



Nonlinear elliptic problems in weighted variable exponent Sobolev spaces by topological degree

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Abstract:

In this paper, we prove the existence of solutions for the nonlinear $p(\cdot)$ -degenerate problems involving nonlinear operators of the form

- $\operatorname{div} a(x, \nabla u) = f(x, u, \nabla u)$ where a and f are Carathéodory functions satisfying some nonstandard growth conditions.

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1. Introduction

A class of nonlinear problems with variable exponents is a new search field which reflects a new type of physical phenomena. For a number of materials with inhomogenies, modeling using classical Lebesgue and Sobolev spaces has shown its limitation. For example, in the context of fluid mechanics, much attention has been given to the study of electrorological fluids that have the ability to change their mechanical properties when an electric field is applied to them. Recently, Rajagopal and M. Růžička developed in [15](see also [16]) a very interesting model for these fluids taking into account the delicate interaction between the electric field $E(x)$ and the moving liquid. The energy associated with this kind of problems is given by $\int_{\Omega} |\nabla u|^{p(x)} dx$ or rather by $\int_{\Omega} w(x) |\nabla u|^{p(x)} dx$. This type of energy also appears in the problems of elasticity [22].

The natural energy space in which such problems can be studied is the weighted variable exponent Sobolev space $W_0^{1,p(x)}(\Omega, w)$. For more information about these spaces and their properties, we refer to articles [1, 19, 20] and references therein.

Let $\Omega \subset \mathbf{R}^N$ be an open bounded set with boundary $\partial\Omega$ of class C^1 . In this paper we study the existence of solutions of the following nonlinear degenerated $p(x)$ elliptic problem

$$(1.1) \quad \begin{cases} -\operatorname{div} a(x, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where

$$p \in \mathcal{C}_+(\Omega) := \{\text{measurable function } h(\cdot) : \overline{\Omega} \rightarrow \mathbf{R}^+ / 1 < h^- \leq h(x) \leq h^+ < +\infty\},$$

whith

$$h^- = \operatorname{ess\,inf}\{h(x)/x \in \Omega\} \text{ and } h^+ = \operatorname{ess\,sup}\{h(x)/x \in \Omega\}.$$

We also assume that p satisfies the log-Hölder continuity condition, i.e. there is a constant $C > 0$ such that for every $x, y \in \overline{\Omega}$, $x \neq y$ with $|x-y| \leq \frac{1}{2}$ one has

$$(1.2) \quad |p(x) - p(y)| \leq \frac{C}{-\log|x-y|},$$

The operator $Au \equiv -\operatorname{div} a(x, \nabla u)$ is a Leray-Lions operator defined on $W_0^{1,p(x)}(\Omega, w)$, where w is a weight function, satisfying some integrability conditions and f is Carathéodory functions satisfying some nonstandard growth condition.

By studying the problem (1.1), we then extend both a class problems that involve Leray-Lions operators with variable exponents (see [2]) and another of degenerate problems that involve special weights [2, 7, 10].

In this paper, the method used to treat the problem (1.1) is the topological degree which is widely used in the study of nonlinear equations, especially elliptic. Since the construction of the first topological degree by Brouwer in 1912 for continuous maps in finite dimensional Euclidean spaces [8], this theory has undergone several extensions and generalizations. Leray and Schauder generalized it in 1934 for compact operators in Banach spaces of infinite dimension [14]. Later, the theory was built by Berkovits and Mustonen [3, 4, 5, 6].

This paper is divided into three sections, organized as follows: in section 2, we introduce some basic properties of the space $W^{1,p(x)}(\Omega, w)$ and an outline of Berkovits degree theory. In section 3, we prove the existence of solutions of (1.1).

2. Mathematical Preliminaries

2.1. Weighted variable Lebesgue and Sobolev spaces

In this subsection, we recall some definitions and fundamental properties of weighted variable Lebesgue and Sobolev spaces (see [1, 19, 20] for more details).

Let Ω an open bounded subset of \mathbf{R}^N ($N \geq 1$) and $p(\cdot) \in \mathcal{C}_+(\Omega)$ satisfying the log-Hölder continuity condition (1.2).

Definition 2.1. A function w defined on Ω is called a weigh function if it is measurable and strictly positive a.e. in Ω .

