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Some fixed point theorems for generalized Kannan type mappings in *b*-metric spaces

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Abstract:

In this paper, we prove some fixed point theorems in b-metric spaces using subadditive altering distance function. Some of these results generalize many existing fixed point theorems for Kannan type mappings. The results are justified with suitable examples.

Keywords: *b*-metric space; Subadditive altering distance function; Kannan type mappings.

MSC (2010): 47H10, 54E50.

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1. Introduction

One of the most prominent fixed point theorem since the famous "Banach contraction principle" [5] in 1922 is undeniably the Kannan fixed point theorem. It is a well known fact that every Banach contraction mapping is continuous. In 1968, Kannan [21] showed that a contractive mapping with a fixed point need not be necessarily continuous in proving the following result:

Theorem 1.1. *[17] Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx,Ty) \leq k\Big\{d(x,Tx)+d(y,Ty)\Big\}$$

for all $x,y \in X$ and $k \in \Big[0,1/2\Big)$. Then T has a unique fixed point $z \in X$, and for any $x \in X$ the sequence of iterates $\Big\{T^n x\Big\}$ converges to z.

The importance of the above result lies in the fact that Kannan's theorem characterizes the completeness of the metric space. This was proved by Subrahmanyam [29] in 1975.

Theorem 1.1 is one of the several generalizations of the Banach contraction principle which were derived either by changing the contraction condition or by changing the space to a more generalized space (refer to [2], [10], [11], [12], [26], [30],among others). In this regard, Bakhtin [4] in 1989 introduced *b-metric spaces* to generalize Banach fixed point theorem. In 1993, Czerwik [9] formally defined the notion of b-metric spaces as follows.

Definition 1.1. *[9] Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0,\infty)$ is called b-metric if it satisfies the following properties.*

1. $d(x,y)=0$ *if and only if* $x=y$;
2. $d(x,y)=d(y,x)$; *and*
3. $d(x,z) \leq s[d(x,y)+d(y,z)]$, *for all* $x,y,z \in X$.

The pair (X,d) is called a b-metric space with coefficient s.

Since then many authors have generalized Banach fixed point theorem in b-metric spaces (refer to [1], [18], [19], [22], [23], [28] and the references therein).

Example 1. *It is evident from the definition that every metric is also a b-metric with coefficient 1. A few more examples (refer [6], [27]) are given below.*

1. *The set* $l_p(\mathbf{R}) = \left\{ \{x_n\} \subset \mathbf{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$ *with* $0 < p < 1$, *together with the function* $d : l_p(\mathbf{R}) \times l_p(\mathbf{R}) \longrightarrow \mathbf{R}$ *given by*

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

   is a b-metric space with coefficient $2^{\frac{1}{p}}$.

2. *The set* $L_p[0, 1]$, $(0 < p < 1)$ *of all real functions* $x(t)$, $t \in [0, 1]$ *where* $\int_0^1 |x(t)|dt < \infty$ *is a b-metric space with coefficient* $2^{\frac{1}{p}}$ *if we define the b-metric* $d : L_p[0, 1] \times L_p[0, 1] \longrightarrow \mathbf{R}$ *by*

$$d(x, y) = \left(\int_0^1 \left| x(t) - y(t) \right|^p dt \right)^{\frac{1}{p}}.$$

3. *Let* (X, d') *be a metric space and define* $d(x, y) = d'(x, y)^p$, *where* $p > 1$ *is a real number. Then* (X, d) *is a b-metric space with coefficient* 2^{p-1}.

It may be noted here that a b-metric need not be always continuous in the topology generated by it (refer Example 2.6 of [24]). Moreover, the notion of convergent sequence, Cauchy sequence, completeness, etc. may as well be defined accordingly in b-metric spaces.

Kannan's fixed point theorem got its due attention and some authors gave an attempt to extend his result (refer to [15], [20], [25], [26], [31]). In this paper, we also try to extend the result of Kannan using the following class of subadditive altering distance functions.

Definition 1.2. *A function* $\phi : [0, \infty) \longrightarrow [0, \infty)$ *is said to be a subadditive altering distance function if*

(i) ϕ *is an altering distance function [15], (i.e.,* ϕ *is continuous, strictly increasing and* $\phi(t) = 0$ *if and only if* $t = 0$*)*

(ii) $\phi(x + y) \leq \phi(x) + \phi(y) \quad \forall \; x, y \in [0, \infty)$

Example 2. *It can be easily seen that the functions* $\phi_1(x) = kx$ *for some* $k \geq 1$, $\phi_2(x) = \sqrt[n]{x},\ n \in \mathbf{N}$, $\phi_3(x) = \log(1+x),\ x \geq 0$ *and* $\phi_4(x) = \tan^{-1} x$ *are such subadditive altering distance functions.*

Here we note, if ϕ is sub-additive, then for any non-negative real number $k < 1$

$$\phi\left(d(x, y)\right) \leq k\phi\left(d(a, b)\right) \qquad\qquad d(x, y) \leq k' d(a, b)$$

for some $k' < 1$.

