Further results on 3-product cordial labeling

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Abstract

A mapping $f : V(G) \to \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i$, $e_f(i)$ denotes the number of edges $xy$ with $f(x)f(y) \equiv i \pmod{3}$. A graph with 3-product cordial labeling is called 3-product cordial graph. In this paper we establish that switching of an apex vertex in closed helm, double fan, book graph $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, I)$ are 3-product cordial graphs.

Key Words. cordial labeling, product cordial labeling, 3-product cordial labeling, 3-product cordial graph.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected. For basic notations and terminology, we follow [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a complete survey of graph labeling is available in [2]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in [1]. Let \( f \) be a function from the vertices of \( G \) to \( \{0, 1\} \) and for each edge \( xy \) assign the label \( |f(x) - f(y)| \). \( f \) is called a cordial labeling of \( G \) if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Sundaram et al. introduced the concept of product cordial labeling in [10]. Let \( f \) be a function from \( V(G) \) to \( \{0, 1\} \). For each edge \( uv \), assign the label \( f(u)f(v) \). Then \( f \) is called product cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \) where \( v_f(i) \) and \( e_f(i) \) denotes the number of vertices and edges respectively labeled with \( i(i = 0, 1)\). The same authors have introduced the concept of EP-cordial labeling in [11]. A vertex labeling \( f : V(G) \rightarrow \{-1, 0, 1\} \) is said to be an EP-cordial labeling if it induces the edge labeling \( f^* \) defined by \( f^*(uv) = f(u)f(v) \) for each \( uv \in E(G) \) and if \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for any \( i \neq j \) \( i, j \in \{-1, 0, 1\} \), where \( v_f(x) \) and \( e_f(x) \) denotes the number of vertices and edges of \( G \) having the label \( x \in \{-1, 0, 1\} \). In [11] it is remarked that any EP-cordial labeling is a 3-product cordial labeling. A mapping \( f : V(G) \rightarrow \{0, 1, 2\} \) is called 3-product cordial labeling if \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for any \( i, j \in \{0, 1, 2\} \), where \( v_f(i) \) denotes the number of edges \( xy \) with \( f(x)f(y) = i(\text{mod} 3) \). A graph with 3-product cordial labeling is called a 3-product cordial graph. Jeyanthi and Maheswari [4]-[8] proved that the graphs \( (B_{n,n} : w), C_n \cup P_n, C_m \circ K_n \) if \( m \geq 3 \) and \( n \geq 1 \), \( P_m \circ K_n \) if \( m, n \geq 1 \), duplicating arbitrary vertex in cycle \( C_n \), duplicating arbitrary edge in cycle \( C_n \), duplicating arbitrary vertex in wheel \( W_n \), middle graph of \( P_n \), the splitting graph of \( P_n \), the total graph of \( P_n, P_n[2], P_n^2, K_2, n \), vertex switching of \( C_n \), ladder \( L_n \), triangular ladder \( TL_n \), the graph \( \{w_n^{(1)} : w_n^{(2)} : \ldots : w_n^{(k)}\} \), the splitting graphs \( S'(K_{1,n}), S'(B_{m,n}) \), the shadow graph \( D_2(B_{n,n}) \), the square graph \( B_{n,n}^2 \), triangular snake, double alternate triangular snake and alternate triangular snake graphs are 3-product cordial graphs. Also they proved that a complete graph \( K_n \) is a 3-product cordial graph if and only if \( n \leq 2 \).

In addition, they proved that if \( G(p, q) \) is a 3-product cordial graph \( (i) \)
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\[ p \equiv 1 \text{ (mod 3)} \] then \( q \leq \frac{p^2 - 2p + 7}{3} \); (ii) \( p \equiv 2 \text{ (mod 3)} \) then \( q \leq \frac{p^2 - p + 4}{3} \)

(iii) \( p \equiv 0 \text{ (mod 3)} \) then \( q \leq \frac{p^2 - 3p + 6}{3} \) and if \( G_1 \) is a 3-product cordial graph with 3m vertices and 3n edges and \( G_2 \) is any 3-product cordial graph then \( G_1 \cup G_2 \) is also 3-product cordial graph. In this paper we establish that switching of an apex vertex in closed helm, double fan, \( K_{1,n} \times K_2 \) and permutation graph \( P(K_2 + mK_1, I) \) are 3-product cordial graphs. We use the following definitions in the subsequent section.

