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(p, q)-Lucas polynomials and their applications to bi-univalent functions

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Abstract:

In the present paper, by using the $L_{p,q,n}(x)$ functions, our methodology intertwine to yield the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, we aim at introducing a new class of bi-univalent functions defined through the (p, q)-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szegö problem for this new function class.

Keywords: *(p, q)-Lucas polynomials; Coefficient bounds; Bi-univalent functions.*

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1. Introduction and definitions

Fibonacci polynomials, Lucas polynomials, Lucas-Lehmer polynomials, Chebychev polynomials, Pell polynomials, Morgan-Voyce polynomials, Orthogonal polynomials and the other special polynomials and their generalizations are of wide spectra in a variety of branches such as Physics, Engineering, Architecture, Nature, Art, Number Theory, Combinatorics and Numerical analysis (see, for example, [8], [10], [11], [12], [14], [15], [16] and [17]).

The well-known (p,q)-Lucas polynomials are defined by the following definition:

Definition 1.1. *(see [7]) Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p,q)-Lucas polynomials $L_{p,q,n}(x)$ are established by the recurrence relation*

$$L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2),$$

from which the first few Lucas polynomials can be found as

$$\begin{aligned} &L_{p,q,0}(x) = 2,\ L_{p,q,1}(x) = p(x),\ L_{p,q,2}(x) = p^2(x) + 2q(x), \\ &L_{p,q,3}(x) = p^3(x) + 3p(x)q(x),\ \ldots . \end{aligned} \tag{1.1}$$

For the special cases of $p(x)$ and $q(x)$, we can get the polynomials given in Table 1.

Table 1: Special cases of the $L_{p,q,n}(x)$ with given initial conditions are given.

$p(x)$	$q(x)$	$L_{p,q,n}(x)$
x	1	Lucas polynomials $L_n(x)$
2x	1	Pell-Lucas polynomials $D_n(x)$
1	2x	Jacobsthal-Lucas polynomials $j_n(x)$
3x	-2	Fermat-Lucas polynomials $f_n(x)$
2x	-1	Chebyshev polynomials first kind $T_n(x)$

Theorem 1.1. *(see [7]) Let $\mathcal{G}_{\{L_{p,q,n}(x)\}}(z)$ be the generating function of the (p,q)-Lucas polynomial sequence $L_{p,q,n}(x)$. Then*

$$\mathcal{G}_{\{L_{p,q,n}(x)\}}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x)z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1.2}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}$ and normalized under the condition $f(0) = f'(0) - 1 = 0$. Further, by S we represent the class of all functions in A which are univalent in Δ.

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Given functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0, \ \ |w(z)| < 1, \ z \in \Delta\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \Delta).$$

We show this subordination by

$$f \prec g \ \text{ or } \ f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\Delta) \subset g(\Delta).$$

According to the Koebe-One Quarter Theorem [4], it ensures that the image of Δ under every univalent function $f \in A$ contains a disc of radius $1/4$. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ $\left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{1.3}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ indicate the class of bi-univalent functions in Δ given by (1.2). For a brief history and interesting examples in the class Σ, see [13] (see also [1], [2], [3], [6] and [9]).

In the present paper, by using the $L_{p,q,n}(x)$ functions, our methodology intertwine to yield the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, we aim at introducing a new class of bi-univalent functions defined through the (p,q)-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szeg problem for this new function class.

Definition 1.2. *A function $f \in \Sigma$ is said to be in the class*

$$W_{\Sigma}(\tau,\mu,\eta;x) \quad (\tau \in \mathbf{C}\backslash\{0\}, \mu \geq 0, \eta \geq 0;\ z,w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[1+\frac{1}{\tau}\left((1-\mu+2\eta)\frac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta z f''(z)-1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(z)-1$$

and

$$\left[1+\frac{1}{\tau}\left((1-\mu+2\eta)\frac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta w g''(w)-1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(w)-1$$

where the function g is given by (1.3).