Let introduce the integrability conditions used on the framework of these spaces

$$(2.1) \quad w \in L_{loc}^1(\Omega),$$

$$(2.2) \quad w^* \in L_{loc}^1(\Omega),$$

where $w^*(x) = w(x)^{1-p'(x)}$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We define the weighted variable exponent Lebesgue space as

$$L^{p(x)}(\Omega, w) = \left\{ u : \Omega \rightarrow \mathbf{R} \text{ measurable, } \int_{\Omega} |u(x)|^{p(x)} w(x) dx < \infty \right\}.$$

The space $L^{p(x)}(\Omega, w)$ endowed with the norm :

$$\|u\|_{p(x),w} = \inf \left\{ \lambda > 0, \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} w(x) dx \leq 1 \right\}$$

is a uniformly convex Banach space, and therefore reflexive. Let us denote

$$I_{p,w}(u) = \int_{\Omega} |u|^{p(x)} w(x) dx, \quad \forall u \in L^{p(x)}(\Omega, w).$$

Note that if $w \equiv 1$, then $L^{p(x)}(\Omega, w) = L^{p(x)}(\Omega)$ and $I_{p,w}(u) = I_{p,1}(u) = \rho_{p(x)}$ (the convex modular, see [11]).

$L^{p'(x)}(\Omega, w^*)$ is the conjugate space of $L^{p(x)}(\Omega, w)$.

By taking

$$(2.3) \quad I_{p,w}(u) = \rho_{p(x)}(w^{\frac{1}{p(x)}} u) \quad \text{and} \quad \|w^{\frac{1}{p(x)}} u\|_{p(x)} = \|u\|_{p(x),w},$$

we have, as in [11], the Hölder inequality

$$(2.4) \quad \forall u \in L^{p(x)}(\Omega, w), \forall v \in L^{p'(x)}(\Omega, w^*), \left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|u\|_{p(x),w} \|v\|_{p'(x),w^*}.$$

We have also the continuous embedding $L^{p_2(x)}(\Omega, w) \hookrightarrow L^{p_1(x)}(\Omega, w)$ for all

$p_1, p_2 \in \mathcal{C}_+(\Omega)$ where $p_1(x) \leq p_2(x)$ a.e. $x \in \overline{\Omega}$.

We can prove also, by the identifications id, as in [11], the following important result

Proposition 2.2. *for each $u \in L^{p(x)}(\Omega, w)$,*

1. $\|u\|_{p(x),w} < 1$ (resp. $= 1$ or > 1) $\Leftrightarrow I_{p,w}(u) < 1$ (resp. $= 1$ or > 1),
2. $\|u\|_{p(x),w} > 1 \Rightarrow \|u\|_{p(x),w}^{p_-} \leq I_{p,w}(u) \leq \|u\|_{p(x),w}^{p_+}$,
 $\|u\|_{p(x),w} < 1 \Rightarrow \|u\|_{p(x),w}^{p_+} \leq I_{p,w}(u) \leq \|u\|_{p(x),w}^{p_-}$,

$$3. I_{p,w}(u) \leq \|u\|_{p(x),w}^{p_+} + \|u\|_{p(x),w}^{p_-},$$

$$4. \|u\|_{p(x),w} \leq I_{p,w}(u) + 1,$$

$$5. \|u\|_{p(x),w} \rightarrow 0 \Leftrightarrow I_{p,w}(u) \rightarrow 0 \text{ and } \|u\|_{p(x),w} \rightarrow \infty \Leftrightarrow I_{p,w}(u) \rightarrow \infty.$$

The weighted variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega, w) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega, w)\}.$$

equipped with the following norm

$$\|u\|_{1,p(x),w} = \|u\|_{p(x)} + \|\nabla u\|_{p(x),w} \quad \text{for all } u \in W^{1,p(x)}(\Omega, w).$$

We denote by $W_0^{1,p(x)}(\Omega, w)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, w)$ with respect to the norm $\|u\|_{1,p(x),w}$, we have the following compact embedding

Proposition 2.3. [13, Proposition 2.3.] $W_0^{1,p(x)}(\Omega, w) \hookrightarrow L^{p(x)}(\Omega).$

The dual space of $W_0^{1,p(\cdot)}(\Omega, w)$, denoted $W^{-1,p'(\cdot)}(\Omega, w^*)$, is equipped with the norm

$$\|v\|_{-1,p'(x),w^*} = \inf \left\{ \|v_0\|_{p'(x),w^*} + \sum_{i=1}^N \|v_i\|_{p'(x),w^*} \right\}$$

where the infimum is taken on all possible decompositions $v = v_0 - \operatorname{div} F$.