**Definition 1.1.** The vertex switching \( G_v \) of a graph \( G \) is the graph obtained by taking a vertex \( v \) of \( G \), by removing all the edges incident with \( v \) and joining the vertex \( v \) to every vertex which is not adjacent to \( v \) in \( G \).

**Definition 1.2.** The helm \( H_n \) is the graph obtained from a wheel \( W_n \) by attaching a pendant edge to each rim vertex.

**Definition 1.3.** The closed helm \( CH_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to from a cycle.

**Definition 1.4.** The graph \( P_n + 2K_1 \) is called a double fan \( DF_n \).

**Definition 1.5.** For any permutation \( f \) on \( 1, 2, 3, \ldots, n \), the \( f \)-permutation graph on a graph \( G \), \( P(G, f) \) consists of two disjoint copies of \( G \), say \( G_1 \) and \( G_2 \), each of which has vertices labeled \( v_1, v_2, \ldots, v_n \) with \( n \) edges obtained by joining each \( v_i \) in \( G_1 \) to \( v_{f(i)} \) in \( G_2 \). We denote the identity permutation by \( I \).

For any real number \( n \), \([n]\) denotes the smallest integer \( \geq n \) and \([n]\) denotes the greatest integer \( \leq n \).

2. Main Results

**Theorem 2.1.** The graph obtained by switching of an apex vertex in closed helm \( CH_n \) admits 3-product cordial labeling if and only if \( n \equiv 2 \text{(mod 3)} \).

**Proof.** Let \( v \) be the apex vertex \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of inner cycle and \( u_1, u_2, u_3, \ldots, u_n \) be the vertices of outer cycle \( CH_n \). Let \( G_v \) denotes graph obtained by switching of an apex vertex \( v \) of \( G = CH_n \). Then \( |V(G_v)| = 2n + 1 \) and \( |E(G_v)| = 4n \). We define \( f : V(G_v) \rightarrow \{0, 1, 2\} \) as follows:
\[ f(v) = 2, \text{ for } 1 \leq i \leq \left\lfloor \frac{2n+1}{3} \right\rfloor, f(v_i) = 0. \]

For \( n \equiv 0, 2, 3 \pmod{4}, 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor, \]

\[ f\left( v_{\left\lfloor \frac{2n+1}{3} \right\rfloor + i} \right) = \begin{cases} 
1 & \text{for } i \equiv 1, 2 \pmod{4} \\
2 & \text{for } i \equiv 0, 3 \pmod{4}
\end{cases} \]

For \( n \equiv 1 \pmod{4}, n > 5, 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 2, \]

\[ f\left( v_{\left\lfloor \frac{2n+1}{3} \right\rfloor + i} \right) = \begin{cases} 
1 & \text{for } i \equiv 1, 2 \pmod{4} \\
2 & \text{for } i \equiv 0, 3 \pmod{4}
\end{cases} \]

\[ f(v_{n-1}) = 1 \text{ and } f(v_n) = 2. \]

For \( n = 5, f(v_4) = 1 \text{ and } f(v_5) = 2. \]

For \( n \equiv 0, 1, 3 \pmod{4}, 1 \leq i \leq n, f(u_i) = \begin{cases} 
1 & \text{if } i \equiv 1, 2 \pmod{4} \\
2 & \text{if } i \equiv 0, 3 \pmod{4}
\end{cases} \]

For \( n \equiv 2 \pmod{4}, 1 \leq i \leq n - 2, f(u_i) = \begin{cases} 
1 & \text{if } i \equiv 1, 2 \pmod{4} \\
2 & \text{if } i \equiv 0, 3 \pmod{4}
\end{cases} \]

\[ f(u_{n-1}) = 1 \text{ and } f(u_n) = 2. \]

In view of the above labeling pattern we have \( v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{2n+1}{3} \right\rfloor, e_f(0) = e_f(1) = e_f(2) + 1 = \left\lfloor \frac{4n}{3} \right\rfloor \text{ if } n \equiv 0, 1 \pmod{4} \)

and \( e_f(0) = e_f(1) + 1 = e_f(2) = \left\lfloor \frac{4n}{3} \right\rfloor \text{ if } n \equiv 2, 3 \pmod{4}. \)

Thus we have \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for all \( i, j = 0, 1, 2. \) Hence \( f \) is a 3-product cordial labeling of \( G_v \) if \( n \equiv 2 \pmod{3}. \)

Conversely, we assume that \( n \equiv 0 \pmod{3} \) and take \( n = 3k. \) Then \( |V(G_v)| = 6k + 1 \) and \( |E(G_v)| = 12k. \)