2. Main results

Consider ϕ as a subadditive altering distance function and the b-metric d is assumed to be continuous in the topology generated by it.
We derive some fixed point results among which one of them is a generalization of a result given by Górnicki in [17].

Theorem 2.1. *Let* (X, d) *be a complete* b*-metric space with coefficient* $s \geq 1$ *and let* $T : X \longrightarrow X$ *be a mapping such that there exists* $p < \frac{1}{2s+1}$ *satisfying*

$$\phi\left(d(Tx, Ty)\right) \leq p\Big\{\phi\left(d(x, y)\right) + \phi\left(d(x, Tx)\right) + \phi\left(d(y, Ty)\right)\Big\} \tag{2.1}$$

for all $x, y \in X$*. Then* T *has an unique fixed point* $z \in X$*, and for any* $x \in X$ *the sequence of iterates* $\{T^n x\}$ *converges to* z *and for* $q = \frac{2p}{1-p} < 1$*,*

$$d(T^{n+1}x, T^n x) \leq q^n d(x, Tx), \quad n = 0, 1, 2, \ldots$$

Proof. For an arbitrary element $x \in X$, let $u = Tx$. Then

$$\phi\left(d(u, Tu)\right) = \phi\left(d(Tx, Tu)\right) \leq p\Big\{\phi\left(d(x, u)\right) + \phi\left(d(x, Tx)\right) + \phi\left(d(u, Tu)\right)\Big\}$$

that is,

$$\phi\left(d(u, Tu)\right) \leq q\phi\left(d(x, Tx)\right) \quad \text{where} \quad q = \frac{2p}{1-p} < 1.$$

Thus

$$d(u, Tu) \leq q' d(x, Tx) \tag{2.2}$$

for some $q' < 1$. Without loss of generality, we assume $q' = q$.

Now, for an arbitrary point $x_0 \in X$ consider the sequence $\{x_n\}$ where $x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$. For $m, n \in \mathbf{N}$ with $m > n$, we have
$d(\mathrm{x}_n, x_m) \leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{m-n+1} d(x_{m-1}, x_m)$
$\leq d(Tx_0, x_0)\left(sq^n + s^2 q^{n+1} + \ldots + s^{m-n+1} q^m\right)$
$\leq q^{n-1} d(Tx_0, x_0)\left(sq + (sq)^2 + \ldots + (sq)^m\right)$
$\leq q^{n-1} d(Tx_0, x_0) \frac{1}{1-sq}$, since $sq < 1$
$\longrightarrow 0$, as $n \to \infty$, showing that $\{x_n\}$ is a Cauchy sequence in H, which is complete. Therefore, there exists $z \in H$ such that

$$\lim_{n\to\infty} x_n = z\ .$$

Now, from (2.1) we get $\phi\left(d(Tz, z)\right) \leq \phi\left(sd(Tz, Tx_n) + s^2 d(Tx_n, x_n) + s^2 d(x_n, z)\right)$
$\leq s\phi\left(d(Tz, Tx_n)\right) + s^2\phi\left(d(Tx_n, x_n)\right) + s^2\phi\left(d(x_n, z)\right)$
$\leq sp\left\{\phi\left(d(z, x_n)\right) + \phi\left(d(z, Tz)\right) + \phi\left(d(x_n, Tx_n)\right)\right\}$
$+ s^2\phi\left(d(Tx_n, x_n)\right) + s^2\phi\left(d(x_n, z)\right)$,

or, $(1 - sp)\,\phi\left(d(Tz, z)\right) \leq (sp + s^2)\left\{\phi\left(d(z, x_n)\right) + \phi\left(d(Tx_n, x_n)\right)\right\}$
$\leq (sp{+}s^2)\left\{\phi\left(d(z, x_n)\right) + \phi\left(q^n d(Tx_0, x_0)\right)\right\}$ Since the above relation is true for all $n \in \mathbf{N}$ and $1 - sp \neq 0$, we have

$$\phi\left(d(Tz, z)\right) \longrightarrow 0, \qquad \text{as} \quad n \to \infty\ ,$$

showing that $d(Tz, z) = 0$. To show the uniqueness of the fixed point z, let $w \in X$ be another fixed point of T. Then $\phi\left(d(z, w)\right) = \phi\left(d(Tz, Tw)\right) \leq p\left\{\phi\left(d(z, w)\right) + \phi\left(d(z, Tz)\right) + \phi\left(d(w, Tw)\right)\right\}$
$\leq p\phi\left(d(z, w)\right)$. Since ϕ is strictly increasing and $p < \frac{1}{2s+1}$, this will be true iff $d(z, w) = 0$.
Finally, from (2.2) we have $\mathrm{d}(\mathrm{T}^{n+1}x, T^n x) \leq qd(T^{n-1}x, T^n x)$, where $q = \frac{2p}{1-p} < 1$ that is,

