



On a class of a Boundary value problems involving the $p(x)$ -Biharmonic operator

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Abstract:

Our aim is to establish the existence of weak solution for a class of Robin problems involving fourth order operator. The nonlinearity is superlinear but does not satisfy the usual Ambrosetti-Rabinowitz condition. The proof is made with and without variational structure.

Keywords: $p(x)$ -biharmonic; Topological degree; Variational methods.

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1. Introduction

Elliptic equations with variable exponent growth have been extensively studied in the last decade, they can model various phenomena which are motivated by the fact that this type of equations can serve as models in the theory of electrorheological fluids, image processing, and the theory of nonlinear elasticity. We refer to the overview papers [6, 14, 16, 18] for the advances and the references in this area.

Our purpose is to study the following variable exponent equation

$$(\mathcal{P}) \quad \begin{cases} \Delta_{p(x)}^2 u = f(x, u) & \text{in } \Omega, \\ |\Delta u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in \mathbf{R}^N with smooth boundary $\partial\Omega$, $\Delta_{p(x)}^2 u = \Delta(|\Delta|^{p(x)-2} \Delta u)$ is the $p(x)$ -biharmonic with $p \in C(\overline{\Omega})$, $p(x) > 1$ for every $x \in \overline{\Omega}$, $\beta \in L^\infty(\Omega)$ with $\text{ess inf}_{x \in \Omega} \beta(x) > 0$ and ν is the outward normal vector on $\partial\Omega$. We define

$$F(x, t) = \int_0^t f(x, s) ds,$$

we denote by

$$p^- := \inf_{x \in \overline{\Omega}} p(x), \quad p^+ := \sup_{x \in \overline{\Omega}} p(x).$$

Throughout this paper, we suppose the following assumption,

(f_0) $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a Carathéodory function and satisfies

$$|f(x, t)| \leq C_1 + C_2 |t|^{\alpha(x)-1}, \quad \forall (x, t) \in \Omega \times \mathbf{R},$$

where

$$\alpha \in C(\overline{\Omega}), \alpha(x) > 1, C_1, C_2 > 0$$

and

$$1 < \alpha^+ = \sup_{x \in \overline{\Omega}} \alpha(x) \leq p_2^*(x),$$

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } 2p(x) < N, \\ +\infty & \text{if } 2p(x) \geq N. \end{cases}$$

Many authors consider the existence of nontrivial solutions for some fourth order problems such as [1, 2, 3, 5, 4, 8, 11, 12, 13]..., which represent

a generalization of the classical p -biharmonic operator obtained in the case when p is a positive constant.

Here we point out that the $p(x)$ -biharmonic operator possesses more complicated nonlinearities than p -biharmonic, for example, it is inhomogeneous and usually it does not have the so-called first eigenvalue. Meanwhile, the methods used in this paper are also applicable for the cases of other boundary value conditions, for example, Navier and Neumann boundary value conditions. We borrow some ideas from [15] and we extend them to the case of $p(x)$ -biharmonic equation with Robin boundary condition.

It is well known that elliptic problem like (\mathcal{P}) involving the $p(x)$ -biharmonic operator without the **(AR)** type condition becomes a very difficult task to get the boundedness of the Palais-Smale type sequences of the corresponding functional, to overcome this difficulty, we use the assumption (f_3) below. That is why, at our best knowledge, the present paper is a first contribution in this direction. It is known that (f_3) is much weaker than the **(AR)** condition in the constant exponent case. Where in (f_3) , it is assumed the existence of two positive constants c_1 and c_2 such that

$$\psi_1(x, t) \leq c_1 \psi_1(x, s) \leq c_2 \psi_2(x, s), \text{ for all } 0 \leq t \leq s,$$