Let \( f \) be a 3-product cordial labeling of \( G_v. \) Hence we have \( v_f(0) = v_f(1) - 1 = v_f(2) = 2k \) or \( v_f(0) = v_f(1) = v_f(2) - 1 = 2k \) and \( e_f(0) = e_f(1) = e_f(2) = 4k. \) If \( f(u_i) = 0 \) if \( 1 \leq i \leq 2k, \) then \( e_f(0) = 6k + 1. \) If \( f(v) = 0 \) if \( 1 \leq i \leq 2k - 1 \) and \( f(v) = 0, \) then \( e_f(0) = 7k - 1. \) If \( f(v) = 0 \) if \( 1 \leq i \leq 2k - 1 \) and \( f(v) = 0, \) then \( e_f(0) = 7k - 1. \) Thus, none of \( f(u_i), f(v_i) \) and \( f(v) \) is 0. From the above argument, we get a contradiction to, \( f \) is a 3-product cordial labeling. Hence \( G_v \) is not a 3-product cordial graph if \( n \equiv 0 \pmod{3}. \)

We assume that \( n \equiv 1 \pmod{3} \) and take \( n = 3k + 1. \) Then \( |V(G_v)| = 6k + 3 \) and \( |E(G_v)| = 12k + 4. \)

Let \( f \) be a 3-product cordial labeling of \( G_v. \) Hence we have \( v_f(0) = v_f(1) = v_f(2) = 2k + 1 \) and \( e_f(0) = 4k + 1 \) or \( 4k + 2. \) If \( v_f(0) = 2k + 1, \) we assign 0 to \( 2k + 1 \) vertices of degree 3. We get \( e_f(0) = 4k + 3. \) we assign
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0 to $2k + 1$ vertices of degree 4. We get $e_f(0) = 6k + 4$. If $f(u_i) = 0$ if $1 \leq i \leq 2k$ and $f(v) = 0$, then $e_f(0) = 7k + 2$. If $f(v_i) = 0$ if $1 \leq i \leq 2k$ and $f(v) = 0$, then $e_f(0) = 7k + 2$. Thus, none of $f(u_i), f(v_i)$ and $f(v)$ is 0. From the above argument, we get a contradiction to, $f$ is a 3-product cordial labeling. Hence $G_v$ is not a 3-product cordial graph if $n \equiv 1(\text{mod } 3)$.

□ An example for the 3-product cordial labeling of a closed helm $CH_8$ by switching of an apex vertex is shown in Figure 1.

![Figure 1](image)

**Theorem 2.2.** The double fan graph $DF_n$ is a 3-product cordial graph if and only if $n \equiv 0(\text{mod } 3)$.

**Proof.** Let $DF_n$ be the double fan with apex vertices $u, v$ and $v_1, v_2, \ldots, v_n$ be the vertices of common path. Then $|V(DF_n)| = n + 2$ and $|E(DF_n)| = 3n - 1$. To define $f : V(DF_n) \rightarrow \{0, 1, 2\}$ as follows:

- $f(u) = 1, f(v) = 2, f(v_i) = 0$ if $1 \leq i \leq \frac{n}{3}$.
- $f(v_{n-1}) = 1, f(v_n) = 2$.

For $n$ is even, $i = \frac{n}{3} + j, 1 \leq j \leq \frac{2n}{3}$, $f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 2(\text{mod } 4) \\ 2 & \text{if } j \equiv 0, 3(\text{mod } 4) \end{cases}$

For $n$ is odd, $i = \frac{n}{3} + j, 1 \leq j \leq \frac{2n}{3} - 2$, $f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 2(\text{mod } 4) \\ 2 & \text{if } j \equiv 0, 3(\text{mod } 4) \end{cases}$

In view of the above labeling pattern we have $v_f(0) + 1 = v_f(1) = v_f(2) = \frac{n}{3} + 1$ and $e_f(0) = e_f(1) = e_f(2) + 1 = n$ if $n$ is even, $e_f(0) = e_f(1) + 1 = e_f(2) = n$ if $n$ is odd.

Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Thus, $f$ is a 3-product cordial labeling. Therefore, $DF_n$ is a 3-product cordial graph if $n \equiv 0(\text{mod } 3)$.