$$d(T^{n+1}x, T^n x) \leq q^n d(x, Tx), \qquad n = 0, 1, 2, \ldots$$

Example 3. *Consider the complete b-metric space* (X, d) *with* $X = [0, 1]$ *and* $d(x, y) = |x - y|$ *for all* $x, y \in X$. *Let* $T : X \longrightarrow X$ *be given by* $Tx = \frac{x}{2}$ *for all* $x \in X$. *Then for* $\phi(t) = \sqrt{t}$, *we have* $\phi\left(d(Tx, Ty)\right) < \frac{1}{3}\left\{\phi\left(d(x, y)\right) + \phi\left(d(x, Tx)\right) + \phi\left(d(y, Ty)\right)\right\}$
$|\frac{x}{2} - \frac{y}{2}| < \frac{1}{3}\left\{|x - y| + |x - \frac{x}{2}| + |y - \frac{y}{2}|\right\}$
$\frac{1}{6}|x - y| < \frac{1}{6}\left\{|x| + |y|\right\}$, *which is true for all* $x, y \in X$. *Thus* T *is a*

continuous map satisfying (2.1) and 0 *is its fixed point, which is unique. Also, if* x_0 *is any point of* X*, then the sequence* $\{T^n x_0\} = \{\frac{x_0}{2^n}\}$ *converges to* 0.
Consider the function

$$Tx = \left\{ \begin{array}{l} x2, \quad 0 \leq x < 10, \quad x = 1 \end{array} \right.$$

which has a discontinuity at $x = 1$*. Similar calculation shows that* T *is a map satisfying (2.1) and* 0 *is its fixed point, which is unique. And if* x_0 *is any point of* X*, then the sequence* $\{T^n x_0\} = \{\frac{x_0}{2^n}\}$ *converges to* 0.

Corollary 2.2. *Let* (X, d) *be a complete* b*-metric space and* $T : X \longrightarrow X$ *be a mapping such that*

$$d(Tx, Ty) \leq p\Big\{d(x, y) + d(x, Tx) + d(y, Ty)\Big\} \qquad \forall\ x, y \in X$$

where $p < \frac{1}{2s+1}$*. Then* T *has a unique fixed point* $z \in X$ *and for every* $x_0 \in X$*, the sequence* $\{T^n x_0\}$ *converges to* z.

Proof. The result follows from Theorem 2.1 on taking $\phi(x) = x$, $x \in X$.

Corollary 2.3. *Let* (X, d) *be a complete* b*-metric space and* $T : X \longrightarrow X$ *be a continuous mapping such that for some positive integer* k

$$\phi\left(d(T^k x, T^k y)\right) \leq p\left\{\phi\left(d(x, y)\right) + \phi\left(d(x, T^k x)\right) + \phi\left(d(y, T^k y)\right)\right\}$$

for some $p < \frac{1}{2s+1}$ *and for all* $x, y \in X$*. Then there exists an unique fixed point of* T.

Proof. Applying Theorem 2.1 to the self mapping $S = T^k$, we get that S has an unique fixed point, say z, so that $T^k z = Sz = z$.
Since $T^{k+1} z = Tz$,

$$STz = T^k(Tz) = T^{k+1} z = Tz\ ,$$

and so Tz is a fixed point of S. By the uniqueness of the fixed point of S, we get $Tz = z$. Taking $\phi(x) = \log(1 + x)$, we get the following result as

a particular case of Theorem 2.1.

Corollary 2.4. *Let (X,d) be a complete b-metric space and let $T : X \longrightarrow X$ be a mapping such that for $p < \frac{1}{2s+1}$, the relation*

$$(2.3)\quad \{1 + d(Tx,Ty)\}^{\frac{1}{p}} < e\,(1 + d(x,y))\,(1 + d(x,Tx))\,(1 + d(y,Ty))$$

holds for all $x, y \in X$. Then T has a unique fixed point $z \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to z and for $q = \frac{2p}{1-p}$,

$$d(T^{n+1}x, T^n x) \leq q^n d(x,Tx), \quad n = 0,1,2,\dots$$

Example 4. *Consider the b-metric space (X,d), where $X = [0,1]$ and $d(x,y) = |x-y|^2$ for all $x, y \in X$. Define the mapping $T : X \longrightarrow X$ by*

$$Tx = \frac{x}{k} \qquad \forall\, x \in X$$

for some $k \in \mathbf{N}$. Then $p < \frac{1}{5}$ and

$$\{1 + d(Tx,Ty)\}^5 = \left(1 + \frac{|x-y|^2}{k^2}\right)^5 \leq \left(1 + \frac{1}{k^2}\right)^5$$

and

$$e\,(1 + d(x,y))\,(1 + d(x,Tx))\,(1 + d(y,Ty)) \geq e.$$

Condition (2.3) is satisfied for $k \geq 3$ and by Corollary 2.4 T has an unique fixed point, which is 0 here. Moreover, for an arbitrary (but fixed) point $x_0 \in X$, the sequence of iterates $\{\frac{x_0}{k^n}\}$ converges to the fixed point 0.
On the other hand, if $d(x,y) = |x-y|$, then T satisfies (2.3) for $k \geq 2$.