where,

$$\psi_1(x, t) = f(x, t)t - p^- F(x, t),$$

$$\psi_2(x, t) = f(x, t)t - p^+ F(x, t)$$

and Ambrosetti-Rabinowitz type conditions

(AR) there exist $\theta > p^+$, $M > 0$ such that for any $x \in \Omega$ and $t \geq M$ we have

$$0 \leq \theta F(x, t) \leq f(x, t)t.$$

This paper is organized as four sections. In section 2, we introduce some basic properties of the variable exponent Lebesgue and Sobolev spaces. In section 3, under the conditions that (\mathcal{P}) has variational structure, by relying on variational argument we give the existence of at least a nontrivial solution. In section 4, when (\mathcal{P}) does not have variational structure, the existence of a solution is established .

2. Preliminaries

In order to deal with the problem (\mathcal{P}) , we need some theory of variable exponent Sobolev Space. For convenience, we only recall some basic facts which will be used later. Suppose that $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Let $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \text{ and } \text{ess inf}_{x \in \overline{\Omega}} p(x) > 1\}$ for any $p(x) \in C_+(\overline{\Omega})$, denote by $p^- = \min_{x \in \overline{\Omega}} p(x)$, $p^+ = \max_{x \in \overline{\Omega}} p(x)$ and

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u : \Omega \rightarrow \mathbf{R} \text{ measurable} : \int_{\Omega} |u|^{p(x)} dx < \infty\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\{\lambda > 0 : \int_{\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\},$$

becomes a separable and reflexive Banach space (see [10]).

Proposition 2.1. (cf.[8, 10]) Set, $\rho(u) = \int_{\Omega} |u|^{p(x)} dx$, if $u \in L^{p(x)}(\Omega)$ we have :

- (1) $|u|_{p(x)} \geq 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$.
- (2) $|u|_{p(x)} \leq 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.

Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $D^{\alpha}u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \dots \partial^{\alpha_N}x_N}$ with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ with the norm $\|u\| = \sum_{|\alpha| \leq k} |D^{\alpha}u|_{p(x)}$ is a Banach separable and reflexive space.

Proposition 2.2. (cf.[8, 10]) For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

Since $\beta^- > 0$, similar to Theorem 2.1. in [7], for any $u \in W^{2,p(x)}(\Omega)$ we have that $|\Delta u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)}$ is a norm in $W^{2,p(x)}(\Omega)$ which is equivalent to standard one $|\Delta u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\Omega)}$.

Proposition 2.3. (cf.[19]) Set $\varrho(u) = \int_{\Omega} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} dx$. For $u, u_n \in W^{2,p(x)}(\Omega)$ we have,

- (1) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \varrho(u) \leq \|u\|^{p^-}$.
- (2) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \varrho(u) \leq \|u\|^{p^+}$.
- (3) $\|u_n\| \rightarrow 0 \Leftrightarrow \varrho(u_n) \rightarrow 0$.
- (4) $\|u_n\| \rightarrow +\infty \Rightarrow \varrho(u_n) \rightarrow +\infty$.

Proposition 2.4. (cf.[10]) For any $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$, we have

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(x)} \|v\|_{q(x)},$$

where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1.$$

Lemma 2.1. (cf.[10]) If $f : \bar{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$ is a Carathéodory function and

$$|f(x, s)| \leq a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \forall (x, s) \in \bar{\Omega} \times \mathbf{R},$$

where $p_1(x), p_2(x) \in C(\bar{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $p_1(x) > 1$, $p_2(x) > 1$, $a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $N_f(u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.5. (cf.[17]) Let X be a real Banach space, \tilde{B} be a bounded open subset of X , $A : \tilde{B} \rightarrow X$ is compact continuous, I is the identity mapping on X , then the Leray-Schauder degree defined by $\deg(I - A, \tilde{B}, 0)$ of $I - A$ verifies the following assertions:

- (i) $\deg(I, \tilde{B}, 0) = 1$;
- (ii) $\deg(I - A, \tilde{B}, 0) \neq 0$ then $Ax = x$ has a solution;
- (iii) If $L : \bar{\tilde{B}} \times [0, 1] \rightarrow X$ is compact continuous mapping with $L(x, \lambda) \neq x$ for $x \in \partial\tilde{B}$ and $\lambda \in [0, 1]$ then $\deg(I - L(\cdot, \lambda), \tilde{B}, 0)$ does not depend on the choice of λ .