Conversely, we assume that $n \equiv 2(\text{mod } 3)$ and take $n = 3k + 2$. Then $|V(DF_n)| = 3k + 4$ and $|E(DF_n)| = 9k + 5$.
Let $f$ be a 3-product cordial labeling of $DF_n$. Hence we have $v_f(0) = v_f(1) = v_f(2) = k + 1$ or $v_f(0) = v_f(1) = v_f(2) - 1 = k + 1$ and $e_f(0) = 3k + 1$ or $3k + 2$. If $f(u) = 0, f(v) = 0$ and the remaining $k - 1$ vertices are $f(v_i) = 0$ then we get, $e_f(0) > 7k + 3$. If $f(u) = 0$ or $f(v) = 0$ and the remaining $k$ vertices are $f(v_i) = 0$ for $1 \leq i \leq k$ then we get, $e_f(0) = 5k + 2$. If $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then we get $e_f(0) = 3k + 3$.

From the above argument, we get a contradiction to, $f$ is a 3-product cordial labeling. Hence $DF_n$ is not a 3-product cordial graph if $n \equiv 2 \pmod{3}$.

We assume that $n \equiv 1 \pmod{3}$ and take $n = 3k + 1$. Then $|V(DF_n)| = 3k + 3$ and $|E(DF_n)| = 9k + 2$.

Let $f$ be a 3-product cordial labeling of $DF_n$. Hence we have $v_f(0) = v_f(1) = v_f(2) = k + 1$ and $e_f(0) = 3k + 1 + 3k + 1$. If $f(u) = 0, f(v) = 0$ and the remaining $k - 1$ vertices are $f(v_i) = 0$ then we get, $e_f(0) > 7k + 1$. If $f(u) = 0$ or $f(v) = 0$ and the remaining $k$ vertices are $f(v_i) = 0$ for $1 \leq i \leq k$ then we get, $e_f(0) = 5k + 1$. If $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then we get $e_f(0) = 3k + 3$. From the above argument, we get a contradiction to, $f$ is a 3-product cordial labeling. Hence $DF_n$ is not a 3-product cordial graph if $n \equiv 1 \pmod{3}$. □

An example for the 3-product cordial labeling for the graph $DF_6$ is shown in Figure 2.

![Figure 2](image_url)

**Theorem 2.3.** The book graph $K_{1,n} \times K_2$ is a 3-product cordial graph.

**Proof.** Let the vertices of $K_{1,n} \times K_2$ be $\{u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edges are $\{uv\} \cup \{u_i v_i/1 \leq i \leq n\} \cup \{u_i u_i/1 \leq i \leq n\}$. Clearly $K_{1,n} \times K_2$ has $2n + 2$ vertices and $3n + 1$ edges. Define $f : V(K_{1,n} \times K_2) \rightarrow \{0, 1, 2\}$ by the following cases.

$f(u) = 1, f(v) = 2$. 

Case (i): $n \equiv 0 \pmod{3}, n = 3k, k > 1$.

For $k$ is even, $f(u_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq k \\
2 & \text{if } k + 1 \leq i \leq \frac{3k}{2} \\
1 & \text{if } \frac{3k}{2} + 1 \leq i \leq 3k 
\end{cases}$

and $f(v_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq k \\
1 & \text{if } k + 1 \leq i \leq \frac{3k}{2} \\
2 & \text{if } \frac{3k}{2} + 1 \leq i \leq 3k 
\end{cases}$

For $k$ is odd, $f(u_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq k \\
2 & \text{if } k + 1 \leq i < \frac{3k}{2} \\
1 & \text{if } \frac{3k}{2} \leq i \leq 3k 
\end{cases}$

and $f(v_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq k \\
1 & \text{if } k + 1 \leq i < \frac{3k}{2} \\
2 & \text{if } \frac{3k}{2} \leq i \leq 3k 
\end{cases}$

For $k = 1$, $f(u_1) = f(v_1) = 0$, $f(u_2) = f(u_3) = 1$, $f(v_2) = f(v_3) = 2$.

From the above labeling we have, $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{6k + 2}{3} \right\rfloor$, $e_f(0) = e_f(1) = e_f(2) - 1 = 3k$ if $k$ is even and $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{6k + 2}{3} \right\rfloor$, $e_f(0) = e_f(1) - 1 = e_f(2) = 3k$ if $k$ is odd.

Case (ii): $n \equiv 1 \pmod{3}, n = 3k + 1, k > 1$.