It is interesting to note that the special values of τ, μ and η lead the class $W_{\Sigma}(\tau,\mu,\eta;x)$ to various subclasses, we illustrate the following subclasses:

1. For $\mu = 1+2\eta$, we get the class $W_{\Sigma}(\tau,1+2\eta,\eta;x) = W_{\Sigma}(\tau,\eta;x)$. A function $f \in \Sigma$ is said to be in the class

$$W_{\Sigma}(\tau,\eta;x) \quad (\tau \in \mathbf{C}\backslash\{0\}, \mu \geq 0,\ z,w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[1+\frac{1}{\tau}\left(f'(z)+\eta z f''(z)-1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(z)-1$$

and

$$\left[1+\frac{1}{\tau}\left(g'(w)+\eta w g''(w)-1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(w)-1$$

where the function g is given by (1.3).

2. For $\eta = 0$, we obtain the class $W_\Sigma(\tau,\mu,0;x) = W_\Sigma(\tau,\mu;x)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma(\tau,\mu;x) \quad (\tau \in \mathbf{C}\backslash\{0\}, \mu \geq 0;\ z,w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[1 + \frac{1}{\tau}\left((1-\mu)\frac{f(z)}{z} + \mu z f'(z) - 1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left[1 + \frac{1}{\tau}\left((1-\mu)\frac{g(w)}{w} + \mu g'(w) - 1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(w) - 1$$

where the function g is given by (1.3).

3. For $\eta = 0$ and $\mu = 1$, we get the class $W_\Sigma(\tau,1,0;x) = W_\Sigma(\tau;x)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma(\tau,\mu;x) \quad (\tau \in \mathbf{C}\backslash\{0\};\ z,w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[1 + \frac{1}{\tau}\left(f'(z) - 1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left[1 + \frac{1}{\tau}\left(g'(w) - 1\right)\right] \prec \mathcal{G}_{\{L_{p,q,n}(x)\}}(w) - 1$$

where the function g is given by (1.3).

2. Coefficient bounds

In this section, we shall make use of the (p,q)-Lucas polynomials to get the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $W_\Sigma(\tau,\mu,\eta;x)$ proposed by Definition 1.2.

Theorem 2.1. *Let f given by (1.2) be in the class $W_{\Sigma}(\tau,\mu,\eta;x)$. Then*

$$|a_2| \leq \frac{|\tau|\,|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|\left[(1+2\mu+2\eta)\tau-(1+\mu)^2\right]p^2(x)-2\,(1+\mu)^2\,q(x)\right|}}$$

and

$$|a_3| \leq \frac{|\tau|^2\,p^2(x)}{(1+\mu)^2}+\frac{|\tau|\,|p(x)|}{1+2\mu+2\eta}.$$

Proof. Let $f \in W_{\Sigma}(\tau,\mu,\eta;x)$. From Definition 1.2, for some analytic functions Φ, Ψ such that $\Phi(0)=\Psi(0)=0$ and $|\Phi(z)|<1$, $|\Psi(w)|<1$ for all $z,w\in\Delta$, we can write

$$1+\frac{1}{\tau}\left((1-\mu+2\eta)\frac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta z f''(z)-1\right)=\mathcal{G}_{\{L_{p,q,n}(x)\}}(\Phi(z))-1,$$

$$1+\frac{1}{\tau}\left((1-\mu+2\eta)\frac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta w g''(w)-1\right)=\mathcal{G}_{\{L_{p,q,n}(x)\}}(\Psi(w))-1,$$

or equivalently

$$\begin{aligned}&1+\frac{1}{\tau}\left((1-\mu+2\eta)\tfrac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta z f''(z)-1\right)\\ &=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\Phi(z)+L_{p,q,2}(x)\Phi^2(z)+\cdots,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&1+\frac{1}{\tau}\left((1-\mu+2\eta)\tfrac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta w g''(w)-1\right)\\ &=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\Psi(w)+L_{p,q,2}(x)\Psi^2(w)+\cdots.\end{aligned} \tag{2.2}$$