Remark 2.4. [19, Theorem 7 and Corollary 2] The Poincaré inequality holds for the weighted Sobolev spaces $W_0^{1,p(\cdot)}(\Omega, w)$. In particular, this space has a norm $\|\cdot\|_{1,p(x)}$ given by

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x),w} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega, w),$$

which equivalent to $\|\cdot\|_{1,p(x),w}$.

2.2. An outline of Berkovits degree theory

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle \cdot, \cdot \rangle$, Ω be a nonempty subset of X and Y be a real Banach space. The symbol \rightarrow (\rightharpoonup) stands for strong (weak) convergence.

In this subsection, we give an outline of the Berkovits degree we will use to study the problem (1.1). We begin by defining the appropriate operator classes that Berkovits used to build this degree. The reader can find the definition of these classes in [3, 4, 5, 6].

We recall that a mapping $F : \Omega \subset X \rightarrow Y$ is

- *bounded*, if it takes any bounded set into a bounded set.
- *demicontinuous*, if for any $(u_n) \subset \Omega$, $u_n \rightarrow u$ implies $F(u_n) \rightharpoonup F(u)$.
- *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \rightarrow X^*$ is said to be

- *of class (S_+)* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup \langle Fu_n, u_n - u \rangle \leq 0$, we have $u_n \rightarrow u$.
- *quasimonotone*, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup \langle Fu_n, u_n - u \rangle \geq 0$.

For any operator $F : \Omega \subset X \rightarrow X$ and any bounded operator $T : \Omega_1 \subset X \rightarrow X^*$ such that $\Omega \subset \Omega_1$, we say that F is *of class $(S_+)_T$* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup \langle Fu_n, y_n - y \rangle \leq 0$, we have $u_n \rightarrow u$.

Let \mathcal{O} be the collection of all bounded open set in X . For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned} \mathcal{F}_1(\Omega) &:= \{F : \Omega \rightarrow X^* \mid F \text{ is bounded, demicontinuous and of class } (S_+)\}, \\ \mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \rightarrow X \mid F \text{ is bounded, demicontinuous and of class } (S_+)_T\}, \\ \mathcal{F}_T(\Omega) &:= \{F : \Omega \rightarrow X \mid F \text{ is demicontinuous and off class } (S_+)_T\}, \\ \mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \mid G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G})\}. \end{aligned}$$

Here, $T \in \mathcal{F}_1(\bar{G})$ is called an *essential inner map* to F .

Definition 2.5. Let G be a bounded open subset of a real reflexive Banach space X , $T \in \mathcal{F}_1(\bar{G})$ be continuous and let $F, S \in \mathcal{F}_T(\bar{G})$. We define a class $\mathcal{H}_T(\bar{G})$ (called a class of admissible affine homotopy with the common continuous essential inner map T) as the class of affine homotopy $H : [0, 1] \times \bar{G} \rightarrow X$ defined by

$$H(t, u) := (1 - t)Fu + tSu \text{ for } (t, u) \in [0, 1] \times \bar{G}.$$

Remark 2.6. [4]

$$\mathcal{H}_T(\bar{G}) \subset (S_+)_T.$$

We introduce the topological degree for the class $\mathcal{F}_B(X)$ due to Berkovits [4].

Theorem 2.7. There exists a unique degree function

$$d : \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_{T,B}(\bar{G}), h \notin F(\partial G)\} \rightarrow \mathbf{Z}$$

that satisfies the following properties

1. (Existence) if $d(F, G, h) \neq 0$, then the equation

$$Fu = h$$

has a solution in G .

2. (Additivity) Let $F \in \mathcal{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

3. (Homotopy invariance) If $H \in \mathcal{H}_T(\bar{G})$ is bounded and $h : [0, 1] \rightarrow X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0, 1]$, then the value of $d(H(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.

4. (Normalization) For any $h \in G$, we have

$$d(I, G, h) = 1$$

where I denotes the identity operator.

3. Main result

3.1. Basic assumptions and technical lemmas

Throughout the paper, we assume that the following assumptions hold. Let Ω be a bounded open set of \mathbf{R}^N ($N \geq 1$) and $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. And let w a weight function in Ω which satisfies the integrability conditions (2.1) and (2.2).