For $k$ is even, $f(u_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq k + 1 \\
2 & \text{if } k + 2 \leq i \leq \frac{3k + 1}{2} \\
1 & \text{if } \frac{3k + 1}{2} + 1 \leq i \leq 3k + 1 
\end{cases}$

and $f(v_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq k \\
1 & \text{if } k + 2 \leq i \leq \frac{3k + 1}{2} \\
2 & \text{if } i = k + 1 \text{ and } \frac{3k + 1}{2} + 1 \leq i \leq 3k + 1 
\end{cases}$
For $k$ is odd, $f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k + 1 \\ 2 & \text{if } k + 2 \leq i \leq \left\lceil \frac{3k + 1}{2} \right\rceil \\ 1 & \text{if } \left\lceil \frac{3k + 1}{2} \right\rceil + 1 \leq i \leq 3k + 1 \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k + 1 \leq i \leq \left\lceil \frac{3k + 1}{2} \right\rceil \\ 2 & \text{if } \left\lceil \frac{3k + 1}{2} \right\rceil + 1 \leq i \leq 3k + 1 \end{cases}$

For $k = 1$, $f(u_1) = f(u_2) = f(v_1) = 0$, $f(v_2) = f(u_4) = 1$, $f(u_3) = f(v_3) = f(v_4) = 2$.

From the above labeling we have, $v_f(0) + 1 = v_f(1) = v_f(2) + 1 = \left\lceil \frac{6k + 4}{3} \right\rceil$, $e_f(0) - 1 = e_f(1) = e_f(2) = 3k + 1$ if $k$ is odd and $v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \left\lceil \frac{6k + 4}{3} \right\rceil$, $e_f(0) - 1 = e_f(1) = e_f(2) = 3k + 1$ if $k$ is even and $k = 1$.

**Case (iii):** $n \equiv 2 \pmod{3}, n = 3k + 2$ and $k$ is even.

$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k + 1 \\ 2 & \text{if } k + 2 \leq i \leq \frac{3k + 2}{2} \\ 1 & \text{if } \frac{3k + 2}{2} + 1 \leq i \leq 3k + 2 \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k + 1 \\ 1 & \text{if } k + 2 \leq i \leq \frac{3k + 2}{2} \\ 2 & \text{if } \frac{3k + 2}{2} + 1 \leq i \leq 3k + 2 \end{cases}$

From the above labeling we have, $v_f(0) = v_f(1) = v_f(2) = 2k + 2, e_f(0) - 1 = e_f(1) = e_f(2) = 3k + 2$. Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i,j = 0,1,2$. Thus, $f$ is a 3-product cordial labeling. Therefore, $K_{1,n} \times K_2$ is a 3-product cordial graph. □

An example for the 3-product cordial labeling for the book graph $K_{1,7} \times K_2$ is shown in Figure 3.
Theorem 2.4. The graph $P(K_2 + mK_1, I)$ is a 3-product cordial graph if and only if $m \equiv 2 \pmod{3}$.

Proof. Let $V(P(K_2 + mK_1, I)) = \{u, u', v, v', u_i, v_i / 1 \leq i \leq m\}$ and $E(P(K_2 + mK_1, I)) = \{uu_i, u'u_i, vv_i, v'v_i, uu', vv', uv', uv', u_iv, u_v, v'u, v'u / 1 \leq i \leq m\}$. Then $|V(P(K_2 + mK_1, I))| = 2m+4$ and $|E(P(K_2 + mK_1, I))| = 5m+4$.

Define a vertex labeling $f : V(P(K_2 + mK_1, I)) \rightarrow \{0, 1, 2\}$ by $f(u) = 1$, $f(u') = 2$, $f(v) = 2$, $f(v') = 1$. $

f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \left\lfloor \frac{m}{3} \right\rfloor \\ 1 & \text{for } i = \left\lfloor \frac{m}{3} \right\rfloor + j \text{ if } j \equiv 1 \pmod{2} \\ 2 & \text{for } i = \left\lfloor \frac{m}{3} \right\rfloor + j \text{ if } j \equiv 0 \pmod{2} \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \left\lfloor \frac{m}{3} \right\rfloor \\ 1 & \text{for } i = \left\lfloor \frac{m}{3} \right\rfloor + j \text{ if } j \equiv 0, 3 \pmod{4} \\ 2 & \text{for } i = \left\lfloor \frac{m}{3} \right\rfloor + j \text{ if } j \equiv 1, 2 \pmod{4} \end{cases}$

In view of the above labeling pattern, we have $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{2m+4}{3} \right\rfloor$, $e_f(0) = e_f(1) + 1 = e_f(2) = \left\lfloor \frac{5m+4}{3} \right\rfloor$ if $m$ is even and $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{2m+4}{3} \right\rfloor$, $e_f(0) = e_f(1) = e_f(2) + 1 = \left\lfloor \frac{5m+4}{3} \right\rfloor$ if $m$ is odd. Thus, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence, $f$ is a 3-product cordial labeling of $P(K_2 + mK_1, I)$ if $m \equiv 2 \pmod{3}$. 


