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I-statistical limit points and I-statistical cluster points

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Abstract:

In this paper we have extended the notion of statistical limit point as introduced by Fridy[8] to I-statistical limit point of sequences of real numbers and studied some basic properties of the set of all Istatistical limit points and I-statistical cluster points of real sequences including their interrelationship. Also introducing additive property of I-asymptotic density zero sets we establish I-statistical analogue of some completeness theorems of **R**.

Keywords: I-statistical convergence; I-statistical limit point; I-statistical cluster point; I-asymptotic density; I- statistical boundedness.

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1. Introduction:

The usual notion of convergence of real sequences was extended to statistical convergence independently by Fast[12] and Schoenberg[24] based on the notion of natural density of subsets of $\mathbf{N}$, the set of all positive integers. Since then a lot of works have been done in this area (in particular after the seminal works of Salat[22] and Fridy[7]). Following the notion of statistical convergence in [8] Fridy introduced and studied the notions of statistical limit points and statistical cluster points of real sequences. More primary work on this convergence can be found from[1, 2, 3, 9, 21, 25] where other references can be found.

The concept of statistical convergence was further extended to I-convergence by Kostyrko et. al.[13] using the notion of ideals of $\mathbf{N}$. Using this notion of ideals the concepts of statistical limit point and statistical cluster point were naturally extended to I-limit point and I-cluster point respectively by Kostyrko et. al. in [14]. More works in this line can be found in [6, 10, 11] and many others.

Recently in [4] the notion of I-statistical convergence and I-statistical cluster point of real sequences have been introduced by Das et. al. using ideals of $\mathbf{N}$, which naturally extends the notions of statistical convergence and statistical cluster point. Further works on such summability method can be found in [5, 19, 23] and many others.

In this paper using the notion of I-statistical convergence we extend the concept of statistical limit point to I-statistical limit point of sequences of real numbers and study some properties of I-statistical limit points and I-statistical cluster points of sequences of real numbers. We also study the sets of I-statistical limit points and I-statistical cluster points of sequences of real numbers and relationship between them. In section 4 of this paper we introduce the condition APIO which is similar to the APO condition used in [2] and using it in section 5 we establish I-statistical analogue of some completeness theorems of $\mathbf{R}$.

2. Basic Definitions and Notations

In this section we recall some basic definitions and notations. Throughout the paper $\mathbf{N}$ denotes the set of all positive integers, $\mathbf{R}$ denotes the set of all real numbers and x denotes the sequence $\{x_k\}_{k\in\mathbf{N}}$ of real numbers.

Definition 2.1. *[20] A subset K of $\mathbf{N}$ is said to have natural density (or asymptotic density) $d(K)$ if*

$$d(K) = \lim_{n\to\infty} \tfrac{|K(n)|}{n}$$

where $K(n) = \{j \in K : j \le n\}$ and $|K(n)|$ represents the number of elements in $K(n)$.

If $\{x_{k_j}\}_{j\in\mathbf{N}}$ is a subsequence of the sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers and $A = \{k_j : j \in \mathbf{N}\}$, then we abbreviate $\{x_{k_j}\}_{j\in\mathbf{N}}$ by $\{x\}_A$. In case $d(A) = 0$, $\{x\}_A$ is called a subsequence of natural density zero or a thin subsequence of x. On the other hand, $\{x\}_A$ is a non-thin subsequence of x if $d(A)$ does not have natural density zero i.e., if either $d(A)$ is a positive number or A fails to have natural density.

Definition 2.2. *[7] Let $x = \{x_k\}_{k\in\mathbf{N}}$ be a sequence of real numbers. Then x is said to be statistically convergent to ξ if for any $\varepsilon > 0$*

$$d(\{k : |x_k - \xi| \ge \varepsilon\}) = 0.$$

In this case we write $st-\lim_{k\to\infty} x_k = \xi$.

Definition 2.3. *[21] A sequence $x = \{x_k\}_{k\in\mathbf{N}}$ is said to be statistically bounded if there exists a compact set C in $\mathbf{R}$ such that $d(\{k : x_k \notin C\}) = 0$.*

Definition 2.4. *[8] A real number l is a statistical limit point of the sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers, if there exists a nonthin subsequence of x that converges to l.*

A real number L is an ordinary limit point of a sequence x if there is a subsequence of x that converges to L. The set of all ordinary limit points and the set of all statistical limit points of the sequence x are denoted by L_x and Λ_x respectively. Clearly $\Lambda_x \subset L_x$.

Definition 2.5. *[8] A real number y is a statistical cluster point of the sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers, if for any $\varepsilon > 0$ the set $\{k \in \mathbf{N} : |x_k - y| < \varepsilon\}$ does not have natural density zero.*

The set of all statistical cluster points of x is denoted by Γ_x. Clearly $\Gamma_x \subset L_x$.

We now recall definitions of ideal and filter in a non-empty set.

Definition 2.6. *[13] Let $X \neq \phi$. A class I of subsets of X is said to be an ideal in X provided, I satisfies the conditions:*
(i)$\phi \in I$,
(ii)$A, B \in I \Rightarrow A \cup B \in I$,
(iii)$A \in I, B \subset A \Rightarrow B \in I$.

An ideal I in a non-empty set X is called non-trivial if $X \notin I$ and $I \neq \{\phi\}$.

Definition 2.7. *[13] Let $X \neq \phi$. A non-empty class $\mathbf{F}$ of subsets of X is said to be a filter in X provided that:*
(i)$\phi \notin \mathbf{F}$,
(ii) $A, B \in \mathbf{F} \Rightarrow A \cap B \in \mathbf{F}$,
(iii)$A \in \mathbf{F}, B \supset A \Rightarrow B \in \mathbf{F}$.