Let $X = W^{2,p(x)}(\Omega)$, endowed with the induced norm $\|\cdot\|$, is also reflexive separable space and there is a continuous and compact embedding from X into $L^r(x)$ for $1 < r(x) < p_2^*(x)$.

Definition 2.1. We say that $u \in X$ is a **weak solution** of problem (\mathcal{P}) if

$$\int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} u v dx = \int_{\Omega} f(x, u) v dx,$$

for all $v \in X$.

3. The variational method

We define the functional associated to problem (\mathcal{P}) by

$$\phi(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} F(x, u) dx.$$

A standard argument shows that the functional ϕ is of class $C^1(X, \mathbf{R})$, (cf. [9]).

By the famous Mountain Pass lemma we state the following Theorem,

Theorem 3.1. Suppose (f_0) with $p^+ < \alpha^-$ hold.

Assume the following hypotheses,

(f_1) The following limit holds uniformly for a.e $x \in \Omega$

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)t}{|t|^{p^+}} = +\infty.$$

(f_2) $f(x, t) = o(t^{p(x)-1})$ as $t \rightarrow 0$ uniformly x in Ω .

(f_3) There exist two positive constants c_1 and c_2 such that

$$\psi_1(x, t) \leq c_1 \psi_1(x, s) \leq c_2 \psi_2(x, s), \text{ for all } 0 \leq t \leq s.$$

Where,

$$\psi_1(x, t) = f(x, t)t - p^- F(x, t),$$

$$\psi_2(x, t) = f(x, t)t - p^+ F(x, t).$$

Then problem (\mathcal{P}) admits at least a nontrivial solution in X .

Noting that ϕ' is the sum of a (S_+) type map and a weakly-strongly continuous map, so ϕ' is of (S_+) type. To see that the Cerami condition (C) holds, it is enough to verify that any Cerami sequence is bounded.

Proof of Theorem 3.1: We check the geometric assumptions and of compactness of the Mountain Pass Theorem as in the following lemmas.

Lemma 3.1. Suppose that $(f_0) - (f_3)$ hold. If $c \in \mathbf{R}$, then any sequence of Cerami $(C)_c$ of ϕ is bounded.

Proof. Let $(u_n)_n$ be a $(C)_c$ sequence of ϕ . We claim that $(u_n)_n$ is bounded, otherwise, up to a subsequence we may assume that

$$\phi(u_n) \rightarrow c, \quad \|u_n\| \rightarrow +\infty, \quad \phi'(u_n) \rightarrow 0.$$

Putting $\omega_n = \frac{u_n}{\|u_n\|}$, up to a subsequence we have $\omega_n \rightharpoonup \omega$ in X , $\omega_n \rightarrow \omega$ in $L^{p(x)}(\Omega)$, $\omega_n(x) \rightarrow w(x)$ a.e. $x \in \Omega$.

Here, two cases appear:

When $\omega \not\equiv 0$. From the fact that $\phi'(u_n)u_n = 0$, which means,

$$(3.1) \quad \int_{\Omega} |\Delta u_n|^{p(x)} dx + \int_{\partial\Omega} |\beta(x)u_n|^{p(x)} dx - \int_{\Omega} f(x, u_n)u_n dx = 0.$$

Dividing (3.1) by $\|u_n\|^{p^+}$, so

$$\int_{\Omega} \frac{f(x, u_n)u_n}{\|u_n\|^{p^+}} < \infty,$$

however, using (f_1) and lemma of Fatou we obtain

$$\int_{\Omega} \frac{f(x, u_n)u_n}{\|u_n\|^{p^+}} dx = \int_{\Omega} \frac{f(x, u_n)u_n}{|u_n|^{p^+}} |\omega|^{p^+} dx \rightarrow \infty,$$

which is contradictory.

