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Strongly convexity on fractal sets and some inequalitiess

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Abstract:

We introduce a generalization of the concept of a strongly convex function on a fractal set, study some algebraic properties and establish Jensen-type and Hermite-Hadamard-type inequalities.

Keywords: Convex function; Generalized convex function; Strongly convex function; Fractal set.

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1. Introduction and preliminaries

The concept of convex function and its generalizations have been the subject of numerous studies that have provided interesting results in some branches related to mathematics, such as geometric functional analysis, mathematical economics, convex analysis, nonlinear optimization, linear programming, theory of control and dynamic systems. On the other hand, the concept of local fractional calculus (also called fractal calculus) introduced by Kolwankar and Gangal [3] has received considerable attention for its application in non-differentiable problems of science and engineering. Motivated by these applications, in 2012, Yang [11] established the analysis of local fractional functions on fractal sets systematically, which included local fractional calculus and the monotonicity of functions. Recently, the fractal calculus has been used by Mo, Sui and Yu [6] to introduce a generalization of the concept of convex function on fractal sets and to establish inequalities of Jensen and Hermite-Hadamard for generalized convex functions. Similarly, Sun [10] introduced the concept of generalized harmonically convex function on fractal sets and established the respective Hermite-Hadamard inequalities for this class of functions. This work continues in the same line of investigation of the two works mentioned above, but in our case we introduce a generalization of the concept of a strongly convex function on a fractal set, study some algebraic properties and establish Jensen-type and Hermite-Hadamard-type inequalities.

Strongly convex functions have been introduced by Polyak in 1966 [8] and has been studied and generalized by different authors [1, 4, 5, 9]. Recall the definition of this class of functions.

Definition 1.1. *Let $I \subset \mathbf{R}$ be an interval and c be a positive real number. A function $f : I \to \mathbf{R}$ is said to be strongly convex with modulus c if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2,$$

for all $x, y \in I$ and $t \in [0,1]$.

Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics. [7, 9]

Recenty, the theory of Yang's fractional sets [11] was introduced as follows. For $0 < \alpha \leq 1$, we have following α-type set of element sets:

$\mathbf{Z}^\alpha = \{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$ (integer numbers α-type).

$\mathbf{Q}^\alpha = \left\{ m^\alpha = \left(\frac{p}{q}\right)^\alpha : p,q \in \mathbf{Z}, q \neq 0 \right\}$ (rationals numbers α-type).
$\mathbf{J}^\alpha = \left\{ m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p,q \in \mathbf{Z}, q \neq 0 \right\}$ (irrational numbers α-type).
$\mathbf{R}^\alpha = \mathbf{Q}^\alpha \cup \mathbf{J}^\alpha$ (real line numbers $\alpha - type$).

We call fractal set to $\mathbf{R}^\alpha$ and any subset of it. The following facts are found in [2], [11] and [12].

If a^α, b^α and c^α belong to the set $\mathbf{R}^\alpha$ of real line numbers, then one has the following:

1. $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set $\mathbf{R}^\alpha$.
2. $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$.
3. $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$.
4. $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$.
5. $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$.
6. $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$.
7. $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ y $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

It is important to note that in this theory the number $(a^2)^\alpha \in \mathbf{R}^\alpha$ will be represented by $a^{2\alpha}$.

Now we introduce some basic definitions about the local factional calculus.

Definition 1.2. *[11] A non-differentiable function $f : \mathbf{R} \to \mathbf{R}^\alpha$, $x \to f(x)$ is called local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbf{R}$. If a function f is local fractional continuous on an interval I, we denote $f \in C_\alpha(I)$.

Definition 1.3. *[11] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha)(f(x) - f(x_0))$ and Γ is the familiar Gamma function.

Let $f^{(\alpha)}(x) = D_x^\alpha f(x)$. If there exists $f^{((k+1)\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbf{R}$, then we denote $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$.

