is controllable.

Example 3.5. *Let $L(G)_{3,3}$ be a free nilpotent Lie algebra generated by the vector fields $\{X_1, X_2, X_3\}$ with nilpotency degree 3. It is assumed that the generator vector fields X_1, X_2, X_3 can be arranged with respect to their* *heights, i.e.,* $X_1 < X_2 < X_3$ *to obtain elements of* $L(G)_{3,3}$ *with the help of Hall basis. The elements of* $L(G)_{3,3}$ *can be shown in the following related with their heights:*

Height 1: $X_1, X_2, X_3,$
Height 2: $X_4 = [X_2, X_1], X_5 = [X_3, X_1], X_6 = [X_3, X_2];$
Height 3: $X_7 = [X_4, X_1] = [[X_2, X_1], X_1], X_8 = [X_4, X_2] = [[X_2, X_1], X_2],$
$X_9 = [X_4, X_3] = [[X_2, X_1], X_3], X_{10} = [X_5, X_1] = [[X_3, X_1], X_1],$
$X_{11} = [X_5, X_2] = [[X_3, X_1], X_2], X_{12} = [X_5, X_3] = [[X_3, X_1], X_3],$
$X_{13} = [X_6, X_2] = [[X_3, X_2], X_2], X_{14} = [X_6, X_3] = [[X_3, X_2], X_3].$

It was mentioned that Grayson suggests a model for free nilpotent Lie algebras, [5]. Thus, for the generators X_1, X_2, X_3 *of* $L(G)_{3,3}$*, we found the alternative basis* $\{E_1, E_2, E_3\}$ *for* $L(G)_{3,3}$ *in Example (3.1). Therefore,*
$E_1 = \frac{\partial}{\partial x_1},$
$E_2 = \frac{\partial}{\partial x_2} + (-x_1)\frac{\partial}{\partial x_4} + (\frac{x_1^2}{2!})\frac{\partial}{\partial x_7} + (x_1x_2)\frac{\partial}{\partial x_8} + (x_1x_3)\frac{\partial}{\partial x_9},$
$E_3 = \frac{\partial}{\partial x_3} + (-x_1)\frac{\partial}{\partial x_5} + (-x_2)\frac{\partial}{\partial x_6} + (\frac{x_1^2}{2!})\frac{\partial}{\partial x_{10}} + (x_1x_2)\frac{\partial}{\partial x_{11}} + (x_1x_3)\frac{\partial}{\partial x_{12}} + (\frac{x_2^2}{2!})\frac{\partial}{\partial x_{13}} + (x_2x_3)\frac{\partial}{\partial x_{14}}.$

From well-known Lie brackets of vector fields, it is obtain that $X_7 = [[X_2, X_1], X_1] = [[E_2, E_1], E_1] = \frac{\partial}{\partial x_7}$*. If the same calculations are proceeded, then they can be viewed as*
$X_8 = \frac{\partial}{\partial x_8}$, $X_9 = \frac{\partial}{\partial x_9}$, $X_{10} = \frac{\partial}{\partial x_{10}}$, $X_{11} = \frac{\partial}{\partial x_{11}}$,
$X_{12} = \frac{\partial}{\partial x_{12}}$, $X_{13} = \frac{\partial}{\partial x_{13}}$, $X_{14} = \frac{\partial}{\partial x_{14}}$ *which are the elements of kernel of* $L(G)_{3,3}$*. Let* $(G_{3,3}, \circ)$ *be the simply connected and connected Lie group of* $L(G)_{3,3}$*. The product "*$\circ$*" on* $G_{3,3}$ *is defined as*

$$
x \circ y = \left\{ \begin{array}{c}
x_1 + y_1 \\
x_2 + y_2 \\
x_3 + y_3 \\
x_4 + y_4 + 1/2(x_2 y_1 - y_2 x_1) \\
x_5 + y_5 + 1/2(x_3 y_1 - y_3 x_1) \\
x_6 + y_6 + 1/2(x_3 y_2 - y_3 x_2) \\
x_7 + y_7 + 1/2(x_4 y_1 - y_4 x_1) + 1/12(x_2 y_1 - y_2 x_1)(x_1 - y_1) \\
x_8 + y_8 + 1/2(x_4 y_2 - y_4 x_2) + 1/12(x_2 y_1 - y_2 x_1)(x_2 - y_2) \\
x_9 + y_9 + 1/2(x_4 y_3 - y_4 x_3) + 1/12(x_2 y_1 - y_2 x_1)(x_3 - y_3) \\
x_{10} + y_{10} + 1/2(x_5 y_1 - y_5 x_1) + 1/12(x_3 y_1 - y_3 x_1)(x_1 - y_1) \\
x_{11} + y_{11} + 1/2(x_5 y_2 - y_5 x_2) + 1/12(x_3 y_1 - y_3 x_1)(x_2 - y_2) \\
x_{12} + y_{12} + 1/2(x_5 y_3 - y_5 x_3) + 1/12(x_3 y_1 - y_3 x_1)(x_3 - y_3) \\
x_{13} + y_{13} + 1/2(x_6 y_2 - y_6 x_2) + 1/12(x_3 y_2 - y_3 x_2)(x_2 - y_2) \\
x_{14} + y_{14} + 1/2(x_6 y_3 - y_6 x_3) + 1/12(x_3 y_2 - y_3 x_2)(x_3 - y_3).
\end{array} \right\}
$$

Let $\Sigma(G_{3,3}, \mathcal{D})$ be the affine control system on the free nilpotent Lie group $G_{3,3}$ where
$\mathcal{D} = \{(D + X) + \sum_{i=1}^{d} u_i(D^i + X^i) | D, D^i \in DerL(G)_{3,3}; X, X^i \in G_{3,3}\}$
is the dynamic of the system. The automorphism orbit of $G_{3,3}$ can be seen as the following form:
$Aut(G_{3,3}) = \exp(L(G)_{3,3} - Z(L(G)_{3,3})) = G_{3,3} - [[G_{3,3}, G_{3,3}], G_{3,3}]$.
As it can be indicated above, each of the elements of kernel of the free nilpotent Lie algebra $Z(L(G)_{3,3}$ defines a line, then removing these elements from the Lie algebra and , of course, from the Lie group doesn't change the dimension of the automorphism orbit which is same as that of $G_{3,3}$. From Lemma 3.3, the automorphism orbit is dense. The Lie algebra automorphism
$\xi : af(L(G)_{3,3}) \to af(L(G)_{3,3})$
$\xi(D + X) = D + \varphi_\lambda(X)$,

can be defined via the Lie algebra automorphism φ_λ defined in Lemma 3.2. Since φ_λ approaches zero as λ approaches zero, the affine control system consists of only vector fields from the derivation algebra of the Lie algebra. Thus, the system can be converted the bilinear control system and we can characterize the controllability of affine control system via that of its bilinear control system.

Conclusion 3.6. *In this work, we study the controllability of affine systems on free nilpotent Lie groups. Controllability problem is one of the most important classical problem in Control Theory and its applications* *are wide. The system which we consider here is the most general case for the tecnique to establish controllability. We characterize the controllability by finding an automorphism in the algebra level and on the automorphism orbit which we prove that it is dense.*

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