In the case when $\omega \equiv 0$, we choose a sequence $t_n \in [0, 1]$ satisfying $\phi(t_n u_n) = \max_{t \in [0, 1]} \phi(t u_n)$.

In virtue of $w_n \rightarrow 0$ in $L^{\alpha(x)}(\Omega)$, $|F(x, t)| \leq C(1 + |t|^{\alpha(x)})$, and by the continuity of the Nemitskii operator, we see that $F(\cdot, w_n) \rightarrow 0$ in $L^1(\Omega)$ as $n \rightarrow +\infty$, so we entail that

$$(3.2) \quad \lim_{n \rightarrow \infty} \int_{\Omega} F(x, w_n) dx = 0.$$

Given $m > 0$, for n large enough we have $\|u_n\|^{-1}(2mp^+)^{\frac{1}{p^+}} \in]0, 1[$, taking into account (3.2) with $R = (2mp^+)^{\frac{1}{p^+}}$, it yields

$$\begin{aligned} \phi(t_n u_n) &\geq \phi\left(\frac{R}{\|u_n\|} u_n\right) = \phi(Rw_n) \\ &\geq \int_{\Omega} \frac{R^{p(x)}}{p(x)} |\Delta w_n|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |w_n|^{p(x)} dx - \int_{\Omega} F(x, Rw_n) dx \\ &\geq \frac{R^{p^+}}{p^+} - \int_{\Omega} F(x, Rw_n) dx \geq m. \end{aligned}$$

Thereby,

$$\phi(t_n u_n) \rightarrow +\infty.$$

On the other hand, we know that $\phi(0) = 0$, $\phi(u_n) \rightarrow c$, thus $t_n \in]0, 1[$ and $\langle \phi'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt} \phi(t u_n)|_{t=t_n} = 0$.

It follows,

$$\phi(t_n u_n) - \frac{1}{p^-} \phi'(t_n u_n)(t_n u_n) \rightarrow +\infty.$$

Therefore,

$$\int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^-} \right) |t_n \Delta u_n|^{p(x)} dx + \int_{\partial\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^-} \right) \beta(x) |t_n u_n|^{p(x)} dx +$$

$$\int_{\Omega} \left(\frac{1}{p^-} f(x, t_n u_n)(t_n u_n) - F(x, t_n u_n) \right) dx \rightarrow +\infty,$$

so we have,

$$\int_{\Omega} \left(\frac{1}{p^-} f(x, t_n u_n)(t_n u_n) - F(x, t_n u_n) \right) dx \rightarrow +\infty,$$

accordingly we have

$$\begin{aligned} \phi(u_n) &= \phi(u_n) - \frac{1}{p^+} \phi'(u_n) \cdot u_n \\ &\geq \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) |\Delta u_n|^{p(x)} dx + \int_{\partial\Omega} \left(\frac{1}{p(x)} - \frac{1}{p^+} \right) \beta(x) |u_n|^{p(x)} dx \\ &\quad + \int_{\Omega} \left(\frac{1}{p^+} f(x, u_n) u_n - F(x, u_n) \right) dx \\ &\geq \int_{\Omega} \left(\frac{1}{p^+} f(x, u_n) u_n - F(x, u_n) \right) dx. \end{aligned}$$

From (f₃), there exist two constants c_1 and c_2 such that

$$\begin{aligned} \phi(u_n) &\geq \int_{\Omega} \left(\frac{1}{p^+} f(x, u_n) u_n - F(x, u_n) \right) dx \\ &\geq c_1 \int_{\Omega} \left(\frac{1}{p^-} f(x, u_n)(u_n) - F(x, u_n) \right) dx \\ (3.3) \quad &\geq c_1 c_2 \int_{\Omega} \left(\frac{1}{p^-} f(x, t_n u_n)(t_n u_n) - F(x, t_n u_n) \right) dx. \end{aligned}$$

Hence

$$\phi(u_n) \rightarrow +\infty,$$

which is impossible and thus (u_n) is bounded in X .