Definition 2.8. *[13] Let I be a non-trivial ideal in a non-empty set X. Then the class $\mathbf{F}(I)= \{M \subset X : \exists A \in I \text{ such that } M = X \setminus A\}$ is a filter on X. This filter $\mathbf{F}(I)$ is called the filter associated with I.*

A non-trivial ideal I in $X(\neq \phi)$ is called admissible if $\{x\} \in I$ for each $x \in X$. Throughout the paper we take I as a non-trivial admissible ideal in $\mathbf{N}$ unless otherwise mentioned.

Definition 2.9. *[13] Let $x = \{x_k\}_{k \in \mathbf{N}}$ be a sequence of real numbers. Then x is said to be I-convergent to ξ if for any $\varepsilon > 0$*

$$\{k : |x_k - \xi| \geq \varepsilon\} \in I.$$

In this case we write $I-\lim_{k \to \infty} x_k = \xi$.

Definition 2.10. *[6] A sequence $x = \{x_k\}_{k \in \mathbf{N}}$ of real number is said to be I-bounded if there exists a real number $G > 0$ such that $\{k : |x_k| > G\} \in I$.*

Definition 2.11. *[4] Let $x = \{x_k\}_{k \in \mathbf{N}}$ be a sequence of real numbers. Then x is said to be I-statistically convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$*

$$\{n \in \mathbf{N} : \tfrac{1}{n}|\{k \leq n : |x_k - \xi| \geq \varepsilon\}| \geq \delta\} \in I.$$

In this case we write I-st-$\lim_{k \to \infty} x_k = \xi$.

3. I-statistical limit points and I-statistical cluster points

In this section, following the line of Fridy [8] and Pehlivan et. al. [21], we introduce the notion of I-statistical limit point of real sequences and present an I-statistical analogue of some results in those papers.

Definition 3.1. *[5] A subset K of $\mathbf{N}$ is said to have I-natural density (or, I-asymptotic density) $d_I(K)$ if*

$$d_I(K) = I - \lim_{n\to\infty} \tfrac{|K(n)|}{n}$$

where $K(n) = \{j \in K : j \leq n\}$ and $|K(n)|$ represents the number of elements in $K(n)$.

Note 3.1. From the above definition, it is clear that, if $d(A) = r, A \subset \mathbf{N}$, then $d_I(A) = r$ for any nontrivial admissible ideal I in $\mathbf{N}$.

In case $d_I(A) = 0$, $\{x\}_A$ is called a subsequence of I-asymptotic density zero, or an I-thin subsequence of x. On the other hand, $\{x\}_A$ is an I-nonthin subsequence of x, if A does not have I-asymptotic density zero i. e., if either $d_I(A)$ is a positive number or A fails to have I-asymptotic density.

Definition 3.2. *A real number l is an I-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers, if there exists an I-nonthin subsequence of x that converges to l. The set of all I-statistical limit points of the sequence x is denoted by $\Lambda_x^S(I)$.*

Definition 3.3. *[5] A real number y is an I-statistical cluster point of a sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers, if for every $\varepsilon > 0$, the set $\{k \in \mathbf{N} : |x_k - y| < \varepsilon\}$ does not have I-asymptotic density zero. The set of all I-statistical cluster points of x is denoted by $\Gamma_x^S(I)$.*

Note 3.2. If $I = I_{fin} = \{A \subset \mathbf{N} : |A| < \infty\}$, then the notions of I-statistical limit points and I-statistical cluster points coincide with the notions of statistical limit points and statistical cluster points respectively.

We first present an I-statistical analogous of some results in [8].

Theorem 3.1. *Let $x = \{x_k\}_{k\in\mathbf{N}}$ be a sequence of real numbers. Then $\Lambda_x^S(I) \subset \Gamma_x^S(I)$.*

Proof. Let $\alpha \in \Lambda_x^S(I)$. So we have a subsequence $\{x_{k_j}\}_{j\in\mathbf{N}}$ of x with $\lim_{j\to\infty} x_{k_j} = \alpha$ and $d_I(K) \neq 0$, where $K = \{k_j : j \in \mathbf{N}\}$. Let $\varepsilon > 0$ be given. Since $\lim_{j\to\infty} x_{k_j} = \alpha$, so $B = \{k_j : \left|x_{k_j} - \alpha\right| \geq \varepsilon\}$ is a finite set. Thus

$$\{k \in \mathbf{N} : |x_k - \alpha| < \varepsilon\} \supset \{k_j : j \in \mathbf{N}\} \setminus B$$

$$\Rightarrow K = \{k_j : j \in \mathbf{N}\} \subset \{k \in \mathbf{N} : |x_k - \alpha| < \varepsilon\} \cup B.$$

Now if $d_I(\{k \in \mathbf{N} : |x_k - \alpha| < \varepsilon\}) = 0$, then we have $d_I(K) = 0$, which is a contradiction. Thus α is an I-statistical cluster point of x. Since $\alpha \in \Lambda_x^S(I)$ is arbitrary, so $\Lambda_x^S(I) \subset \Gamma_x^S(I)$.

Note 3.3. The set $\Lambda_x^S(I)$ of all I-statistical limit points of a sequence x may not be equal to the set $\Gamma_x^S(I)$ of all I-statistical cluster points of x. To show this we cite the following example.