Definition 1.4. *[11] Let $f \in C_\alpha[a,b]$. The local fractional integral of f on the interval $[a,b]$ of order α (denoted by ${}_aI_b^{(\alpha)} f$) is defined by*

$$ {}_aI_b^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, $$

with $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_{N-1}\}$ and $\Delta t_j = t_{j+1} - t_j$ for $j = 0, 1, \ldots,$ $N-1$, where $a = t_0 < t_1 < \cdots < t_i < \cdots < t_{N-1} < t_N = b$ is a partition of the interval $[a,b]$.

Here, it follows that ${}_aI_b^{(\alpha)} f = 0$ if $a = b$ and ${}_aI_b^{(\alpha)} f = -\,{}_bI_a^{(\alpha)} f$ if $a < b$. If ${}_aI_x^{(\alpha)} f$ there exits for any $x \in [a,b]$, then it is denoted by $f \in I_x^{(\alpha)}[a,b]$.

In 2014, H. Mo et al. [6] used the local fractional calculus to introduce the following generalized convex function.

Definition 1.5. *[6] Let $f : I \to \mathbf{R}^\alpha$. For any $x, y \in I$ and $t \in [0,1]$, if the following inequality*

$$ f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t)^\alpha f(y), $$

holds, then f is called a generalized convex function on I.

We will denote by $GC_\alpha(I)$ to the set of the generalized convex functions on I, that is to say,

$$ GC_\alpha(I) = \{f : I \to \mathbf{R}^\alpha | f \text{ is a generalized convex function on } I\}. $$

2. Main Results

In this section we introduce the concept of a strongly convex generalized function on a fractal set with modulo c, study some algebraic properties and establish Jensen-type and Hermite-Hadamard-type inequalities.

Definition 2.1. *Let $I \subset \mathbf{R}$ be an interval and $c \in \mathbf{R}_+$. A function $f : I \to \mathbf{R}^\alpha$ is called generalized strongly convex with modulus c if*

$$ f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t)^\alpha f(y) - c^\alpha t^\alpha (1-t)^\alpha (x-y)^{2\alpha}, $$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 2.2. Note that for particular cases of the numbers $0 < \alpha \leq 1$ and $c \in \mathbf{R}_+$, we recover well known classical concepts of convex functions as is shown.

1. *If $\alpha = 1$ then the generalized strongly convex functions are the strongly convex functions.*
2. *If $c = 0$ then the generalized strongly convex functions are the generalized convex functions.*
3. *If $\alpha = 1$ and $c = 0$ then the generalized strongly convex functions are the convex functions.*

The family of all generalized strongly convex functions with modulus c is denoted by $GSC^c_\alpha(I)$; that is,

$$GSC^c_\alpha(I) = \{f : I \to \mathbf{R}^\alpha | f \text{ is generalized strongly convex with modulus } c\}.$$

Note that if $f \in GSC^c_\alpha(I)$ and $f(0) = 0^\alpha$ then $f(tx) \leq t^\alpha f(x)$. Also, if $f \in GSC^c_\alpha(I)$ then $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2^\alpha} - c^\alpha(\frac{x-y}{2})^{2\alpha}$.

Theorem 2.3. *If $f, g \in GSC^{\frac{c}{2}}_\alpha(I)$, then $f + g \in GSC^c_\alpha(I)$.*

Proof. Let $f, g \in GSCC^{\frac{c}{2}}_\alpha(I)$, $c > 0$, $x, y \in I$ and $t \in [0, 1]$. Then

$$\begin{aligned}
(f+g)(tx+(1-t)y) &= f(tx+(1-t)y) + g(tx+(1-t)y) \\
&\leq t^\alpha f(x) + (1-t)^\alpha f(y) - \frac{c^\alpha}{2^\alpha} t^\alpha (1-t)^\alpha (x-y)^{2\alpha} \\
&\quad + t^\alpha g(x) + (1-t)^\alpha g(y) - \frac{c^\alpha}{2^\alpha} t^\alpha (1-t)^\alpha (x-y)^{2\alpha} \\
&= t^\alpha \left(f(x) + g(x)\right) + (1-t)^\alpha \left(f(y) + g(y)\right) \\
&\quad - 2^\alpha \frac{c^\alpha}{2^\alpha} t^\alpha (1-t)^\alpha (x-y)^{2\alpha} \\
&= t^\alpha (f+g)(x) + (1-t)^\alpha (f+g)(y) \\
&\quad - c^\alpha t^\alpha (1-t)^\alpha (x-y)^{2\alpha}.
\end{aligned}$$