Lemma 3.2. *Under the conditions of Theorem 3.1, ϕ verifies the following:*

- (a) *There exist $\rho > 0$ and $\beta > 0$ such that $\phi(u) > \beta$ when $\|u\| = \rho$.*
- (b) *There exists $v \in X$ such that $\|v\| < \rho$ and $\phi(v) < 0$.*

Proof. In view of (f_0) and (f_2) , there is a constant $C_1 > 0$ such that

$$|F(x, t)| \leq \frac{1}{2p^+} |t|^{p(x)} + C_1 |t|^{\alpha(x)}, \quad \text{for } (x, t) \in \Omega \times \mathbf{R}.$$

Therefore, for $\|u\| \leq 1$ we have

$$\begin{aligned} \phi(u) &\geq \frac{1}{p^+} [\int_{\Omega} (|\Delta u|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)} dx) - \frac{1}{2p^+} \int_{\Omega} |u|^{p(x)} dx - C_1 \int_{\Omega} |u|^{\alpha(x)} dx] \\ &\geq \frac{1}{2p^+} [\int_{\Omega} (|\Delta u|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)} dx) - C_1 \int_{\Omega} |u|^{\alpha(x)} dx] \\ &\geq \frac{1}{2p^+} \|u\|^{p^+} - C_2 \|u\|^{\alpha^-} \\ &\geq \|u\|^{p^+} \left(\frac{1}{2p^+} - C_2 \|u\|^{\alpha^- - p^+} \right). \end{aligned}$$

Since $p^+ < \alpha^-$, the function $t \mapsto \left(\frac{1}{2p^+} - C_2 t^{\alpha^- - p^+} \right)$ is strictly positive in a neighborhood of zero. It follows that there exist $\rho > 0$ and $\beta > 0$ such that

$$\phi(u) \geq \beta \quad \forall u \in X : \|u\| = \rho.$$

To apply the Mountain Pass Theorem, it suffices to show that

$$\phi(tu) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty,$$

for a certain $u \in X$.

Let $u \in X \setminus \{0\}$, by (f_1) , we may choose a constant

$$A > \frac{\int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} dx}{\int_{\Omega} |u|^{p^+} dx}, \text{ such that}$$

$$F(x, t) \geq A |t|^{p^+} \quad \text{uniformly in } x \in \Omega.$$

Let $t > 1$ large enough, we have

$$\begin{aligned} \phi(tu) &\leq \int_{\Omega} \frac{t^{p(x)}}{p(x)} [|\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |tu|^{p(x)} dx] - \int_{\Omega} F(x, tu) dx \\ &\leq t^{p^+} [\int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} dx] - \int_{|tu| > C_A} F(x, tu) dx - \int_{|tu| \leq C_A} F(x, tu) dx \\ &\leq t^{p^+} [\int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} dx] - A t^{p^+} \int_{\Omega} |u|^{p^+} dx - \int_{|tu| \leq C_A} F(x, tu) dx \\ &\quad + A t^{p^+} \int_{|tu| \leq C_A} |u|^{p^+} dx \\ &\leq t^{p^+} [\int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)} dx] - A t^{p^+} \int_{\Omega} |u|^{p^+} dx + C_1, \end{aligned}$$

where $C_1 > 0$ is a constant, which implies that

$$\phi(tu) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

It follows that there exists $e \in X$ such that $\|e\| > \rho$ and $\phi(e) < 0$. According to the Mountain Pass Theorem, ϕ admits a critical value $\tau \geq C'$ which is characterized by