Example 3.1. *Let $x = \{x_k\}_{k\in\mathbf{N}}$ be a sequence of real numbers, defined by $x_k = \frac{1}{m}$, where $k = 2^{m-1}(2t+1)$; i.e., $m-1$ is the power of 2 in the prime factorization of k.*

Clearly for each m, $d(\{k : x_k = \frac{1}{m}\}) = \frac{1}{2^m} > 0$. Now since I is admissible, we have $d_I(\{k : x_k = \frac{1}{m}\}) = \frac{1}{2^m} > 0$. Thus $\frac{1}{m} \in \Lambda_x^S(I)$. Also, $d_I(\{k : 0 < x_k < \frac{1}{m}\}) = 2^{-m}$, so $0 \in \Gamma_x^S(I)$ and we have $\Gamma_x^S(I) = \{0\} \cup \{\frac{1}{m}\}_{m=1}^{\infty}$. Now we claim that $0 \notin \Lambda_x^S(I)$. To establish our claim, it is sufficient to show that, if $\{x\}_M$ is a subsequence converging to zero, then $d_I(M) = 0$. For this, note that for each $m \in \mathbf{N}$, we have

$$|M(n)| = \left|\{k \in M : k \le n, x_k \ge \tfrac{1}{m}\}\right| + \left|\{k \in M : k \le n, x_k < \tfrac{1}{m}\}\right| \le$$
$$O(1) + \left|\{k \in M : k \le n, x_k < \tfrac{1}{m}\}\right| \le O(1) + \tfrac{n}{2^m}.$$

Thus $d_I(M) \le \frac{1}{2^m}$ and since m is arbitrary, so we have $d_I(M) = 0$. Thus $\Lambda_x^S(I)\Gamma_x^S(I)$.

Theorem 3.2. *If $x = \{x_k\}_{k\in\mathbf{N}}$ and $y = \{y_k\}_{k\in\mathbf{N}}$ are sequences of real numbers such that $d_I(\{k : x_k \ne y_k\}) = 0$, then $\Lambda_x^S(I) = \Lambda_y^S(I)$ and $\Gamma_x^S(I) = \Gamma_y^S(I)$.*

Proof. Let $\gamma \in \Gamma_x^S(I)$ and $\varepsilon > 0$ be given. Then $\{k : |x_k - \gamma| < \varepsilon\}$ does not have I-asymptotic density zero. Let $A = \{k : x_k = y_k\}$. Since $d_I(A) = 1$ so $\{k : |x_k - \gamma| < \varepsilon\} \cap A$ does not have I-asymptotic density zero. Thus $\gamma \in \Gamma_y^S(I)$. Since $\gamma \in \Gamma_x^S(I)$ is arbitrary, so $\Gamma_x^S(I) \subset \Gamma_y^S(I)$. By symmetry we have $\Gamma_y^S(I) \subset \Gamma_x^S(I)$. Hence $\Gamma_x^S(I) = \Gamma_y^S(I)$.

Also let $\beta \in \Lambda_x^S(I)$. Then x has an I-nonthin subsequence $\{x_{k_j}\}_{j\in\mathbf{N}}$ that converges to β. Let $K = \{k_j : j \in \mathbf{N}\}$. Since $d_I(\{k_j : x_{k_j} \ne y_{k_j}\}) = 0$, we have $d_I(\{k_j : x_{k_j} = y_{k_j}\}) \ne 0$. Therefore from the latter set we have an I-nonthin subsequence $\{y\}_{K'}$ of $\{y\}_K$ that converges to β. Thus $\beta \in \Lambda_y^S(I)$.

Since $\beta \in \Lambda_x^S(I)$ is arbitrary, so $\Lambda_x^S(I) \subset \Lambda_y^S(I)$. By symmetry we have $\Lambda_x^S(I) \supset \Lambda_y^S(I)$. Hence $\Lambda_x^S(I) = \Lambda_y^S(I)$.

We now investigate some topological properties of the set $\Gamma_x^S(I)$ of all I-statistical cluster points of x.

Theorem 3.3. *Let A be a compact set in $\mathbf{R}$ and $A \cap \Gamma_x^S(I) = \emptyset$. Then the set $\{k \in \mathbf{N} : x_k \in A\}$ has I-asymptotic density zero.*

Proof. Since $A \cap \Gamma_x^S(I) = \emptyset$, so for any $\xi \in A$ there is a positive number $\varepsilon = \varepsilon(\xi)$ such that

$$d_I(\{k : |x_k - \xi| < \varepsilon(\xi)\}) = 0.$$

Let $B_{\varepsilon(\xi)}(\xi) = \{y : |y - \xi| < \varepsilon(\xi)\}$. Then the set of open sets $\left\{B_{\varepsilon(\xi)}(\xi) : \xi \in A\right\}$ form an open covers of A. Since A is a compact set so there is a finite sub-cover of $\{B_{\varepsilon(\xi)}(\xi) : \xi \in A\}$ for A, say $\left\{A_i = B_{\varepsilon(\xi_i)}(\xi_i) : i = 1, 2, ..q\right\}$. Then $A \subset \bigcup\limits_{i=1}^{q} A_i$ and

$$d_I(\{k : |x_k - \xi_i| < \varepsilon(\xi_i)\}) = 0 \text{ for } i = 1, 2, ...q.$$

We can write, for any $n \in \mathbf{N}$,

$$|\{k : k \leq n; x_k \in A\}| \leq \sum_{i=1}^{q} |\{k : k \leq n; |x_k - \xi_i| < \varepsilon(\xi_i)\}|,$$

and by the property of I-convergence,

$$I\text{-}\lim_{n\to\infty} \tfrac{|\{k : k \leq n; x_k \in A\}|}{n} \leq \sum_{i=1}^{q} I\text{-}\lim_{n\to\infty} \tfrac{|\{k : k \leq n; |x_k - \xi_i| < \varepsilon(\xi_i)\}|}{n} = 0.$$

Which gives $d_I(\{k : x_k \in A\}) = 0$ and this completes the proof.