From the equalities (2.1) and (2.2), we obtain that

$$\begin{aligned}&1+\frac{1}{\tau}\left((1-\mu+2\eta)\tfrac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta z f''(z)-1\right)\\ &=1+L_{p,q,1}(x)t_1 z+\left[L_{p,q,1}(x)t_2+L_{p,q,2}(x)t_1^2\right]z^2+\cdots,\end{aligned} \tag{2.3}$$

and

$$1+\frac{1}{\tau}\left((1-\mu+2\eta)\tfrac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta wg''(w)-1\right) \tag{2.4}$$
$$=1+L_{p,q,1}(x)s_1w+\left[L_{p,q,1}(x)s_2+L_{p,q,2}(x)s_1^2\right]w^2+\cdots.$$

It is fairly well-known that if

$$|\Phi(z)|=\left|t_1z+t_2z^2+t_3z^3+\cdots\right|<1 \quad (z\in\Delta)$$

and

$$|\Psi(w)|=\left|s_1w+s_2w^2+s_3w^3+\cdots\right|<1 \quad (w\in\Delta),$$

then

$$|t_k|\leq 1 \quad \text{and} \quad |s_k|\leq 1 \quad (k\in\mathbf{N}). \tag{2.5}$$

Thus, upon comparing the corresponding coefficients in (2.3) and (2.4), we have

$$\frac{1}{\tau}(1+\mu)a_2=L_{p,q,1}(x)t_1, \tag{2.6}$$

$$\frac{1}{\tau}(1+2\mu+2\eta)a_3=L_{p,q,1}(x)t_2+L_{p,q,2}(x)t_1^2, \tag{2.7}$$

$$-\frac{1}{\tau}(1+\mu)a_2=L_{p,q,1}(x)s_1 \tag{2.8}$$

and

$$\frac{1}{\tau}(1+2\mu+2\eta)\left(2a_2^2-a_3\right)=L_{p,q,1}(x)s_2+L_{p,q,2}(x)s_1^2. \tag{2.9}$$

From the equations (2.6) and (2.8), we can easily see that

$$t_1=-s_1, \tag{2.10}$$

$$\frac{2}{\tau^2}(1+\mu)^2a_2^2=L_{p,q,1}^2(x)\left(t_1^2+s_1^2\right). \tag{2.11}$$

If we add (2.7) to (2.9), we get

$$\frac{2}{\tau}(1+2\mu+2\eta)a_2^2 = L_{p,q,1}(x)\,(t_2+s_2) + L_{p,q,2}(x)\left(t_1^2+s_1^2\right). \tag{2.12}$$

Clearly, by using (2.11) in the equality (2.12), we have

$$\frac{2\left[(1+2\mu+2\eta)\tau L_{p,q,1}^2(x) - (1+\mu)^2 L_{p,q,2}(x)\right]}{\tau^2 L_{p,q,1}^2(x)} a_2^2 = L_{p,q,1}(x)\,(t_2+s_2). \tag{2.13}$$

which gives

$$|a_2| \le \frac{|\tau|\,|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|\left[(1+2\mu+2\eta)\tau - (1+\mu)^2\right]p^2(x) - 2(1+\mu)^2 q(x)\right|}}.$$

Moreover, if we subtract (2.9) from (2.7), we obtain

$$\frac{2}{\tau}(1+2\mu+2\eta)(a_3-a_2^2) = L_{p,q,1}(x)\,(t_2-s_2) + L_{p,q,2}(x)\left(t_1^2-s_1^2\right). \tag{2.14}$$

Then, in view of (2.10) and (2.11), (2.14) becomes

$$a_3 = \frac{\tau^2 L_{p,q,1}^2(x)\left(t_1^2+s_1^2\right)}{2(1+\mu)^2} + \frac{\tau L_{p,q,1}(x)\,(t_2-s_2)}{2(1+2\mu+2\eta)}.$$

It is seen from (1.1) and (2.5) that

$$|a_3| \le \frac{|\tau|^2 p^2(x)}{(1+\mu)^2} + \frac{|\tau|\,|p(x)|}{1+2\mu+2\eta}.$$