The function $a : \Omega \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a Carathéodory function satisfying the following conditions: For all $\xi \in \mathbf{R}^N$ and for almost every $x \in \Omega$,

$$(3.1) \quad |a(x, \xi)| \leq \beta w^{\frac{1}{p(x)}} (k(x) + w^{\frac{1}{p'(x)}} |\xi|^{p(x)-1}),$$

$$(3.2) \quad [a(x, \xi) - a(x, \eta)](\xi - \eta) > 0 \quad \forall \xi \neq \eta,$$

$$(3.3) \quad a(x, \xi)\xi \geq \alpha w(x) |\xi|^{p(x)},$$

where $k(x)$ is a positive function in $L^{p'(x)}(\Omega)$ and α and β are a positive constants.

Let $f(x, s, \xi) : \Omega \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbf{R}$, $\xi \in \mathbf{R}^N$, the growth condition

$$(3.4) \quad |f(x, s, \xi)| \leq \gamma w^{\frac{1}{p(x)}} (h(x) + w^{\frac{1}{p'(x)}} |s|^{q(x)-1} + w^{\frac{1}{p'(x)}} |\xi|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, where γ is a positive constant, h is a positive function in $L^{p'(x)}(\Omega)$ and $1 < q^- \leq q(x) \leq q^+ < p^-$.

Lemma 3.1. [13] Let $g \in L^{r(x)}(\Omega, w)$ and $g_n \in L^{r(x)}(\Omega, w)$ with $\|g_n\|_{r(x), w} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \rightarrow g(x)$ a.e on Ω , then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega, w)$.

Let define the operator $A : W_0^{1,p(x)}(\Omega, w) \rightarrow W^{-1,p'(x)}(\Omega, w^*)$ by

$$(3.5) \quad Au = -\operatorname{div}(x, \nabla u)$$

then,

$$\langle Au, v \rangle = \int_{\Omega} a(x, \nabla u) \nabla v dx,$$

for all v in $W_0^{1,p(x)}(\Omega, w)$.

Lemma 3.2. If the assumptions (3.1)-(3.3) holds, then A is bounded, continuous, coercive and of class (S_+) .

Proof. Step 1:

Let's show that the operator A is bounded.

By using the Hölder's inequality (2.4), we have for all $u, v \in W_0^{1,p(x)}(\Omega, w)$

$$\begin{aligned} |\langle Au, v \rangle| &= \left| \int_{\Omega} a(x, \nabla u) \nabla v dx \right| \\ &= \left| \int_{\Omega} a(x, \nabla u) w(x)^{\frac{-1}{p(x)}} \cdot \nabla v w(x)^{\frac{1}{p(x)}} dx \right| \\ &\leq C \|a(x, \nabla u) w(x)^{\frac{-1}{p(x)}}\|_{p'(x)} \cdot \|\nabla v w(x)^{\frac{1}{p(x)}}\|_{p(x)}. \end{aligned}$$

Thanks to (2.1), (3.1) and Proposition 2.2 we can easily prove that

$\|a(x, \nabla u) w(x)^{\frac{-1}{p(x)}}\|_{p'(x)}$ is bounded for all $u \in W_0^{1,p(x)}(\Omega, w)$. Therefore

$$\begin{aligned} |\langle Au, v \rangle| &\leq \text{const} \|\nabla v w(x)^{\frac{1}{p(x)}}\|_{p(x)} \\ &= \text{const} \|\nabla v\|_{p(x), w} \\ &= \text{const} \|v\|_{1, p(x), w}, \end{aligned}$$

which means that the operator A is bounded.

Step 2:

To show that A is continuous, let $u_n \rightarrow u$ in $W_0^{1,p(x)}(\Omega, w)$. Then $\nabla u_n \rightarrow \nabla u$ in $L^{p(x)}(\Omega, w)$. Hence there exist a subsequence (u_k) of (u_n) and measurable function g in $(L^{p(x)}(\Omega, w))^N$ such that

$$\nabla u_k(x) \rightarrow \nabla u(x) \text{ and } |\nabla u_k(x)| \leq g(x)$$

for a.e. $x \in \Omega$ and all $k \in \mathbf{N}$. Since a satisfies the Carathodory condition, we have

$$a(x, \nabla u_k(x)) \rightarrow a(x, \nabla u(x)) \text{ a.e. } x \in \Omega.$$

and from (3.1), we get

$$|a(x, \nabla u_k(x))| \leq \beta w^{\frac{1}{p(x)}} (k(x) + w^{\frac{1}{p'(x)}} |g(x)|^{p(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbf{N}$. Since

$$\int_{\Omega} [w^{\frac{1}{p(x)}} (k(x) + w^{\frac{1}{p'(x)}} |g(x)|^{p(x)-1})]^{p'(x)} w^*(x) dx \leq C \int_{\Omega} (|k(x)|^{p'(x)} + w |g(x)|^{p(x)}) < \infty,$$