Conversely, we assume that \( m \equiv 0(\text{mod } 3) \) and take \( m = 3k \). Then \(|V(P(K_2 + mK_1, I))| = 6k + 4\) and \(|E(P(K_2 + mK_1, I))| = 15k + 4\). Let \( f \) be a vertex 3-product cordial labeling of \( P(K_2 + mK_1, I) \). Then \( v_f(1) = 2k + 1 \) or \( v_f(0) = v_f(1) = 2k + 1 \) and \( e_f(0) = 5k + 1 \) or \( 5k + 2 \). If \( f(u), f(u'), f(v), f(v') \) are zero and the remaining \( 2k - 3 \) vertices are either \( f(u_i) = 0 \) for \( 1 \leq i \leq k - 2 \) and \( f(v_i) = 0 \) for \( 1 \leq i \leq k - 1 \) or \( f(u_i) = 0 \) for \( 1 \leq i \leq k - 2, k > 2 \) and \( f(v_i) = 0 \) for \( 1 \leq i \leq k - 1, k > 2 \) then \( e_f(0) = 13k + 3 \). If one of \( f(u) \) or \( f(u') \) and \( f(v) \) or \( f(v') \) is zero and the remaining \( 2k - 1 \) vertices are either \( f(u_i) = 0 \) for \( 1 \leq i \leq k - 1, k > 1 \) and \( f(v_i) = 0 \) for \( 1 \leq i \leq k, k > 1 \) or \( f(u_i) = 0 \) for \( 1 \leq i \leq k, k > 1 \) and \( f(v_i) = 0 \) for \( 1 \leq i \leq k - 1, k > 1 \) then \( e_f(0) = 7k + 3 \) or \( 7k + 4 \). If all the \( 2k + 1 \) vertices are either \( f(u_i) = 0 \) for \( 1 \leq i \leq k + 1 \) and \( f(v_i) = 0 \) for \( 1 \leq i \leq k \) or \( f(u_i) = 0 \) for \( 1 \leq i \leq k + 1 \) and \( f(v_i) = 0 \) for \( 1 \leq i \leq k + 1 \) then \( e_f(0) = 5k + 3 \). Hence none of \( f(u), f(u'), f(v) \) and \( f(v') \) is zero. From the above arguments, we get a contradiction to, \( f \) is a 3-product cordial labeling. Hence, \( P(K_2 + mK_1, I) \) is not a 3-product cordial graph if \( m \equiv 0(\text{mod } 3) \).

We assume that \( m \equiv 1(\text{mod } 3) \) and take \( m = 3k + 1 \). Then \(|V(P(K_2 + mK_1, I))| = 6k + 6\) and \(|E(P(K_2 + mK_1, I))| = 15k + 9\). Let \( f \) be a vertex 3-product cordial labeling of \( P(K_2 + mK_1, I) \). Hence we have \( v_f(0) = v_f(1) = 2k + 2 \) and \( e_f(0) = e_f(1) = e_f(2) = 5k + 3 \). If \( f(u), f(u'), f(v), f(v') \) are zero and the remaining \( 2k - 2 \) vertices are \( f(u_i) = f(v_i) = 0 \) for \( 1 \leq i \leq k - 1, k > 1 \) then \( e_f(0) = 13k + 7 \). If one of \( f(u) \) or \( f(u') \) and \( f(v) \) or \( f(v') \) is zero and the remaining \( 2k \) vertices are \( f(u_i) = f(v_i) = 0 \) for \( 1 \leq i \leq k \) then \( e_f(0) = 9k + 6 \) or \( 9k + 5 \). If all the \( 2k + 2 \) vertices are \( f(u_i) = f(v_i) = 0 \) for \( 1 \leq i \leq k + 1 \) then \( e_f(0) = 5k + 5 \). Hence none of \( f(u), f(u'), f(v) \) and \( f(v') \) is zero. From the above arguments, we get a contradiction to, \( f \) is a 3-product cordial labeling. Hence, \( P(K_2 + mK_1, I) \) is not a 3-product cordial graph if \( m \equiv 1(\text{mod } 3) \).

An example for the 3-product cordial labeling for the graph \( P(K_2 + 5K_1, I) \) is shown in Figure 4.
Further results on 3-product cordial labeling

References


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