Theorem 2.5. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let $T : X \longrightarrow X$ be a mapping such that there exists p_1, p_2, p_3 with $p_1 + p_2 + p_3 < 1$ and $sp_2 < 1$ satisfying*

$$(2.4)\quad \phi\,(d(Tx,Ty)) \leq p_1 \phi\,(d(x,y)) + p_2 \phi\,(d(x,Tx)) + p_3 \phi\,(d(y,Ty))$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to z and for $q = \frac{p_1+p_2}{1-p_3} < 1$,

$$d(T^{n+1}x, T^n x) \leq q^n d(x,Tx), \quad n = 0,1,2,\dots$$

Proof. The proof is similar to the proof of Theorem 2.1. On considering (X, d), a metric space and $\phi(x) = x$ in the above result, we get the result given in [17] as a particular case.

Following [16], we get a characterization for the completeness of (X, d) using the mapping T, with the help of the the properties of the subadditive altering distance function ϕ.

Theorem 2.6. *For a b-metric space (X, d), if every mapping $T : X \longrightarrow X$ satisfying (2.1) for some $0 \leq p < \frac{1}{2s+1}$ has an unique fixed point, then X is complete.*

It is worth mentioning that if (X, d) is a complete b-metric space and T is a self map on X such that for some $0 \leq p < \frac{1}{2s+1}$

$$\phi\left(d(Tx, Ty)\right) \leq p\Big\{\phi\left(d(x, Tx)\right) + \phi\left(d(y, Ty)\right)\Big\} \qquad \forall \;\; x, y \in X$$

then from Theorem 2.1, T has a unique fixed point $z \in X$ and for every $x_0 \in X$, the sequence $\Big\{T^n x_0\Big\}$ converges to z.

Following the proof of Theorem 2.1, we get the following result and derive the Kannan fixed point theorem as a consequence.

Theorem 2.7. *Let (X, d) be a complete b-metric space and let $T : X \longrightarrow X$ be a mapping such that there exists $p < \frac{1}{2s}$ satisfying*

$$\phi\left(d(Tx, Ty)\right) \leq p\Big\{\phi\left(d(x, Tx)\right) + \phi\left(d(y, Ty)\right)\Big\} \tag{2.5}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to z and for $q = \frac{p}{1-p} < 1$,

$$d(T^{n+1}x, z) \leq q^n d(x, Tx), \quad n = 0, 1, 2, \ldots$$

We note that when (X, d) is a complete metric space and $\phi(x) = x$ in the above theorem, we get Theorem 1.1, the Kannan fixed point theorem.

Example 5. *Consider the complete b-metric space (X, d) where*

$X = [0, 1] \cup [2, \infty)$ *and* $d(x, y) = \{\min\{x+y, 2\}, \;\; x \neq y 0, \qquad x = y$

If $\phi(t) = \log(1+t)$*, then condition (2.5) reduces to*

$$\{1 + d(Tx, Ty)\}^2 < e\,\{1 + d(x, Tx)\}\,\{1 + d(y, Ty)\} \tag{2.6}$$

Let $T : X \longrightarrow X$ *be defined by*

$$Tx = \{\ 12\ ,\quad 0 \le x \le 1\frac{1}{2} - \frac{1}{x}\ ,\quad x \ge 2$$

If $x, y \in [0, 1]$*, then (2.6) is trivially satisfied.*
For $2 < x < y$*, we have*

$$[1 + d(Tx, Ty)]^2 = \left[1 + \min\left\{1 - \frac{1}{x} - \frac{1}{y}, 2\right\}\right]^2 < 4$$

and

$$e\,\{1 + d(x, Tx)\}\,\{1 + d(y, Ty)\} \ge 9\ e\ .$$

When $x \in [0, 1]$ *and* $y \ge 2$*, then*

$$[1 + d(Tx, Ty)]^2 = \left[1 + \min\left\{1 - \frac{1}{y}, 2\right\}\right]^2 < 4$$

and

$$e\,\{1 + d(x, Tx)\}\,\{1 + d(y, Ty)\} = 3e\left(1 + x + \frac{1}{2}\right) \ge \frac{9}{2}e\ .$$

Thus T *satisfies (2.6) and by Theorem 2.7,* T *has a unique fixed point which in this case is* $x = \frac{1}{2}$*. For an arbitrary* $x_0 \in X$*, the sequence of iterates* $\{T^n x_0\}$ *converges to* $\frac{1}{2}$*. In fact,* $T^2 x = \frac{1}{2}$ *for all* $x \in X$*.*
We note that $s = 1$*,* $p < \frac{1}{2}$ *and* $sp < 1$*. If we consider the b-metric defined by*

$$d(x, y) = \{\ \min\{x + y, 2\}^2,\quad x \ne y0,\qquad x = y$$

then $s = 2$ *and we still have* $sp < 1$*. Similar calculation shows that* T *satisfies the conditions of Theorem 2.7 and we get the result.*

Theorem 2.8. *For a b-metric space* (X, d)*, if every mapping* $T : X \longrightarrow X$ *satisfying (2.5) for some* $p < \frac{1}{2s}$ *has a unique fixed point, then* X *is complete.*

Proof. Following the proof of Theorem 2.6, we get the result.