Note 3.4. If the set A is not compact then the above result may not be true. To show this we cite the following example.

Example 3.2. *Let us consider the sequence $x = \{x_k\}_{k \in \mathbf{N}}$ in $\mathbf{R}$ defined by*

$$x_k = \begin{cases} 0.5, & \text{if } k \text{ is odd} \\ k, & \text{if } k \text{ is even.} \end{cases}$$

Then $\Gamma_x^S(I) = \{0.5\}$. Now if we take $A = [1, \infty)$, then $A \cap \Gamma_x^S(I) = \emptyset$, but $d_I(\{(k : x_k \in A\}) = \frac{1}{2} \neq 0$.

Theorem 3.4. *If a sequence $x = \{x_k\}_{k\in\mathbf{N}}$ has a bounded I-non-thin subsequence, then the set $\Gamma_x^S(I)$ is a non-empty closed set.*

Proof. Let $x = \left\{x_{k_q}\right\}_{q\in\mathbf{N}}$ is a bounded I-non-thin subsequence of x and A be a compact set such that $x_{k_q} \in A$ for each $q \in \mathbf{N}$. Let $P = \{k_q : q \in \mathbf{N}\}$. Clearly $d_I(P) \neq 0$. Now if $\Gamma_x^S(I) = \emptyset$, then $A\cap\Gamma_x^S(I) = \emptyset$ and so by Theorem 3.3 we have

$$d_I(\{k : x_k \in A\}) = 0.$$

But for any $n \in \mathbf{N}$,

$$|\{k : k \leq n,\ k \in P\}| \leq |\{k : k \leq n,\ x_k \in A\}|,$$

which implies that $d_I(P) = 0$, which is a contradiction. So $\Gamma_x^S(I) \neq \emptyset$.

Now to show that $\Gamma_x^S(I)$ is closed, let ξ be a limit point of $\Gamma_x^S(I)$. Then for every $\varepsilon > 0$ we have $B_\varepsilon(\xi) \cap (\Gamma_x^S(I) \setminus \{\xi\}) \neq \emptyset$. Let $\beta \in B_\varepsilon(\xi) \cap (\Gamma_x^S(I) \setminus \{\xi\})$. Now we can choose $\epsilon' > 0$ such that $B_{\epsilon'}(\beta) \subset B_\varepsilon(\xi)$. Since $\beta \in \Gamma_x^S(I)$ so

$$d_I(\{k : |x_k - \beta| < \epsilon'\}) \neq 0$$
$$\Rightarrow d_I(\{k : |x_k - \xi| < \varepsilon\}) \neq 0.$$

Hence $\xi \in \Gamma_x^S(I)$.

Definition 3.4. *(a) A sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers is said to be I- statistically bounded above if, there exists $L \in \mathbf{R}$ such that $d_I(\{k \in \mathbf{N} : x_k > L\}) = 0$.*
(b) A sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers is said to be I- statistically bounded below if, there exists $l \in \mathbf{R}$ such that $d_I(\{k \in \mathbf{N} : x_k < l\}) = 0$.

Definition 3.5. *A sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real numbers is said to be I-statistically bounded if, there exists $l > 0$ such that for any $\delta > 0$, the set*

$$A = \{n \in \mathbf{N} : \tfrac{1}{n}|\{k \in \mathbf{N} : k \leq n, |x_k| > l\}| \geq \delta\} \in I$$

i.e. $d_I(\{k \in \mathbf{N} : |x_k| > l\}) = 0$.
Equivalently, $x = \{x_k\}_{k\in\mathbf{N}}$ is said to be I-statistically bounded if, there exists a compact set C in $\mathbf{R}$ such that for any $\delta > 0$, the set $A = \{n \in \mathbf{N} : \frac{1}{n}|\{k \in \mathbf{N} : k \leq n, x_k \notin C\}| \geq \delta\} \in I$ i.e., $d_I(\{k \in \mathbf{N} : x_k \notin C\}) = 0$.

Note 3.5. If $I = I_{fin} = \{A \subset \mathbf{N} : |A| < \infty\}$, then the notion of I-statistical boundedness coincide with the notion of statistical boundedness.

Corollary 3.5. *If $x = \{x_k\}_{k\in\mathbf{N}}$ is I-statistically bounded. Then the set $\Gamma_x^S(I)$ is non empty and compact.*

Proof. Let C be a compact set such that $d_I(\{k : x_k \notin C\}) = 0$. Then $d_I(\{k : x_k \in C\}) = 1$, which implies that C contains an I-non-thin subsequence of x. Hence by Theorem 3.4, $\Gamma_x^S(I)$ is nonempty and closed.

Now to prove that $\Gamma_x^S(I)$ is compact it is sufficient to prove that $\Gamma_x^S(I) \subset C$. If possible let us assume that $\xi \in \Gamma_x^S(I)$ but $\xi \notin C$. Since C is compact, so there exists $\varepsilon > 0$ such that $B_\varepsilon(\xi) \cap C = \emptyset$. In this case we have

$$\{k : |x_k - \xi| < \varepsilon\} \subset \{k : x_k \notin C\}.$$

Therefore $d_I(\{k : |x_k - \xi| < \varepsilon\}) = 0$, which contradicts the fact that $\xi \in \Gamma_x^S(I)$. Therefore $\Gamma_x^S(I) \subset C$.