So $f + g \in GSC^c_\alpha(I)$. □

Theorem 2.4. *If $f \in GSC^{\frac{c}{\lambda}}_\alpha(I)$, then $\lambda^\alpha f \in GSC^c_\alpha(I)$, when $\lambda \in \mathbf{R}^+$.*

Proof. If $x, y \in I$ and $t \in [0,1]$, then

$$\begin{aligned}
(\lambda^\alpha f)(tx+(1-t)y) &= \lambda^\alpha f\left(tx+(1-t)y\right) \\
&\leq \lambda^\alpha \left[t^\alpha f(x)+(1-t)^\alpha f(y)-\frac{c^\alpha}{\lambda^\alpha}t^\alpha(1-t)^\alpha(x-y)^{2\alpha}\right] \\
&= t^\alpha \lambda^\alpha f\left(x\right)+(1-t)^\alpha \lambda^\alpha f\left(y\right)-c^\alpha t^\alpha(1-t)^\alpha(x-y)^{2\alpha} \\
&= t^\alpha \left(\lambda^\alpha f\right)(x)+(1-t)^\alpha \left(\lambda^\alpha f\right)(y) \\
&\quad -c^\alpha t^\alpha(1-t)^\alpha(x-y)^{2\alpha},
\end{aligned}$$

and hence $\lambda^\alpha f \in GSC^c_\alpha(I)$. □

Theorem 2.5. *If $f_n : I \longrightarrow \mathbf{R}^\alpha$, $n \in \mathbf{N}$, is a sequence of generalized strongly convex functions with modulus c converging pointwise to a function $f : I \longrightarrow \mathbf{R}^\alpha$, then $f \in GSC^c_\alpha(I)$.*

Proof. Let $x, y \in I$, $t \in [0,1]$ and $\lim_{n\to\infty} f_n(x) = f(x)$, then

$$\begin{aligned}
f(tx+(1-t)y) &= \lim_{n\to\infty} f_n(tx+(1-t)y) \\
&\leq \lim_{n\to\infty} \left(t^\alpha f_n(x)+(1-t)^\alpha f_n(y)-c^\alpha t^\alpha(1-t)^\alpha(x-y)^{2\alpha}\right) \\
&= t^\alpha \lim_{n\to\infty} f_n(x)+(1-t)^\alpha \lim_{n\to\infty} f_n(y) \\
&\quad -\lim_{n\to\infty} c^\alpha t^\alpha(1-t)^\alpha(x-y)^{2\alpha} \\
&= t^\alpha f(x)+(1-t)^\alpha f(y)-c^\alpha t^\alpha(1-t)^\alpha(x-y)^{2\alpha};
\end{aligned}$$

that is, $f \in GSC^c_\alpha(I)$. □

Remark 2.6. If $x, y \in I$, $t \in [0,1]$ and $\overline{x} = tx+(1-t)y$, then we have

$$t^\alpha(1-t)^\alpha(x-y)^{2\alpha} = t^\alpha(x-\overline{x})^\alpha+(1-t)^\alpha(y-\overline{x})^\alpha.$$

Hence, a function $f : I \to \mathbf{R}^\alpha$ is generalized strongly convex with modulus c if and only if

$$f(tx+(1-t)y) \leq t^\alpha f(x)+(1-t)^\alpha f(y)-c^\alpha t^\alpha(x-\overline{x})^\alpha+(1-t)^\alpha(y-\overline{x})^\alpha,$$

for all $x, y \in I$ and $t \in [0,1]$. This fact serve as motivation to establish a generalization of the version of the discrete Jensen-type inequality given in [5].