Remark 1. *Since sequentially compact b-metric spaces are complete, the completeness condition in Theorem 2.7 may be replaced by sequential compactness.*

Boundedly compactness and T-orbital compactness of X

A *boundedly compact metric space* ([14], [16]) is a metric space X in which every bounded sequence in X has a convergent subsequence. The same notion may be defined in the case of b-metric spaces. The class of boundedly compact b-metric spaces is larger than that of sequentially compact spaces as the the b-metric space $\mathbf{R}$ of real numbers with the usual metric is not sequentially compact but boundedly compact. In the next result p is independent of the coefficient s of the b-metric space.

Theorem 2.9. *Let (X, d) be a boundedly compact b-metric space and $T : X \longrightarrow X$ be a continuous mapping satisfying (2.5) for some $0 \leq p < \frac{1}{2}$. Then T has a unique fixed point $z \in X$ and for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to z.*

Proof. Let x_0 be an arbitrary point of X. Consider the iterated sequence $\{x_n\}$, where $x_n = T^n x_0$ for every $n \in \mathbf{N}$. We denote $d(x_n, x_{n+1})$ by λ_n and suppose that $\lambda_n > 0$ for all $n \in \mathbf{N}$. Then using (2.5), we have $\phi(\lambda_n) = \phi\left(d(T^n x_0, T^{n+1} x_0)\right) = \phi\left(d\left(T(T^{n-1} x_0), T(T^n x_0)\right)\right)$
$\leq p\Big\{\phi\left(d(T^{n-1} x_0, T^n x_0)\right) + \phi\left(d(T^n x_0, T^{n+1} x_0)\right)\Big\}$
$= p\phi(\lambda_{n-1}) + p\phi(\lambda_n)$ This implies

$$(1 - p)\phi(\lambda_n) < p\phi(\lambda_{n-1}) \qquad \forall\ n \in \mathbf{N}\ . \tag{2.7}$$

Since $1 - p \geq p$, it follows that

$$\lambda_n < \lambda_{n-1} \qquad \forall\ n \in \mathbf{N}$$

showing that the sequence $\{\lambda_n\}$ of positive real numbers is strictly decreasing sequence and hence convergent, say,

$$\lim_{n \to \infty} \lambda_n = \lambda\ .$$

Now, for $m, n \in \mathbf{N}$ with $n < m$, we have $\phi\left(d(x_m, x_n)\right) \leq p\Big\{\phi\left(d(x_{m-1}, x_m)+\right.$
$\left.\phi\left(d(x_{n-1}, x_n)\right)\Big\}$
$= p\Big\{\phi(\lambda_{m-1}) + \phi(\lambda_{n-1})\Big\}$. As $m, n \to \infty$, we have

$$\phi\left(d(x_m, x_n)\right) \leq \phi(\lambda)\ .$$

This implies $d(x_m, x_n) \leq \lambda$ as $m, n \to \infty$, showing that $\{x_n\}$ is a bounded sequence. Therefore, $\{x_n\}$ has a subsequence which converges to, say, z, i.e.,

$$\lim_{k\to\infty} x_{n_k} = z\ .$$

By the continuity of T, we have

$$Tz = T\left(\lim_{k\to\infty} T^{n_k}x_0\right) = \lim_{n\to\infty} T^{n_k+1}x_0 = z\ ,$$

which proves z is a fixed point of T.
Finally, if w is another fixed point of T, then $\phi\left(d(z, w)\right) = \phi\left(d(Tz, Tw)\right)$ $\leq p\left\{\phi\left(d(z, w)\right) + \phi\left(d(z, Tz)\right) + \phi\left(d(w, Tw)\right)\right\}$, that is,

$$(1-p)\phi\left(d(z, w)\right) \leq 0\ ,$$

which shows $z = w$, and thus z is the unique fixed point of T.