(because $k \in L^{p'(x)}(\Omega)$ and $g \in (L^{p(x)}(\Omega, w))^N$), then

$$x \longmapsto \beta w^{\frac{1}{p(x)}} (k(x) + w^{\frac{1}{p'(x)}} |g(x)|^{p(x)-1}) \in L^{p'(x)}(\Omega, w^*),$$

and thanks to the equality

$$I_{p', w^*}(a(x, \nabla u_k) - a(x, \nabla u)) = \int_{\Omega} |a(x, \nabla u_k(x)) - a(x, \nabla u(x))|^{p'(x)} w^*(x) dx,$$

the dominated convergence theorem and the equivalence (5) of Proposition 2.2 allow us to assert that

$$a(x, \nabla u_k) \rightarrow a(x, \nabla u) \text{ in } L^{p'(x)}(\Omega, w^*).$$

Thus the entire sequence $a(x, \nabla u_n)$ converges to $a(x, \nabla u)$ in $L^{p'(x)}(\Omega, w^*)$.

Then, $\forall v \in W_0^{1,p(x)}(\Omega, w); \langle Au_n, v \rangle \rightarrow \langle Au, v \rangle$, which implies that the operator A is continuous.

Step 3

In this step, we show that A is coercive. For that Let $v \in W_0^{1,p(x)}(\Omega, w)$, We have from (3.3), the Proposition 2.2 and Remark 2.4

$$\frac{\langle Av, v \rangle}{\|v\|_{1,p(x),w}} \geq \frac{C}{\|v\|_{1,p(x)}} \alpha I_{p,w}(\nabla v) \geq C' \|v\|_{1,p(x)}^r$$

for some $r > 1$. Therefore

$$\frac{\langle Av, v \rangle}{\|v\|_{1,p(x),w}} \longrightarrow \infty \quad \text{as } \|v\|_{1,p(x),w} \rightarrow \infty.$$

So, A is coercive.

Step 4

It remains to show that the operator A is of class (S_+) . Let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega, w)$ such that

$$(3.6) \quad \begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p(x)}(\Omega, w) \\ \limsup_{k \rightarrow \infty} \langle Au_n, u_n - u \rangle \leq 0. \end{cases}$$

We will prove that

$$u_n \longrightarrow u \text{ in } W_0^{1,p(x)}(\Omega, w).$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega, w)$, then $(u_n)_n$ is a bounded sequence in $W_0^{1,p(x)}(\Omega, w)$. By using the Propositions 2.2 and 2.3, there exist a subsequence still denoted by $(u_n)_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega, w), \\ u_n &\rightarrow u \text{ in } L^{p(x)}(\Omega) \text{ and a.e in } \Omega. \end{aligned}$$

From (3.2) and (3.6), we have

$$(3.7) \quad \lim_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle = \lim_{n \rightarrow \infty} \langle Au_n - Au, u_n - u \rangle = 0.$$

Let $D_n = a(x, \nabla u_n) \cdot \nabla(u_n - u)$. By (3.7) $D_n \rightarrow 0$ in $L^1(\Omega)$ and for a subsequence $D_n \rightarrow 0$ a.e. in Ω .

Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega, w)$ and a.e in Ω , there exists a subset B of Ω , of zero measure, such that for $x \in \Omega \setminus B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $k(x) < \infty$, $u_n(x) \rightarrow u(x)$, $D_n(x) \rightarrow 0$.