Theorem 3.6. *Let $x = \{x_k\}_{k\in\mathbf{N}}$ be an I-statistically bounded sequence. Then for every $\varepsilon > 0$ the set*

$$\left\{k : d(\Gamma_x^S(I), x_k) \geq \varepsilon\right\}$$

has I-asymptotic density zero, where $d(\Gamma_x^S(I), x_k) = inf_{y\in\Gamma_x^S(I)}|y - x_k|$ the distance from x_k to the set $\Gamma_x^S(I)$.

Proof. Let C be a compact set such that $d_I(\{k : x_k \notin C\}) = 0$. Then by Corollary 3.5 we have $\Gamma_x^S(I)$ is non-empty and $\Gamma_x^S(I) \subset C$.

Now if possible let $d_I(\left\{k : d(\Gamma_x^S(I), x_k) \geq \varepsilon'\right\}) \neq 0$ for some $\varepsilon' > 0$.

Now we define $B_{\varepsilon'}(\Gamma_x^S(I)) = \left\{y : d(\Gamma_x^S(I), y) < \varepsilon'\right\}$ and let $A = C \setminus B_{\varepsilon'}(\Gamma_x^S(I))$. Then A is a compact set which contains an I- non-thin subsequence of x. Then by Theorem 3.3 $A \cap \Gamma_x^S(I) \neq \emptyset$, which is absurd, since $\Gamma_x^S(I) \subset B_{\varepsilon'}(\Gamma_x^S(I)$. Hence

$$d_I(\left\{k : d(\Gamma_x^S(I), x_k) \geq \varepsilon\right\}) = 0$$

for every $\varepsilon > 0$.

4. Condition APIO

The condition (APO) was used in [2]. We introduce the condition (APIO) which is similar to the (APO) condition.

Definition 4.1. (Additive property for I-asymptotic density zero sets). *The I-asymptotic density d_I is said to satisfy APIO if, given any countable collection of mutually disjoint sets $\{A_j\}_{j\in\mathbf{N}}$ in $\mathbf{N}$ with $d_I(A_j) = 0$, for each $j \in \mathbf{N}$, there exists a collection of sets $\{B_j\}_{j\in\mathbf{N}}$ in $\mathbf{N}$ with the properties $|A_j \Delta B_j| < \infty$ for each $j \in \mathbf{N}$ and $d_I(B = \bigcup_{j=1}^{\infty} B_j) = 0$.*

Theorem 4.1. *A sequence $x = \{x_k\}_{k\in\mathbf{N}}$ of real number is I-statistically convergent to l implies there exists a subset B with $d_I(B) = 1$ and $\lim\limits_{k\in\mathcal{B},k\to\infty} x_k = l$ if and only if d_I has the property APIO.*

Proof. Suppose x is I-statistically convergent to l implies there exists a subset B of $\mathbf{N}$ with $d_I(B) = 1$ and $\lim\limits_{k\in\mathcal{B},k\to\infty} x_k = l$. We have to prove d_I has the property APIO.

Let $\{A_j\}_{j\in\mathbf{N}}$ be a countable collection of mutually disjoint sets in $\mathbf{N}$ with $d_I(A_j) = 0, \forall j \in \mathbf{N}$. Let us define a sequence $\{x_k\}_{k\in\mathbf{N}}$ as follows

$$x_k = \begin{cases} \frac{1}{j} & \text{if } k \in A_j, \\ 0 & \text{if } k \notin \bigcup\limits_{j=1}^{\infty} A_j. \end{cases}$$

Let $\varepsilon > 0$ be given. Then there exists $i \in \mathbf{N}$ such that $\frac{1}{i+1} < \varepsilon$. Then we have

$$\{k : x_k \geq \varepsilon\} \subset A_1 \cup A_2 \cup ... \cup A_i.$$

Since $d_I(A_j) = 0,\ \forall j, j = 1, 2, ..i$, we have $d_I(\{k : x_k \geq \varepsilon\}) = 0$. Hence $\{x_k\}_{k\in\mathbf{N}}$ is I-statistically convergent to 0. Then by the assumption there exists a set $B \subset \mathbf{N}$, $d_I(B) = 0$ such that $\lim\limits_{k\in\mathbf{N}\setminus\mathcal{B},k\to\infty} x_k = 0$. Therefore for each $j = 1, 2, ...$ we have $n_j \in \mathbf{N}$ such that $n_{j+1} > n_j$ and $x_k < \frac{1}{j}$ for all $k \geq n_j$, $k \in \mathbf{N} \setminus B$. Thus if $x_k \geq \frac{1}{j}$ and $k \geq n_j$ then $k \in B$.

Set $B_j = \{k : k \in A_j, k \geq n_{j+1}\} \cup \{k : k \in B, n_j \leq k < n_{j+1}\}, j \in \mathbf{N}$. Clearly for all $j \in \mathbf{N}$ we have $|A_j \Delta B_j| < \infty$. We now show that $B = \bigcup\limits_{j=1}^{\infty} B_j$. Fix $j \in \mathbf{N}$ and let $k \in B_j$. If $k \in \{k : k \in B, n_j \leq k < n_{j+1}\}$, then we are done. If $k \geq n_{j+1}$ and $k \in A_j$ we have $x_k = \frac{1}{j}$ and so $k \in B$. Therefore $B_j \subset B$ for all $j \in \mathbf{N}$.

Again let $k \in B$. Then there exists $s \in \mathbf{N}$ such that $n_s \leq k < n_{s+1}$, which implies $k \in B_s$. Therefore $B \subset \bigcup\limits_{j=1}^{\infty} B_j$. Thus $B = \bigcup\limits_{j=1}^{\infty} B_j$ and $d_I(B = \bigcup\limits_{j=1}^{\infty} B_j) = 0$. This shows that d_I has the property APIO.