Example 6. *Consider the boundedly compact b-metric space (X, d), where $X = [0, \infty)$ and*

$$d(x, y) = \{\ x + y,\ \ x \neq y0, \qquad x = y$$

Define $T : X \longrightarrow X$ by

$$Tx = \Big\{\ \ 12,\ \ 0 \leq x \leq 2\frac{1}{x},\ \ x > 2$$

For $\phi(t) = t$, we have condition (2.5) as

$$d(Tx, Ty) < \frac{1}{2}\Big\{d(x, Tx) + d(y, Ty)\Big\}. \tag{2.8}$$

Now, for $x \neq y$ and $x, y > 2$, we have

$$d(Tx, Ty) = \frac{1}{x} + \frac{1}{y} < 1$$

and

$$\frac{1}{2}\{d(x,Tx)+d(y,Ty)\}=\frac{1}{2}\left\{x+\frac{1}{x}+y+\frac{1}{y}\right\}\geq 2.$$

Again, for $0 \leq x \leq 2$ and $y > 2$, we have

$$d(Tx,Ty)=\frac{1}{2}+\frac{1}{y}<1$$

and

$$\frac{1}{2}\{d(x,Tx)+d(y,Ty)\}=\frac{1}{2}\left\{x+\frac{1}{2}+y+\frac{1}{y}\right\}>1\ .$$

Thus T satisfies (2.8) and by Theorem 2.9, T has a unique fixed point which is $x=\frac{1}{2}$. Since $T^2x=\frac{1}{2}$, we see that for every $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ converges to $\frac{1}{2}$.

Garai et al. [16] defined *T-orbitally compact metric spaces* and derived a fixed point result for the same. The definition of T-orbitally compactness can be extended to b-metric spaces as follows.

Definition 2.1. *[16] Let (X,d) be a b-metric space and T be a self mapping on X. The orbit of T at the point $x \in X$ is defined as the set*

$$O_x(T)=\left\{x,Tx,T^2x,T^3x,\ldots\right\}$$

and X is said to be T-orbitally compact if every sequence in $O_x(T)$ has a convergent subsequence for all x in X.

As mentioned by Garai et al. [16] a T-orbitally compact metric space need not be complete. For more details of T-orbitally compact metric spaces one may refer to Garai et al. [16].

Theorem 2.10. *Let (X,d) be a T-orbitally compact b-metric space with T satisfying (2.5) with $p<\frac{1}{2}$ and $sp<1$. Then T has a unique fixed point w and for every $x \in X$,*

$$\lim_{n\to\infty} T^n x = w.$$

Proof. Let $x_0 \in X$ be arbitrarily chosen but fixed, and consider the sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \in \mathbf{N}$. Denoting $d(x_n, x_{n+1})$ by μ_n, we have from (2.5)

$$\phi(\mu_n) \le p\Big\{\phi(\mu_{n-1}) + \phi(\mu_n)\Big\}$$

and since ϕ is strictly increasing and $p < \frac{1}{2}$, we get

$$\mu_n < \mu_{n-1},$$

which shows that the sequence $\{\mu_n\}$ of non-negative real numbers is a decreasing sequence and hence convergent. Since X is T-orbitally compact, $\{x_n\}$ has a convergent subsequence, $\{x_{n_k}\}$, which converges to, $w \in X$, say. Now,

$$\lim_{n\to\infty} \mu_{n_k} = \lim_{n\to\infty} d\left(x_{n_k}, x_{n_k+1}\right) = d\left(\lim_{n\to\infty} x_{n_k}, \lim_{n\to\infty} x_{n_k+1}\right) = d(z,z) = 0.$$

This shows that the convergent sequence $\{\mu_n\}$ contains a subsequence $\{\mu_{n_k}\}$ which converges to 0 and therefore

$$\lim_{n\to\infty} \mu_n = 0.$$

For every $m, n \in \mathbf{N}$, we have $\phi\left(d(x_n, x_m)\right) \le p\Big\{\phi\left(d(T^{n-1}x, T^n x)\right) + \phi\left(d(T^{m-1}x, T^m x)\right)\Big\}$
$= p\Big\{\phi\left(\mu_{n-1}\right) + \phi\left(\mu_{m-1}\right)\Big\}$
$\longrightarrow 0$ as $n, m \to \infty$. This implies

$$d(x_n, x_m) \longrightarrow 0 \quad \text{as} \quad n, m \to \infty$$

showing that $\{x_n\}$ is a Cauchy sequence, and therefore

$$\lim_{n\to\infty} x_n = w.$$

Now, $\phi\left(d(w, Tw)\right) \le \phi\Big(sd(w, T^{n+1}x) + sd(T^{n+1}x, Tw)\Big)$
$\le s\phi\left(d(w, x_{n+1})\right) + sp\Big\{\phi\left(d(x_n, x_{n+1})\right) + \phi\left(d(w, Tw)\right)\Big\}$ that is, (1-sp)$\phi\left(d(w, Tw)\right) \le s\phi\left(d(w, x_{n+1})\right) + sp\phi\left(d(x_n, x_{n+1})\right)$
$\longrightarrow 0$ as $n \to \infty$ which implies $d(w, Tw) = 0$, establishing that w is a fixed point of T. The uniqueness of the fixed point is derived from condition (2.5) and the monotonicity of ϕ.