Defining $\xi_n = \nabla u_n(x)$, $\xi = \nabla u(x)$, we have

$$\begin{aligned} D_n(x) &= a(x, \xi_n) \cdot (\xi_n - \xi) \\ &= a(x, \xi_n) \xi_n - a(x, \xi_n) \xi \\ &\geq \alpha w(x) |\xi_n|^{p(x)} - \beta w(x)^{\frac{1}{p(x)}} (k(x) + w(x)^{\frac{1}{p(x)}} |\xi_n|^{p(x)-1}) |\xi| \\ &\geq \alpha |\xi_n|^{p(x)} - C_x [1 + |\xi_n|^{p(x)-1}], \end{aligned}$$

where C_x is a constant which depends on x , but does not depend on n . Since

$u_n(x) \rightarrow u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n , we deduce that

$$D_n(x) \geq |\xi_n|^{p(x)} \left(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} \right).$$

If $|\xi_n| \rightarrow \infty$ (for a subsequence), then $D_n(x) \rightarrow \infty$ which gives a contradiction. Let now ξ^* be a cluster point of ξ_n . We have $|\xi^*| < \infty$ and by the continuity of a we obtain

$$a(x, \xi^*) \cdot (\xi^* - \xi) = 0.$$

In view of (3.2), we have $\xi^* = \xi$, which implies that

$$\nabla u_n(x) \rightarrow \nabla u(x) \quad \text{a.e. in } \Omega.$$

Since the sequence $a(x, \nabla u_n)$ is bounded in $(L^{p'(x)}(\Omega, w^*))^N$, and

$a(x, \nabla u_n) \rightarrow a(x, \nabla u)$ a.e. in Ω , then by Lemma 3.1 we get

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \quad \text{in } (L^{p'(x)}(\Omega, w^*))^N \text{ a.e. in } \Omega.$$

We set $\bar{y}_n = a(x, \nabla u_n) \nabla u_n$ and $\bar{y} = a(x, \nabla u) \nabla u$. We can write

$$\bar{y}_n \rightarrow \bar{y} \quad \text{in } L^1(\Omega).$$

We have

$$a(x, \nabla u_n) \cdot \nabla u_n \geq \alpha w(x) |\nabla u_n|^{p(x)}.$$

Let $z_n = w |\nabla u_n|^{p(\cdot)}$, $z = w |\nabla u|^{p(\cdot)}$, $y_n = \frac{\bar{y}_n}{\alpha}$, and $y = \frac{\bar{y}}{\alpha}$. By Fatou's lemma,

$$\int_{\Omega} 2y \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} y + y_n - |z_n - z| \, dx;$$

i.e., $0 \leq -\limsup_{n \rightarrow \infty} \int_{\Omega} |z_n - z| \, dx$. Then

$$0 \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |z_n - z| \, dx \leq \limsup_{n \rightarrow \infty} \int_{\Omega} |z_n - z| \, dx \leq 0,$$

this implies

$$\nabla u_n \rightarrow \nabla u \quad \text{in } (L^{p(x)}(\Omega, w))^N.$$

Hence $u_n \rightarrow u$ in $W_0^{1,p(x)}(\Omega, w)$, which completes the proof.

Lemma 3.3. *Let $S : W_0^{1,p(x)}(\Omega, w) \rightarrow W^{-1,p'(x)}(\Omega, w^*)$ be the operator setting by*

$$\langle Su, v \rangle = - \int_{\Omega} f(x, u, \nabla u) v \, dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega, w).$$

If the assumption (3.4) holds, then S is compact.

Proof. Let us define an operator $\phi : W_0^{1,p(x)}(\Omega, w) \rightarrow L^{p'(x)}(\Omega, w^*)$ by

$$\phi u(x) := -f(x, u, \nabla u) \text{ for } u \in W_0^{1,p(x)}(\Omega, w) \text{ and } x \in \Omega.$$

We begin by showing that ϕ is bounded and continuous.

For each $u \in W_0^{1,p(x)}(\Omega, w)$, we have by the growth condition (3.4), the inequalities (3) and (4) of Proposition 2.2 that

$$\begin{aligned}
 \|\phi u\|_{p'(x), w^*} &\leq I_{p', w}(\phi u) + 1 \\
 &= \int_{\Omega} |f(x, u(x), \nabla u(x))|^{p'(x)} w^*(x) dx + 1 \\
 &\leq \text{const}(\rho_{p'(x)}(h) + I_{r, w}(u) + I_{r, w}(\nabla u)) + 1 \\
 &\leq \text{const}(\|h\|_{p'(x)}^{p'^+} + \|h\|_{p'(x)}^{p'^-} + \|u\|_{r(x), w}^{r^+} + \|u\|_{r(x), w}^{r^-} \\
 &\quad + \|\nabla u\|_{r(x), w}^{r^+} + \|\nabla u\|_{r(x), w}^{r^-}) + 1,
 \end{aligned}$$

where $r(x) = (q(x) - 1)p'(x) < p(x)$. By the continuous embedding $L^{p(x)} \hookrightarrow L^{r(x)}$ and the Poincaré inequality, we have

$$\|\phi u\|_{p'(x), w^*} \leq \text{const}(\|h\|_{p'(x)}^{p'^+} + \|h\|_{p'(x)}^{p'^-} + \|u\|_{1, p(x), w}^{r^+} + \|u\|_{1, p(x), w}^{r^-}) + 1$$

Which means that that ϕ is bounded on $W_0^{1, p(x)}(\Omega, w)$.