Theorem 2.7 (generalized Jensen-type inequality). *Assume that $f \in GSC_{\alpha}^{c}(I)$. Then for any $x_i \in I$ and $t_i \in [0,1]$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{n} t_i = 1$ and $\bar{x} = \sum_{i=1}^{n} t_i x_i$, we have*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i^{\alpha} f(x_i) - c^{\alpha} \sum_{i=1}^{n} t_i^{\alpha} (x_i - \overline{x})^{2\alpha}.$$

Proof. The proof is by induction on n. When $n = 2$, we have

$$\begin{aligned} f\left(\sum_{i=1}^{2} t_i x_i\right) &= f(t_1 x_1 + t_2 x_2) = f(t_1 x_1 + (1 - t_1) x_2) \\ &\leq t_1^{\alpha} f(x_2) + (1 - t_1)^{\alpha} f(x_2) - c^{\alpha} \left(t_1^{\alpha} (x_1 - \overline{x})^{2\alpha} + (1 - t_1)^{\alpha} (x_2 - \overline{x})^{2\alpha}\right) \\ &= t_1^{\alpha} f(x_2) + (1 - t_1)^{\alpha} f(x_2) - c^{\alpha} \left(t_1^{\alpha} (x_1 - \overline{x})^{2\alpha} + t_2^{\alpha} (x_2 - \overline{x})^{2\alpha}\right) \\ &= \sum_{i=1}^{2} t_i^{\alpha} f(x_i) - c^{\alpha} \sum_{i=1}^{2} t_i^{\alpha} (x_i - \overline{x})^{2\alpha}, \end{aligned}$$

where $\overline{x} = \sum_{i=1}^{2} t_i x_i$.

Assume that for $n = k$ the inequality is also true, i.e. for any $x_1, x_2, ..., x_k \in I$ and $t_i \in [0,1]$ $(i = 1, 2, ..., k)$ with $\sum_{i=1}^{k} t_i = 1$, we have

$$f\left(\sum_{i=1}^{k} t_i x_i\right) \leq \sum_{i=1}^{k} t_i^{\alpha} f(x_i) - c^{\alpha} \sum_{i=1}^{k} t_i^{\alpha} (x_i - \overline{x})^{2\alpha},$$

where $\overline{x} = \sum_{i=1}^{k} t_i x_i$.

Now let us verify that for $n = k+1$ the inequality is true. If $x_1, x_2, ..., x_k, x_{k+1} \in I$ and $t_i \in [0,1]$ $(i = 1, 2, ..., k, k+1)$ with $\sum_{i=1}^{k+1} t_i = 1$, then let $\overline{x} = \sum_{i=1}^{k+1} t_i x_i$ and $\overline{y} = \sum_{i=1}^{k} \lambda_i x_i$, where $\lambda_i = \dfrac{t_i}{1 - t_{k+1}}$. By the generalized strongly convexity of f, we have

$$
\begin{aligned}
f\left(\sum_{i=1}^{k+1} t_i x_i\right) &= f\left(t_{k+1}x_{k+1} + (1-t_{k+1})\overline{y}\right) \\
&\leq t_{k+1}^{\alpha} f\left(x_{k+1}\right) + \left(1-t_{k+1}\right)^{\alpha} f\left(\overline{y}\right) \\
&\quad -c^{\alpha}\left[t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} + \left(1-t_{k+1}\right)^{\alpha}\left(\overline{y}-\overline{x}\right)^{2\alpha}\right].
\end{aligned}
$$