□

Corollary 2.1. *Let f given by (1.2) be in the class $W_{\Sigma}(\tau,\eta;x)$. Then*

$$|a_2| \leq \frac{|\tau|\,|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|\left[3(1+2\eta)\tau-4\,(1+\eta)^2\right]p^2(x)-8\,(1+\eta)^2\,q(x)\right|}}$$

and

$$|a_3| \leq \frac{|\tau|^2\,p^2(x)}{4\,(1+\eta)^2}+\frac{|\tau|\,|p(x)|}{3(1+2\eta)}.$$

Corollary 2.2. *Let f given by (1.2) be in the class $W_{\Sigma}(\tau,\mu;x)$. Then*

$$|a_2| \leq \frac{|\tau|\,|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|\left[(1+2\mu)\tau-(1+\mu)^2\right]p^2(x)-2\,(1+\mu)^2\,q(x)\right|}}$$

and

$$|a_3| \leq \frac{|\tau|^2\,p^2(x)}{(1+\mu)^2}+\frac{|\tau|\,|p(x)|}{1+2\mu}.$$

Corollary 2.3. *Let f given by (1.2) be in the class $W_{\Sigma}(\tau;x)$. Then*

$$|a_2| \leq \frac{|\tau|\,|p(x)|\sqrt{|p(x)|}}{\sqrt{|(3\tau-4)p^2(x)-8q(x)|}}$$

and

$$|a_3| \leq \frac{|\tau|^2\,p^2(x)}{4}+\frac{|\tau|\,|p(x)|}{3}.$$

3. Fekete-Szeg problem

The classical Fekete-Szeg inequality, presented by means of Loewner's method, for the coefficients of $f \in S$ is

$$\left|a_3-\xi a_2^2\right| \leq 1+2\exp(-2\xi/(1-\xi)) \quad \text{for } \xi \in [0,1).$$

As $\xi \to 1^-$, we have the elementary inequality $|a_3-a_2^2| \leq 1$. Moreover, the coefficient functional

$$\Gamma_{\xi}(f)=a_3-\xi a_2^2$$

on the normalized analytic functions f in the unit disk Δ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Gamma_\xi(f)$ is called the Fekete-Szeg problem, see [5].

In this section, we aim to provide Fekete-Szeg inequalities for functions in the class $T_\Sigma^n(\tau;x)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let f given by (1.2) be in the class $W_\Sigma(\tau,\mu,\eta;x)$ and $\xi\in\mathbf{R}$. Then*

$$\left|a_3-\xi a_2^2\right|\le\begin{cases}\dfrac{|p(x)|}{(1+2\mu+2\eta)\,|\tau|}, \\[2ex] |1-\xi|\le\left|\dfrac{1}{\tau^2}-\dfrac{(1+\mu)^2}{(1+2\mu+2\eta)\tau^3}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right| \\[3ex] \dfrac{|\tau|^2\left|p^3(x)\right||1-\xi|}{\left|\left[(1+2\mu+2\eta)\tau-(1+\mu)^2\right]p^2(x)-2\left(1+\mu\right)^2q(x)\right|}, \\[3ex] |1-\xi|\ge\left|\dfrac{1}{\tau^2}-\dfrac{(1+\mu)^2}{(1+2\mu+2\eta)\tau^3}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right|\end{cases}$$

Proof. From (2.13) and (2.14)

$$\begin{aligned}a_3-\xi a_2^2 &=\frac{\tau^2L_{p,q,1}^3(x)\,(1-\xi)\,(t_2+s_2)}{2\left[(1+2\mu+2\eta)\tau L_{p,q,1}^2(x)-(1+\mu)^2L_{p,q,2}(x)\right]}\\ &+\frac{\tau L_{p,q,1}(x)\,(t_2-s_2)}{2(1+2\mu+2\eta)}\\ &=L_{p,q,1}(x)\left[\left(K(\xi,x)+\frac{1}{2(1+2\mu+2\eta)\tau}\right)t_2\right.\\ &\left.+\left(K(\xi,x)-\frac{1}{2(1+2\mu+2\eta)\tau}\right)s_2\right]\end{aligned}$$

where

$$K(\xi,x)=\frac{\tau^2L_{p,q,1}^2(x)\,(1-\xi)}{2\left[(1+2\mu+2\eta)\tau L_{p,q,1}^2(x)-(1+\mu)^2L_{p,q,2}(x)\right]}.$$