To show that ϕ is continuous, let $u_n \rightarrow u$ in $W_0^{1, p(x)}(\Omega, w)$. Then $u_n \rightarrow u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \rightarrow \nabla u$ in $(L^{p(x)}(\Omega, w))^N$. We can thus extract from (u_n) a subsequence (u_k) and there exist measurable functions H in $L^{p(x)}(\Omega)$ and G in $(L^{p(x)}(\Omega, w))^N$ such that

$$u_k(x) \rightarrow u(x) \text{ and } \nabla u_k(x) \rightarrow \nabla u(x),$$

$$|u_k(x)| \leq H(x) \text{ and } |\nabla u_k(x)| \leq |G(x)|$$

for a.e. $x \in \Omega$ and all $k \in \mathbf{N}$. Since f satisfies the Carathodory condition, we get

$$f(x, u_k(x), \nabla u_k(x)) \rightarrow f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

From (3.4), it follows that

$$|f(x, u_k(x), \nabla u_k(x))| \leq \gamma w^{\frac{1}{p(x)}} (h(x) + w^{\frac{1}{p'(x)}} |H(x)|^{q(x)-1} + w^{\frac{1}{p'(x)}} |G(x)|^{q(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbf{N}$.

Since

$$w^{\frac{1}{p(x)}} (h(x) + w^{\frac{1}{p'(x)}} |H(x)|^{q(x)-1} + w^{\frac{1}{p'(x)}} |G(x)|^{q(x)-1}) \in L^{p'(x)}(\Omega, w^*),$$

and by the equality

$$I_{p', w^*}(\phi u_k - \phi u) = \int_{\Omega} |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} w^*(x) dx,$$

we can conclude with the dominated convergence theorem and the equivalence (5) of the Proposition 2.2 that

$$\phi u_k \rightarrow \phi u \text{ in } L^{p'(x)}(\Omega, w^*).$$

Thus, the entire sequence (ϕu_n) converges ϕu in $L^{p'(x)}(\Omega, w^*)$. This means that ϕ is continuous.

Since the embedding $I : W_0^{1,p(x)}(\Omega, w) \hookrightarrow L^{p(x)}(\Omega, w)$ is compact, its adjoint operator $I^* : L^{p'(x)}(\Omega, w^*) \rightarrow W^{-1,p'(x)}(\Omega, w^*)$ is also compact. Finally, the composition

$$S = I^* \circ \phi : W_0^{1,p(x)}(\Omega, w) \rightarrow W^{-1,p'(x)}(\Omega, w^*)$$

is compact.

Lemma 3.4. [4, Lemma 2.2 and 2.4] *Let G be a bounded open set in a real reflexive Banach space X . Suppose that $T \in \mathcal{F}_1(\bar{G})$ is continuous and $S : D_S \subset X^* \rightarrow X$ is demicontinuous such that $T(\bar{G}) \subset D_S$. Then the following statements hold:*

- *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\bar{G})$.*
- *If S is of class (S_+) , then $SoT \in \mathcal{F}_T(\bar{G})$.*

3.2. Existence of weak solution

Let us first define the weak solution of problem (1.1).

Definition 3.5. A weak solution of the problem (1.1) is a measurable function

$u \in W_0^{1,p(x)}(\Omega, w)$ such that

$$\int_{\Omega} a(x, \nabla u) \nabla v dx = \int_{\Omega} f(x, u, \nabla u) v dx$$

for all $v \in W_0^{1,p(x)}(\Omega, w)$.

The main result of this paper is the following theorem.