Since $\sum_{i=1}^{k} \lambda_i = 1$, then by inductive hypothesis it follows that

$$
\begin{aligned}
f\left(\sum_{i=1}^{k+1} t_i x_i\right) &\leq t_{k+1}^{\alpha} f\left(x_{k+1}\right) + \left(1-t_{k+1}\right)^{\alpha}\left[\sum_{i=1}^{k} \lambda_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k} \lambda_i^{\alpha}\left(x_i-\overline{y}\right)^{2\alpha}\right] \\
&\quad -c^{\alpha}\left[t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} + \left(1-t_{k+1}\right)^{\alpha}\left(\overline{y}-\overline{x}\right)^{2\alpha}\right] \\
&= t_{k+1}^{\alpha} f\left(x_{k+1}\right) + \sum_{i=1}^{k} t_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k} t_i^{\alpha}\left(x_i-\overline{y}\right)^{2\alpha} \\
&\quad -c^{\alpha} t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} - c^{\alpha}\left(\sum_{i=1}^{k} t_i\right)^{\alpha}\left(\overline{y}-\overline{x}\right)^{2\alpha} \\
&= \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k} t_i^{\alpha}\left[\left(x_i-\overline{y}\right)^{2\alpha} + \left(\overline{y}-\overline{x}\right)^{2\alpha}\right] \\
&\quad -c^{\alpha} t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} \\
&= \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k} t_i^{\alpha}\left[\left(x_i-\overline{x}\right)^{2\alpha} - 2^{\alpha}\left(x_i-\overline{y}\right)^{\alpha}\left(\overline{y}-\overline{x}\right)^{\alpha}\right] \\
&\quad -c^{\alpha} t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} \\
&= \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k} t_i^{\alpha}\left[\left(x_i-\overline{x}\right)^{2\alpha} - 2^{\alpha}\left(x_i-\overline{y}\right)^{\alpha}\frac{t_{k+1}^{\alpha}}{\left(1-t_{k+1}\right)^{\alpha}}\right. \\
&\quad \left.\left(\overline{x}-x_{k+1}\right)^{\alpha}\right] - c^{\alpha} t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} \\
&= \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k} t_i^{\alpha}\left(x_i-\overline{x}\right)^{2\alpha} - c^{\alpha} t_{k+1}^{\alpha}\left(x_{k+1}-\overline{x}\right)^{2\alpha} \\
&\quad +2^{\alpha} c^{\alpha} t_{k+1}^{\alpha}\left(\overline{x}-x_{k+1}\right)^{\alpha}\sum_{i=1}^{k} \lambda_i^{\alpha}\left(x_i-\overline{y}\right)^{\alpha} \\
&= \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right) - c^{\alpha}\sum_{i=1}^{k+1} t_i^{\alpha}\left(x_i-\overline{x}\right)^{2\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+2^{\alpha}c^{\alpha}t_{k+1}^{\alpha}\left(\overline{x}-x_{k+1}\right)^{\alpha}\left[\left(\sum_{i=1}^{k}\lambda_i x_i\right)^{\alpha}-\overline{y}^{\alpha}\sum_{i=1}^{k}\lambda_i^{\alpha}\right] \\
= \quad & \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right)-c^{\alpha}\sum_{i=1}^{k+1} t_i^{\alpha}\left(x_i-\overline{x}\right)^{2\alpha}+2^{\alpha}c^{\alpha}t_{k+1}^{\alpha}\left(\overline{x}-x_{k+1}\right)^{\alpha}\left(\overline{y}^{\alpha}-\overline{y}^{\alpha}1^{\alpha}\right) \\
= \quad & \sum_{i=1}^{k+1} t_i^{\alpha} f\left(x_i\right)-c^{\alpha}\sum_{i=1}^{k+1} t_i^{\alpha}\left(x_i-\overline{x}\right)^{2\alpha}.
\end{aligned}
$$

Hence the inequality holds for n = k + 1, and thus for all n. $\square$

Theorem 2.8. *A function $f : I \to \mathbf{R}^{\alpha}$ is generalized strongly convex with modulus c if and only if the function $g : I \to \mathbf{R}^{\alpha}$ defined by $g(x) = f(x) - c^{\alpha}x^{2\alpha}$ is generalized convex.*