Conversely suppose that d_I has the property APIO. Let $x = \{x_k\}_{k\in\mathbf{N}}$ be a sequence such that x is I-statistically convergent to l. Then for each $\varepsilon > 0$, the set $\{k : |x_k - l| \geq \varepsilon\}$ has I-asymptotic density zero. Set $A_1 = \{k : |x_k - l| \geq 1\}$, $A_j = \{k : \frac{1}{j-1} > |x_k - l| \geq \frac{1}{j}\}$ for $j \geq 2, j \in \mathbf{N}$. Then

$\{A_j\}_{j\in\mathbf{N}}$ is a countable sequence of mutually disjoint sets with $d_I(A_j)=0$ for all $j\in\mathbf{N}$. Then by assumption there exists a countable sequence of sets $\{B_j\}_{j\in\mathbf{N}}$ with $|A_j\Delta B_j|<\infty$ and $d_I(B=\bigcup\limits_{j=1}^{\infty}B_j)=0$. We claim that $\lim\limits_{k\in\mathbf{N}\setminus\mathcal{B},k\to\infty} x_k=l$. To establish our claim, let $\delta>0$ be given, then there exists an $i\in\mathbf{N}$ such that $\frac{1}{i+1}<\delta$. Then $\{k:|x_k-l|\geq\delta\}\subset\bigcup\limits_{j=1}^{i+1}A_j$. Now since $|A_j\Delta B_j|<\infty$,$j=1,2,...,i+1$, there exists $n'\in\mathbf{N}$ such that $\bigcup\limits_{j=1}^{i+1}A_j\cap(n',\infty)=\bigcup\limits_{j=1}^{i+1}B_j\cap(n',\infty)$. Now if $k\notin B$, $k>n'$, then $k\notin\bigcup\limits_{j=1}^{i+1}B_j$ and consequently $k\notin\bigcup\limits_{j=1}^{i+1}A_j$, which implies $|x_k-l|<\delta$. This completes the proof.

Theorem 4.2. *Let I be an ideal such that d_I has the property APIO, then for any sequence $x=\{x_k\}_{k\in\mathbf{N}}$ of real numbers there exists a sequence $y=\{y_k\}_{k\in\mathbf{N}}$ such that $L_y=\Gamma_x^S(I)$ and the set $\{k:x_k\neq y_k\}$ has I-asymptotic density zero.*

Proof. We first prove that $\Gamma_x^S(I)\subset L_x$. Let $\xi\in\Gamma_x^S(I)$ and $\epsilon>0$ be given. Then $d_I(\{k:|x_k-\xi|<\epsilon\})\neq 0$. Since I is admissible, we have $d(\{k:|x_k-\xi|<\epsilon\})\neq 0$. Thus ξ is a statistical cluster point of x and hence a limit point of x. Thus $\Gamma_x^S(I)\subset L_x$. If $\Gamma_x^S(I)=L_x$ then the proof is trivial, we take $y=\{y_k\}_{k\in\mathbf{N}}=\{x_k\}_{k\in\mathbf{N}}=x$. Now suppose that $\Gamma_x^S(I)$ is a proper subset of L_x. Let $\eta\in L_x\setminus\Gamma_x^S(I)$. Choose an open interval I_η with center at η such that $d_I(\{k:x_k\in I_\eta\})=0$. The collection of all such I_η's is an open cover of $L_x\setminus\Gamma_x^S(I)$ and by the Lindelöf covering lemma there exists a countable subcover, say $\{I_{\eta_j}\}_{j\in\mathbf{N}}$ of $\{I_\eta:\eta\in L_x\setminus\Gamma_x^S(I)\}$ for $L_x\setminus\Gamma_x^S(I)$. Since each η_j is a limit point of x, consequently each I_{η_j} contains an I-thin subsequence of x. Let $I_1=\{k:x_k\in I_{\eta_1}\},I_j=\{k:x_k\in I_{\eta_j}\}\setminus(I_1\cup I_2...\cup I_{j-1}),\forall j\geq 2,j\in\mathbf{N}$. Then $\{I_j\}_{j\in\mathbf{N}}$ is a countable collection of mutually disjoint sets with $d_I(I_j)=0,\forall j\in\mathbf{N}$. Since d_I has the property APIO, so there exists a countable collection of sets $\{B_j\}_{j\in\mathbf{N}}$ such that $|I_j\Delta B_j|<\infty$ for each $j\in\mathbf{N}$ and $d_I(B=\bigcup\limits_{j=1}^{\infty}B_j)=0$. Then $I_j\setminus B$ is a finite set and so $\{k:k\in I_{\eta_j}\}\setminus B$ is a finite set for each $j\in\mathbf{N}$. Let $\mathbf{N}\setminus B=\{j_i<j_2<...\}$ and we define a sequence $y=\{y_k\}_{k\in\mathbf{N}}$ as follows

$$y_k = \begin{cases} x_{j_k} & \text{if } k \in B, \\ x_k & \text{if } k \in N \setminus B. \end{cases}$$

Obviously the set $\{k : x_k \neq y_k\}(\subset B)$ has I-asymptotic density zero and by Theorem 3.2 we have $\Gamma_x^S(I) = \Gamma_y^S(I)$. Now we show that $L_y = \Gamma_y^S(I)$. If possible, let $\Gamma_y^S(I) L_y$ and $l \in L_y \setminus \Gamma_y^S(I)$. Then there exists a subsequence of y converging to l. Note that the subsequence must be I-thin but $\{y\}_B$ has no limit point. Therefore no such l can exist. Hence $L_y = \Gamma_y^S(I)$. Consequently $L_y = \Gamma_x^S(I)$.