Example 7. *Consider the incomplete b-metric space (X, d), where $X = (0, \infty)$ and*

$$d(x, y) = \{\, x + y,\ x \neq y0, \quad x = y$$

Define $T : X \longrightarrow X$ by

$$Tx = \Big\{\ 12, \quad 0 < x < 21, \quad x = 2\frac{1}{x}, \quad x > 2$$

It can be easily seen that T is not continuous and X is T-orbitally compact. For $\phi(x) = \log(1 + x)$, we have condition (2.5) as

$$\{1 + d(Tx, Ty)\}^2 < e\,\{1 + d(x, Tx)\}\,\{1 + d(y, Ty)\}$$

For $x, y > 2$, we have

$$\{1 + d(Tx, Ty)\}^2 = \left\{1 + \frac{1}{x} + \frac{1}{y}\right\}^2 < 4,$$

$$e\,\{1 + d(x, Tx)\}\,\{1 + d(y, Ty)\} = e\left\{1 + x + \frac{1}{x}\right\}\left\{1 + y + \frac{1}{y}\right\} \geq 9e.$$

For $0 < x < 2$ and $y > 2$, we have

$$\{1 + d(Tx, Ty)\}^2 = \left\{1 + \frac{1}{2} + \frac{1}{y}\right\}^2 < 4,$$

$$e\,\{1 + d(x, Tx)\}\,\{1 + d(y, Ty)\} = e\left\{1 + x + \frac{1}{2}\right\}\left\{1 + y + \frac{1}{y}\right\} > 4e.$$

For $0 < x < 2$ and $y = 2$, we have

$$\{1 + d(Tx, T2)\}^2 = \left\{1 + \frac{1}{2} + 1\right\}^2 = \frac{25}{4} < 6,$$

$$e\,\{1 + d(x, Tx)\}\,\{1 + d(2, T2)\} = e\left\{1 + x + \frac{1}{2}\right\}\{1 + 2 + 1\} \geq 6e.$$

For $x > 2$ and $y = 2$, we have

$$\{1 + d(Tx, T2)\}^2 = \left\{1 + \frac{1}{x} + 1\right\}^2 < 6,$$

$$e\,\{1 + d(x, Tx)\}\,\{1 + d(2, T2)\} = e\left\{1 + x + \frac{1}{2}\right\}\{1 + 2 + 1\} > 12e.$$

Thus T satisfies condition (2.5) and therefore, by Theorem 2.10, T has a unique fixed point, $x = \frac{1}{2}$. Also, for an arbitrary $x_0 \in X$, it is easily seen that $T^2 x_0 = \frac{1}{2}$ so that the sequence of iterates $\{T^n x_0\}$ converge to the fixed point $x = \frac{1}{2}$.

Asymptotic regularity of T

In the previous section, Theorem 2.7 does not hold for $p \geq \frac{1}{2}$. Here, we try to raise the bound of p by assuming T to be an asymptotically regular mapping. For a metric space (X, d), a mapping $T : X \longrightarrow X$ is called *asymptotically regular* [7] if

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0 \qquad \text{for all} \quad x \in X.$$

For further details in asymptotic regular mappings we refer to [3, 8] and the references therein.

Theorem 2.11. *Let (X, d) be a complete b-metric space and $T : X \longrightarrow X$ be an asymptotically regular map satisfying (2.5) for some p with $sp < 1$. Then T has a unique fixed point.*

Proof. Let $x \in X$ and consider the sequence $\{x_n\}$ where $x_n = T^n x$, $n \in$ **N**. For $m > n$, since T is asymptotically regular $\phi\left(d(T^{n+1}x, T^{m+1}x)\right) \leq p\Big\{\phi\left(d(T^n x, T^{n+1}x)\right) + \phi\left(d(T^m, T^{m+1})\right)\Big\}$
$\longrightarrow 0 \quad$ as $\quad n \to \infty$ Thus

$$d(T^{n+1}x, T^{m+1}x) \longrightarrow 0 \qquad \text{as} \quad n \to \infty$$

showing that the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} T^n x = z$.
Again, $\phi\left(d(z, Tz)\right) \leq \phi\Big(sd(z, T^{n+1}x) + sd(T^{n+1}x, Tz)\Big)$
$\leq s\phi\left(d(z, T^{n+1}x)\right) + s\phi\left(d(T^{n+1}x, Tz)\right)$
$\leq s\phi\left(d(z, T^n x)\right) + sp\Big\{\phi\left(d(T^n x, T^{n+1}x)\right) + \phi\left(d(z, Tz)\right)\Big\}$ That is,

$$(1 - sp)\phi\left(d(z, Tz)\right) \leq s\phi\left(d(z, T^n x)\right) + sp\phi\left(d(T^n x, T^{n+1}x)\right).$$

Therefore, in the limiting case when $n \to \infty$, we have

$$(1 - sp)\phi\left(d(z, Tz)\right) = 0 \qquad\qquad d(z, Tz) = 0\ .$$

Suppose that $Tw = w$ with $z \neq w$. Then

$$\phi\left(d(Tz, Tw)\right) \leq p\Big\{\phi(d(z, Tz)) + \phi(d(w, Tw))\Big\} = 0\ ,$$

implying $Tz = Tw$. But then we have

$$w = Tw = Tz = z\ ,$$

a contradiction.