Proof. Suppose that $f \in GSC_{\alpha}^{c}(I)$, then

$$
\begin{aligned}
g(tx+(1-t)y) &= f(tx+(1-t)y)-c^{\alpha}(tx+(1-t)y)^{2\alpha} \\
&\le t^{\alpha}f(x)+(1-t)^{\alpha}f(y)-c^{\alpha}t^{\alpha}(1-t)^{\alpha}(x-y)^{2\alpha}-c^{\alpha}(tx+(1-t)y)^{2\alpha} \\
&= t^{\alpha}f(x)+(1-t)^{\alpha}f(y)-c^{\alpha}t^{\alpha}(1-t)^{\alpha}(x^{2\alpha}-2^{\alpha}x^{\alpha}y^{\alpha}+y^{2\alpha}) \\
&\quad -c^{\alpha}(t^{2\alpha}x^{2\alpha}+2^{\alpha}t^{\alpha}x^{\alpha}(1-t)^{\alpha}y^{\alpha}+(1-t)^{2\alpha}y^{2\alpha}) \\
&= t^{\alpha}f(x)+(1-t)^{\alpha}f(y)-c^{\alpha}t^{\alpha}(x^{2\alpha}-2^{\alpha}x^{\alpha}y^{\alpha}+y^{2\alpha}-t^{\alpha}x^{2\alpha}+2^{\alpha}t^{\alpha}x^{\alpha}y^{\alpha} \\
&\quad -t^{\alpha}y^{2\alpha})-c^{\alpha}(t^{2\alpha}x^{2\alpha}+2^{\alpha}t^{\alpha}x^{\alpha}y^{\alpha}-2^{\alpha}t^{2\alpha}x^{\alpha}y^{\alpha}+y^{2\alpha}-2^{\alpha}t^{\alpha}y^{2\alpha}+t^{2\alpha}y^{2\alpha}) \\
&= t^{\alpha}f(x)+(1-t)^{\alpha}f(y)-c^{\alpha}t^{\alpha}x^{2\alpha}+2^{\alpha}c^{\alpha}t^{\alpha}x^{\alpha}y^{\alpha}-c^{\alpha}t^{\alpha}y^{2\alpha}+c^{\alpha}t^{2\alpha}x^{2\alpha} \\
&\quad -2^{\alpha}c^{\alpha}t^{2\alpha}x^{\alpha}y^{\alpha}+c^{\alpha}t^{2\alpha}y^{2\alpha}-c^{\alpha}t^{2\alpha}x^{2\alpha}-2^{\alpha}c^{\alpha}t^{\alpha}x^{\alpha}y^{\alpha}+2^{\alpha}c^{\alpha}t^{2\alpha}x^{\alpha}y^{\alpha} \\
&\quad -c^{\alpha}y^{2\alpha}+2^{\alpha}c^{\alpha}t^{\alpha}y^{2\alpha}-c^{\alpha}t^{2\alpha}y^{2\alpha} \\
&= \left(t^{\alpha}f(x)-c^{\alpha}t^{\alpha}x^{2\alpha}\right)+(1-t)^{\alpha}f(y)-c^{\alpha}t^{\alpha}y^{2\alpha}-c^{\alpha}y^{2\alpha}+2^{\alpha}c^{\alpha}t^{\alpha}y^{2\alpha} \\
&= t^{\alpha}\left(f(x)-c^{\alpha}x^{2\alpha}\right)+(1-t)^{\alpha}f(y)-c^{\alpha}y^{2\alpha}+c^{\alpha}t^{\alpha}y^{2\alpha} \\
&= t^{\alpha}\left(f(x)-c^{\alpha}x^{2\alpha}\right)+(1-t)^{\alpha}f(y)-(1-t)^{\alpha}c^{\alpha}y^{2\alpha} \\
&= t^{\alpha}\left(f(x)-c^{\alpha}x^{2\alpha}\right)+(1-t)^{\alpha}(f(y)-c^{\alpha}y^{2\alpha}) \\
&= t^{\alpha}g(x)+(1-t)^{\alpha}g(y),
\end{aligned}
$$

for all $x, y \in I$ and $t \in [0,1]$. This shows that $g \in GC_{\alpha}^{c}(I)$.