5. I-statistical analogous of Completeness Theorems

In this section following the line of Fridy [8], we formulate and prove an I-statistical analogue of the theorems concerning sequences that are equivalent to the completeness of the real line.

We first consider a sequential version of the least upper bound axiom (in $\mathbf{R}$), namely, Monotone sequence Theorem: every monotone increasing sequence of real numbers which is bounded above, is convergent. The following result is an I-statistical analogue of that Theorem.

Theorem 5.1. *Let $x = \{x_k\}_{k \in \mathbf{N}}$ be a sequence of real numbers and $M = \{k : x_k \leq x_{k+1}\}$. If $d_I(M) = 1$ and x is bounded above on M, then x is I-statistically convergent.*

Proof. Since x is bounded above on M, so let l be the least upper bound of the range of $\{x_k\}_{k \in M}$. Then we have
(i) $x_k \leq l$, $\forall k \in M$
(ii) for a pre-assigned $\varepsilon > 0$, there exists a natural number $k_0 \in M$ such that $x_{k_0} > l - \varepsilon$.

Now let $k \in M$ and $k > k_0$. Then $l - \varepsilon < x_{k_0} \leq x_k < l + \varepsilon$. Thus $M \cap \{k : k > k_0\} \subset \{k : l - \varepsilon < x_k < l + \varepsilon\}$. Since the set on the left hand side of the inclusion is of I-asymptotic density 1, we have $d_I(\{k : l - \varepsilon < x_k < l + \varepsilon\}) = 1$ i.e., $d_I(\{k : |x_k - l| \geq \varepsilon\}) = 0$. Hence x is I-statistically convergent to l.

Theorem 5.2. *Let $x = \{x_k\}_{k \in N}$ be a sequence of real numbers and $M = \{k : x_k \geq x_{k+1}\}$. If $d_I(M) = 1$ and x is bounded below on M, then x is I-statistically convergent.*

Proof. The proof is similar to that of Theorem 5.1 and so is omitted.

Note 5.1. (a) In the Theorem 5.1 if we replace the criteria that 'x is bounded above on M' by 'x is I-statistically bounded above on M' then the result still holds. Indeed if x is a I-statistically bounded above on M, then there exists $l \in \mathbf{R}$ such that $d_I(\{k \in M : x_k > l\}) = 0$ i.e., $d_I(\{k \in M : x_k \leq l\}) = 1$. Let $S = \{k \in M : x_k \leq l\}$ and $l' = sup\{x_k : k \in S\}$. Then (i) $x_k \leq l'$ for all $k \in S$ (ii) for any $\varepsilon > 0$, there exists a natural number $k_0 \in S$ such that $x_{k_0} > l' - \varepsilon$. Then proceeding in a similar way as in Theorem 5.1 we get the result. (b) Similarly, In the Theorem 5.2 if we replace the criteria that 'x is bounded below on M' by 'x is I-statistically bounded below on M' then the result still holds.

Another completeness result for $\mathbf{R}$ is the Bolzano-Weierstrass Theorem, which tells us that, every bounded sequence of real numbers has a cluster point. The following result is an I-statistical analogue of that result.

Theorem 5.3. *Let I be an ideal such that d_I has the property APIO. Let $x = \{x_k\}_{k \in \mathbf{N}}$ be a sequence of real numbers. If x has a bounded I-nonthin subsequence, then x has an I-statistical cluster point.*

Proof. Using Theorem 4.2, we have a sequence $y = \{y_k\}_{k \in \mathbf{N}}$ such that $L_y = \Gamma_x^S(I)$ and $d_I(\{k : x_k = y_k\}) = 1$. Let $\{x\}_K$ be the bounded I-nonthin subsequence of x. Then $d_I(\{k : x_k = y_k\} \cap K) \neq 0$. Thus y has a bounded I-nonthin subsequence and hence by Bolzano-Weierstrass Theorem, $L_y \neq \emptyset$. Thus $\Gamma_x^S(I) \neq \emptyset$.

Corollary 5.4. *Let I be an ideal such that d_I has the property APIO. If x is a bounded sequence of real numbers, then x has an I-statistical cluster point.*

The next result is an I-statistical analogue of the Heine-Börel Covering Theorem.

Theorem 5.5. *Let I be an ideal such that d_I has the property APIO. If $x = \{x_k\}_{k \in \mathbf{N}}$ is a bounded sequence of real numbers, then it has an I-thin subsequence $\{x\}_B$ such that $\{x_k : k \in \mathbf{N} \setminus B\} \cup \Gamma_x^S(I)$ is a compact set.*

Proof. Using Theorem 4.2, we have a sequence $y = \{y_k\}_{k \in \mathbf{N}}$ such that $L_y = \Gamma_x^S(I)$ and $d_I(\{k : x_k = y_k\}) = 1$. Also $d_I(P) = 0$, where $P = \{k : x_k \neq y_k\}$. Then we have $\{x_k : k \in \mathbf{N} \setminus P\} \cup \Gamma_x^S(I) = \{y_k : k \in \mathbf{N}\} \cup L_y$. Since the set on the right hand side is compact, so the set on the left hand side is also compact. This completes the proof.

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References

[1] J. Connor, J. Fridy and J. Kline, “Statistically pre-Cauchy sequences”, *Analysis,* vol. 14, no. 4, pp. 311-317, 1994, doi: 10.1524/anly.1994.14.4.311.