Example 8. *Consider the complete b-metric space (X, d) where $X = [0,1]$ and d is the usual metric on X. It can be easily seen that the function $T : X \longrightarrow X$ defined by $Tx = \frac{x}{2}$ for all $x \in X$ is asymptotically regular. Since $s = 1$, we can take $p < 1$. Then for $\phi(t) = \log(1+t)$, condition (2.5) reduces to*

$$1 + d(Tx, Ty) < e\Big\{1 + d(x, Tx)\Big\}\Big\{1 + d(y, Ty)\Big\}$$

But we have, 1+d(Tx,Ty)=1+$|\frac{x}{2} - \frac{y}{2}| \leq 1 + |\frac{x}{2}| + |\frac{y}{2}| \leq \{1 + |\frac{x}{2}|\}\,\{1 + |\frac{y}{2}|\}$ $\leq e\Big\{1 + d(x, Tx)\Big\}\Big\{1 + d(y, Ty)\Big\}$. By the above theorem, there exists a unique fixed point. Here, $x = 0$ is the unique fixed point.
If we consider the b-metric $d(x, y) = |x - y|^2$, then $s = 2$ and $p < \frac{1}{2}$. Condition (2.5) in this case is

$$\Big\{1 + d(Tx, Ty)\Big\}^2 < e\Big\{1 + d(x, Tx)\Big\}\Big\{1 + d(y, Ty)\Big\}$$

and is satisfied by T and we get the same result as before.

Theorem 2.12. *Let (X, d) be a complete b-metric space and $T : X \longrightarrow X$ be an asymptotically regular map satisfying (2.1) for some p with $sp < 1$. Then T has a unique fixed point.*

Proof. Let $x \in X$ and consider the sequence $\{x_n\}$ where $x_n = T^n x$, $n \in \mathbf{N}$. For $m > n$, since T is asymptotically regular, we have $\phi\Big(d(T^{n+1}x, T^{m+1}x)\Big) \leq$
$p\Big\{\phi\left(d(T^n x, T^m x)\right) + \phi\left(d(T^n x, T^{n+1}x)\right)$
$+ \phi\left(d(T^m x, T^{m+1}x)\right)\Big\}$
$\leq kp\Big\{\phi\left(d(T^n x, T^{n+1}x)\right) + \phi\left(d(T^m x, T^{m+1}x)\right)\Big\}$
$p\phi\Big(d(T^{n+1}x, T^{m+1}x)\Big)$, for some positive integer k, and so, $\phi\Big(d(T^{n+1}x, T^{m+1}x)\Big) \leq$
$\frac{kp}{1-p}\Big\{\phi\left(d(T^n x, T^{n+1}x)\right) + \phi\left(d(T^m x, T^{m+1}x)\right)\Big\}$

$\longrightarrow 0$ as $n \to \infty$, which shows that $\{T^n x\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} T^n x = z.$$

Now, $\phi\Big(d(z,Tz)\Big) \leq \phi\Big(sd(z,T^{n+1}x) + sd(T^{n+1}x,Tz)\Big)$
$\leq s\phi\left(d(z,T^{n+1}x)\right) + sp\Big\{\phi\left(d(T^n x,z)\right) + \phi\left(d(T^n x,T^{n+1}x)\right)$
$+\phi\left(d(z,Tz)\right)\Big\}$ or, (1-sp)$\phi\Big(d(z,Tz)\Big) \leq s\phi\Big(d(z,T^{n+1}x)\Big)$
$+sp\Big\{\phi\left(d(T^n x,z)\right) + \phi\left(d(T^n x,T^{n+1}x)\right)\Big\}$
$\longrightarrow 0$ as $n \to \infty$. Hence $d(z,Tz) = 0$, that is, z is a fixed point of T. If possible, let $w \neq z$ with $Tw = w$. Then $\phi\left(d(Tw,Tz)\right) \leq p\Big\{\phi\left(d(w,z)\right)+$
$\phi\left(d(w,Tw)\right) + \phi\left(d(z,Tz)\right)\Big\}$
$< \phi\left(d(w,z)\right)$ which is a contradiction. Hence the result. As pointed out by Górnicki in [17], a mapping $T : X \longrightarrow X$ satisfying

$$\phi\left(d(Tx,Ty)\right) < \phi\left(d(x,Tx)\right) + \phi\left(d(y,Ty)\right)$$

for all $x, y \in X$ with $x \neq y$, and asymptotically regular may not have a fixed point (one may refer to Example 3.2 of [17]).

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