Conversely, assume that $g \in GC_\alpha^c(I)$. Then

$$g(tx + (1-t)y) \leq t^\alpha g(x) + (1-t)^\alpha g(y),$$

for all $x, y \in I$ and $t \in [0,1]$; that is,

$$f(tx+(1-t)y)-c^\alpha(tx+(1-t)y)^{2\alpha} \leq t^\alpha \left(f(x) - c^\alpha x^{2\alpha}\right)+(1-t)^\alpha(f(y)-c^\alpha y^{2\alpha}),$$

for all $x, y \in I$ and $t \in [0,1]$. From the above formula it follows that

$$\begin{aligned}
f(tx+(1-t)y) &\leq t^\alpha f(x) - c^\alpha t^\alpha x^{2\alpha} + (1-t)^\alpha f(y) - c^\alpha(1-t)^\alpha y^{2\alpha} \\
&\quad + c^\alpha(t^{2\alpha}x^{2\alpha} + 2^\alpha t^\alpha(1-t)^\alpha x^\alpha y^\alpha + (1-t)^{2\alpha}y^{2\alpha}) \\
&= t^\alpha f(x) + (1-t)^\alpha f(y) - c^\alpha t^\alpha x^{2\alpha} - c^\alpha(1-t)^\alpha y^{2\alpha} \\
&\quad + c^\alpha t^{2\alpha}x^{2\alpha} + 2^\alpha c^\alpha t^\alpha(1-t)^\alpha x^\alpha y^\alpha + c^\alpha(1-t)^{2\alpha}y^{2\alpha} \\
&= t^\alpha f(x) + (1-t)^\alpha f(y) - c^\alpha t^\alpha(1-t)^\alpha x^{2\alpha} \\
&\quad + 2^\alpha c^\alpha t^\alpha(1-t)^\alpha x^\alpha y^\alpha - c^\alpha(1-t)^\alpha(1^\alpha - (1-t)^\alpha)y^{2\alpha} \\
&= t^\alpha f(x) + (1-t)^\alpha f(y) - c^\alpha t^\alpha(1-t)^\alpha x^{2\alpha} \\
&\quad + 2^\alpha c^\alpha t^\alpha(1-t)^\alpha x^\alpha y^\alpha - c^\alpha t^\alpha(1-t)^\alpha y^{2\alpha} \\
&= t^\alpha f(x) + (1-t)^\alpha f(y) - c^\alpha t^\alpha(1-t)^\alpha(x^{2\alpha} - 2^\alpha x^\alpha y^\alpha + y^{2\alpha}) \\
&= t^\alpha f(x) + (1-t)^\alpha f(y) - c^\alpha t^\alpha(1-t)^\alpha(x-y)^{2\alpha}
\end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$. Thus, we have $f \in GSC_\alpha^c(I)$.
□

Theorem 2.9. *Let $f : I \to \mathbf{R}^\alpha$. Then $f \in GSC_\alpha^c(I)$ if and only if the inequality*

$$\frac{f(x_1) - f(x_2)}{(x_1 - x_2)^\alpha} \leq \frac{f(x_3) - f(x_2)}{(x_3 - x_2)^\alpha} - c^\alpha(x_3 - x_1)^\alpha$$

holds, for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$.

Proof. Suppose that $f \in GSC_\alpha^c(I)$, then by Theorem 2.8, this is equivalent to saying that the function $g : I \to \mathbf{R}^\alpha$ defined by $g(x) = f(x) - c^\alpha x^{2\alpha}$ is generalized convex. By [6, Theorem 8], the above is equivalent to the fact that the inequality

$$\frac{f(x_1) - c^\alpha x_1^{2\alpha} - f(x_2) + c^\alpha x_2^{2\alpha}}{(x_1 - x_2)^\alpha} \leq \frac{f(x_3) - c^\alpha x_3^{2\alpha} - f(x_2) + c^\alpha x_2^{2\alpha}}{(x_3 - x_2)^\alpha}$$

holds, for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$. Equivalently, we have