[2] J. Connor, “R-type summability methods, Cauchy criteria, P-sets and statistical convergence”, *Proceedings of the American mathematical society*, vol. 115, no. 2, pp. 319-327, 1992, doi: 10.1090/S0002-9939-1992-1095221-7.

[3] J. Connor, “The statistical and strong P-Cesàro convergence of sequences”, *Analysis*, vol. 8, no. 1-2, pp. 47-63, 1988, doi: 10.1524/anly.1988.8.12.47.

[4] P. Das, E. Savas and S. Ghosal, “On generalizations of certain summability methods using ideals”, *Applied mathematics letters*, vol. 24, no. 4, pp. 1509-1514, Sep. 2011, doi: 10.1016/j.aml.2011.03.036.

[5] P. Das and E. Savas, “On I-statistically pre-Cauchy sequences”, *Taiwanese journal of mathematics*, vol. 18, no. 1, pp. 115-126, 2014, doi: 10.11650/tjm.18.2014.3157.

[6] K. Demirci, “I-limit superior and limit inferior”, *Mathematical communications*, vol. 6, no. 2, pp. 165-172, 2001. [On line]. Available: https://bit.ly/2MgNKdF

[7] J. Fridy, “On statistical convergence”, *Analysis*, vol. 5, no. 4, pp. 301-313, 1985, doi:10.1524/anly.1985.5.4.301.

[8] J. Fridy, “Statistical limit points”, *Proceedings of the American mathematical society*, vol. 118, no. 4, pp. 1187-1192, 1993, doi: 10.1090/S0002-9939-1993-1181163-6.

[9] J. Fridy and C. Orhan, “Statistical limit superior and limit inferior”, *Proceedings of the American mathematical society*, vol. 125, no. 12, pp. 3625-3631, 1997, doi: 10.1090/S0002-9939-97-04000-8.

[10] B. Lahiri, P. Das, “I and I∗-convergence in topological spaces”, *Mathematica bohemica*, vol. 130, no. 2, pp. 153-160, 2005. [On line]. Available: https://bit.ly/38L4xPp.

[11] B. Lahiri, P. Das, “I and I∗-convergence of nets”, *Real analysis exchange*, vol. 33, no. 2, pp. 431-442, 2007. [On line]. Available: https://bit.ly/2swO4Ok

[12] H. Fast, “Sur la convergence statistique”, *Colloquium mathematicae*, vol. 2, no. 3-4, pp. 241-244, 1951. [On line]. Available: https://bit.ly/2RZ3hlZ

[13] P. Kostyrko, Wilczyński and T. Šalát, “I-convergence”, *Real analysis exchange*, vol. 26, no. 2, pp. 669-685, 2000. [On line]. Available: https://bit.ly/2qZJ2th

[14] P. Kostyrko, M. Mačaj, T. Šalát, and M. Sleziak, “I-convergence and external, I-limit points”, *Mathematica slovaca*, vol. 55, no. 4, pp. 443-454, 2005. [On line]. Available: https://bit.ly/2Ptjhea

[15] M. Mursaleen, S. Mohiuddine and O. Edely, “On ideal convergence of double sequences in intuitionistic fuzzy normed spaces”, *Computers & mathematics with applications*, vol. 59, no. 2, pp. 603-611, Jan. 2010, doi: 10.1016/j.camwa.2009.11.002.

[16] M. Mursaleen and S. Mohiuddine, “On ideal convergence of double sequences in probabilistic normed spaces”, *Mathematical reports*, vol. 12, no. 4, pp. 359-371, 2010.

[17] M. Mursaleen and A. Alotaibi, “On I-convergence in random 2-normed spaces”, *Mathematica slovaca*, vol. 61, no. 6, pp. 933-940, Dec. 2011, doi: 10.2478/s12175-011-0059-5.

[18] M. Mursaleen and S. Mohiuddine, “On ideal convergence in probabilistic normed spaces”, *Mathematica slovaca*, vol. 62, no. 1, pp. 49-62, Feb. 2012, doi: 10.2478/s12175-011-0071-9.

[19] M. Mursaleen, D. Debnath and D. Rakshit, “I-statistical limit superior and I- statistical limit inferior”, *Filomat*, vol. 31, no. 7, pp. 2103-2108, 2017. [On line]. Available: https://bit.ly/2M3giXT

[20] I. Niven and H. Zuckerman, *An introduction to the theory of numbers*, 4th ed. New York: Wiley, 1980.

[21] S. Pehlivan, A. Güncan and M. A. Mamedov, “Statistical cluster points of sequences in finite dimensional spaces,” *Czechoslovak mathematical journal,* vol. 54, no. 1, pp. 95–102, Mar. 2004, doi: 10.1023/B:CMAJ.0000027250.19041.72.

[22] T. Šalát, “On statistically convergent sequences of real numbers”, *Mathematica slovaca*, vol. 30, no. 2, pp. 139-150, 1980. [On line]. Available: https://bit.ly/34uzcNs

[23] E. Savas and P. Das, "A generalized statistical convergence via ideals", *Applied mathematics letters*, vol. 24, no. 6, pp. 826-830, Jun. 2011, doi: 10.1016/j.aml.2010.12.022.

[24] I. Schoenberg, "The integrability of certain functions and related summability methods", *The American mathematical monthly*, vol. 66, no. 5, pp. 361-375, 1959, doi: 10.1080/00029890.1959.11989303.

[25] H. Steinhus, "Sur la convergence ordinaire et la convergence asymptotique", *Colloquium mathematicum*, vol. 2, pp. 73-74, 1951.