Along the way, in view of (1.1), we conclude that

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \dfrac{|p(x)|}{(1+2\mu+2\eta)\,|\tau|}, & 0 \leq |K(\xi, x)| \leq \dfrac{1}{2(1+2\mu+2\eta)\,|\tau|} \\[2ex] 2\,|p(x)|\,|K(\xi, x)|\,, & |K(\xi, x)| \geq \dfrac{1}{2(1+2\mu+2\eta)\,|\tau|} \end{cases}$$

□

Corollary 3.1. *Let f given by (1.2) be in the class $W_\Sigma(\tau, \eta; x)$ and $\xi \in \mathbf{R}$. Then*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \dfrac{|p(x)|}{3(1+2\eta)\,|\tau|}, \\[2ex] |1-\xi| \leq \left|\dfrac{1}{\tau^2} - \dfrac{4\,(1+\eta)^2}{3(1+2\eta)\tau^3}\left(1 + \dfrac{2q(x)}{p^2(x)}\right)\right| \\[2ex] \dfrac{|\tau|^2\,|p^3(x)|\,|1-\xi|}{\left|\left[3(1+2\eta)\tau - 4\,(1+\eta)^2\right]p^2(x) - 8\,(1+\eta)^2\,q(x)\right|}, \\[2ex] |1-\xi| \geq \left|\dfrac{1}{\tau^2} - \dfrac{4\,(1+\eta)^2}{3(1+2\eta)\tau^3}\left(1 + \dfrac{2q(x)}{p^2(x)}\right)\right| \end{cases}.$$

Corollary 3.2. *Let f given by (1.2) be in the class $W_\Sigma(\tau, \mu; x)$ and $\xi \in \mathbf{R}$. Then*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \dfrac{|p(x)|}{(1+2\mu)\,|\tau|}, \\[2ex] |1-\xi| \leq \left|\dfrac{1}{\tau^2} - \dfrac{(1+\mu)^2}{(1+2\mu)\tau^3}\left(1 + \dfrac{2q(x)}{p^2(x)}\right)\right| \\[2ex] \dfrac{|\tau|^2\,|p^3(x)|\,|1-\xi|}{\left|\left[(1+2\mu)\tau - (1+\mu)^2\right]p^2(x) - 2\,(1+\mu)^2\,q(x)\right|}, \\[2ex] |1-\xi| \geq \left|\dfrac{1}{\tau^2} - \dfrac{(1+\mu)^2}{(1+2\mu)\tau^3}\left(1 + \dfrac{2q(x)}{p^2(x)}\right)\right| \end{cases}$$

Corollary 3.3. *Let f given by (1.2) be in the class $W_\Sigma(\tau; x)$ and $\xi \in \mathbf{R}$. Then*

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \dfrac{|p(x)|}{3\,|\tau|}, \\ |1-\xi| \leq \left|\dfrac{1}{\tau^2}-\dfrac{4}{3\tau^3}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right| \\ \dfrac{|\tau|^2\left|p^3(x)\right||1-\xi|}{\left|(3\tau-4)p^2(x)-8q(x)\right|}, \\ |1-\xi| \geq \left|\dfrac{1}{\tau^2}-\dfrac{4}{3\tau^3}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right| \end{cases}$$

If we choose $\xi=1$, we get the next corollaries.

Corollary 3.4. *If* $f \in W_\Sigma(\tau,\mu,\eta;x)$, *then* $\left|a_3-a_2^2\right| \leq \dfrac{|p(x)|}{(1+2\mu+2\eta)\,|\tau|}$.

Corollary 3.5. *If* $f \in W_\Sigma(\tau,\eta;x)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|p(x)|}{3(1+2\eta)\,|\tau|}.$$

Corollary 3.6. *If* $f \in W_\Sigma(\tau,\mu;x)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|p(x)|}{(1+2\mu)\,|\tau|}.$$

Corollary 3.7. *If* $f \in W_\Sigma(\tau;x)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|p(x)|}{3\,|\tau|}.$$

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