$$\tau = \inf_{h \in \Gamma} \sup_{t \in [0,1]} \phi(h(t))$$

where

$$\Gamma = \{h \in C([0,1], X) : h(0) = 0 \text{ and } h(1) = e\}.$$

4. The non-variational method

We define the operators

$$\langle Au, v \rangle = \int_{\Omega} |\Delta u|^{p(x)-2} u \Delta v \, dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} u v \, dx$$

and

$$\langle Bu, v \rangle = \int_{\Omega} f(x, u) v \, dx, \quad \forall v \in X,$$

where $A, B : X \rightarrow X^*$.

We recall the interesting proposition,

Proposition 4.1. (cf.[8])

- i) $A : X \rightarrow X^*$ is a continuous, bounded and strictly monotone operator.
- ii) A is a mapping of type (S_+) , i.e. if $u_n \rightharpoonup u$ in X and $\limsup_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in X .
- iii) The operator $A : X \rightarrow X^*$ is a bounded homeomorphism.

So, we have the following lemma.

Lemma 4.1. The operator $A^{-1} \circ B$ is compact continuous from X to X where A^{-1} is the inverse operator of A .

Proof. Let $(u_n)_n$ be a bounded sequence of X , and then up to a subsequence denoted also by $(u_n)_n$, there exists $u \in X$ such that $u_n \rightharpoonup u$ in $L^{\alpha(x)}(\Omega)$ therefore, from lemma 2.1 we infer that Bu_n is strong convergent in X^* . Since A^{-1} is a bounded homeomorphism then $A^{-1} \circ B$ is strong convergent in X .

The main result of this section reads as follows:

Theorem 4.1. Suppose that the Carathéodory function f satisfies (f_0) with $\alpha(x) < p^-$, then the problem (\mathcal{P}) has a solution in X .

Set $L_\lambda u = \lambda Bu$.

Proof. We consider the equation

$$(4.1) \quad Au = L_\lambda u.$$

The solutions of (4.1) are uniformly bounded for $\lambda \in [0, 1]$, if not, then there exists a sequence of solutions $(u_n)_n$ of (4.1) such that $\|u_n\| \rightarrow +\infty$ and

$$\int_{\Omega} (|\Delta u_n|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u_n|^{p(x)} dx) = \int_{\Omega} \lambda_n f(x, u_n) u_n dx,$$

with $(\lambda_n)_n \subset [0, 1]$. In view of (f_0) we have

$$\int_{\Omega} \lambda_n f(x, u_n) u_n dx \leq \varepsilon \int_{\Omega} |u_n|^{p(x)} dx + C(\varepsilon),$$

with $\varepsilon > 0$ is small enough, because $\alpha(x) < p(x)$. So we deduce that $(u_n)_n$ is bounded, which is a contradiction.

Let choose a radius $R > 0$ which all solutions of (4.1) are in the ball $\mathcal{B}(0, R)$. Applying the Leray-Schauder degree, proposition 2.5, (because now it is well defined) so we entail that

$$\begin{aligned} \deg(I - A^{-1} \circ L_\lambda, \mathcal{B}(0, R), 0) &= \deg(I - A^{-1} \circ L_0, \mathcal{B}(0, R), 0) \\ &= \deg(I - A^{-1} \circ 0, \mathcal{B}(0, R), 0), \end{aligned}$$

where $L_0 = 0$ and I is the identity mapping on X .

We point out that $I - A^{-1} \circ L_0$ has zero as a unique solution and thus from proposition 2.5, we obtain

$$\deg(I - A^{-1} \circ L_1, \mathcal{B}(0, R), 0) = \deg(I - A^{-1} \circ L_0, \mathcal{B}(0, R), 0) = 1,$$

and consequently there exists $u \in \mathcal{B}(0, R)$ such that

$$Au - Bu = 0$$

has at least a solution.

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