$$\begin{aligned}\frac{f(x_1)-f(x_2)}{(x_1-x_2)^\alpha} &\leq \frac{f(x_3)-f(x_2)}{(x_3-x_2)^\alpha} - c^\alpha\left[\frac{(x_3^\alpha-x_2^\alpha)(x_3^\alpha+x_2^\alpha)}{(x_3^\alpha-x_2^\alpha)} - \frac{(x_2^\alpha-x_1^\alpha)(x_2^\alpha+x_1^\alpha)}{(x_2^\alpha-x_1^\alpha)}\right] \\ &= \frac{f(x_3)-f(x_2)}{(x_3-x_2)^\alpha} - c^\alpha(x_3-x_1)^\alpha.\end{aligned}$$

□

Remark 2.10. Proceeding as in the proof of Theorem 2.4, it is shown that $f \in GSC^c_\alpha(I)$ if and only if

$$\begin{aligned}\frac{f(x_2)-f(x_1)}{(x_2-x_1)^\alpha} - c^\alpha(x_3-x_1)^\alpha &\leq \frac{f(x_2)-f(x_1)}{(x_2-x_1)^\alpha} \leq \frac{f(x_3)-f(x_1)}{(x_3-x_1)^\alpha} - c^\alpha(x_3-x_2)^\alpha \\ &\leq \frac{f(x_3)-f(x_1)}{(x_3-x_1)^\alpha} \leq \frac{f(x_3)-f(x_2)}{(x_3-x_2)^\alpha} - c^\alpha(x_2-x_1)^\alpha \\ &\leq \frac{f(x_3)-f(x_2)}{(x_3-x_2)^\alpha},\end{aligned}$$

for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$.

Theorem 2.11 (generalized Hermite-Hadamard-type inequality). *If $f \in I_x^{(\alpha)}[a,b]$ and $f \in GSC^c_\alpha[a,b]$, then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) - c^\alpha\left(\frac{a+b}{2}\right)^{2\alpha} &\leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\left[\,{}_aI_b^{(\alpha)}f(x) - c^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b^{3\alpha}-a^{3\alpha})\right] \\ &\leq \frac{f(a)+f(b)}{2^\alpha} - c^\alpha\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^\alpha}\right).\end{aligned}$$

Proof. Suppose that $f \in I_x^{(\alpha)}[a,b]$ and $f \in GSC^c_\alpha[a,b]$, then by Theorem 2.8, this is equivalent to saying that the function $g : [a,b] \to \mathbf{R}^\alpha$ defined by $g(x) = f(x) - c^\alpha x^{2\alpha}$ is generalized convex. By [6, Theorem 14], the above implies that the generalized Hermite-Hadamard inequality holds for g; i.e.

$$g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^{(\alpha)}g(x) \leq \frac{g(a)+g(b)}{2^\alpha}.$$

Equivalently, we have

$$f\left(\frac{a+b}{2}\right)-c^{\alpha}\left(\frac{a+b}{2}\right)^{2\alpha} \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{(\alpha)}[f(x)-c^{\alpha}x^{2\alpha}] \leq \frac{f(a)+f(b)}{2^{\alpha}}-c^{\alpha}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right).$$

Consequently,

$$f\left(\frac{a+b}{2}\right)-c^{\alpha}\left(\frac{a+b}{2}\right)^{2\alpha} \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{(\alpha)}f(x)-c^{\alpha}\frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(b^{3\alpha}-a^{3\alpha})}{(b-a)^{\alpha}} \leq \frac{f(a)+f(b)}{2^{\alpha}}-c^{\alpha}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right).$$

Thus, we have

$$f\left(\frac{a+b}{2}\right)-c^{\alpha}\left(\frac{a+b}{2}\right)^{2\alpha} \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\left[{}_aI_b^{(\alpha)}f(x)-c^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b^{3\alpha}-a^{3\alpha})\right] \leq \frac{f(a)+f(b)}{2^{\alpha}}-c^{\alpha}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right).$$

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