Theorem 3.6. *If the assumptions (3.1)-(3.4) holds, then the problem (1.1) admits at least one weak solution.*

Proof. Let A and $S : W_0^{1,p(x)}(\Omega, w) \rightarrow W^{-1,p'(x)}(\Omega, w^*)$ be as in (3.5) and Lemma 3.3 respectively. Then $u \in W_0^{1,p(x)}(\Omega, w)$ is a weak solution of (1.1) if and only if

$$(3.8) \quad Au = -Su$$

Since the operator A is strictly monotone (by assumption (3.2)), thanks to the properties seen in Lemma 3.2 and in view of Minty-Browder Theorem (see [21], Theorem 26A), the inverse operator $T := A^{-1} : W^{-1,p'(x)}(\Omega, w^*) \rightarrow W_0^{1,p(x)}(\Omega, w)$ is bounded, continuous and of class (S_+) . Moreover, note from Lemma 3.3 that the operator S is bounded, continuous and quasi-monotone.

Therefore, equation (3.8) is equivalent to

$$(3.9) \quad u = Tv \text{ and } v + SoTv = 0.$$

To solve equation (3.9), we will apply the theory of Berkovits degree seen in Theorem 2.7. For this, we will first show that the set

$$B := \{v \in W^{-1,p'(x)}(\Omega, w^*) | v + tSoTv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded.

Indeed, let $v \in B$ and set $u := Tv$, then $\|Tv\|_{1,p(x),w} = \|\nabla u\|_{p(x),w}$.

- If $\|\nabla u\|_{p(x),w} \leq 1$, then $\|Tv\|_{1,p(x),w}$ is bounded.
- If $\|\nabla u\|_{p(x),w} > 1$, then we have by the Proposition 2.2

$$(3.10) \quad \|Tv\|_{1,p(x),w}^{p^-} = \|\nabla u\|_{p(x),w}^{p^-} \leq I_{p,w}(\nabla u).$$

By the assumption (3.3), we have

$$a(x, \nabla u) \cdot \nabla u \geq \alpha w(x) |\nabla u|^{p(x)}.$$

Then

$$\begin{aligned} I_{p,w}(\nabla u) &= \int_{\Omega} |\nabla u|^{p(x)} w(x) dx \\ &\leq \frac{1}{\alpha} \int_{\Omega} a(x, \nabla u) \cdot \nabla u \\ &= \frac{1}{\alpha} \langle Au, u \rangle \\ &= \frac{1}{\alpha} \langle v, Tv \rangle \\ &= \frac{-t}{\alpha} \langle SoTv, Tv \rangle. \end{aligned}$$

This implies that

$$(3.11) \quad I_{p,w}(\nabla u) \leq \frac{t}{\alpha} \int_{\Omega} f(x, u, \nabla u) u dx.$$

We get, by the inequalities (3.10), (3.11) and the growth condition (3.4), the estimate

$$\|Tv\|_{1,p(x),w}^{p^-} \leq \text{const} \left(\int_{\Omega} |w^{\frac{1}{p(x)}} h(x) u(x)| dx + I_{q,w}(u) + \int_{\Omega} |\nabla u|^{q(x)-1} |u| w(x) dx \right).$$

Since $h \in L^{p'(x)}(\Omega)$, then $w^{\frac{1}{p(\cdot)}} h \in L^{p'(x)}(\Omega, w^*)$. By the Hölder inequality, we have

$$\int_{\Omega} |w^{\frac{1}{p(x)}} h(x) u(x)| dx \leq 2 \|w^{\frac{1}{p(\cdot)}} h\|_{p'(x),w^*} \|u\|_{p(x),w} = 2 \|h\|_{p'(x)} \|u\|_{p(x),w}.$$

By the Young inequality, we have

$$\begin{aligned} \int_{\Omega} |\nabla u|^{q(x)-1} |u| w(x) dx &= \int_{\Omega} |\nabla u|^{q(x)-1} w^{\frac{1}{q'(x)}} \cdot |u| w^{\frac{1}{q(x)}} dx \\ &\leq \int_{\Omega} \frac{1}{q'(x)} |\nabla u|^{q(x)} w(x) dx + \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} w(x) dx \\ &= \frac{1}{q'^-} I_{q,w}(\nabla u) + \frac{1}{q^-} I_{q,w}(u) \\ &\leq \text{const} (\|\nabla u\|_{q(x),w}^{q^+} + \|u\|_{q(x),w}^{q^+} + \|u\|_{q(x),w}^{q^-}). \end{aligned}$$

We can then deduce

$$\|Tv\|_{1,p(x),w}^{p^-} \leq \text{const} (\|u\|_{p(x),w} + \|u\|_{q(x),w}^{q^+} + \|u\|_{q(x),w}^{q^-} + \|\nabla u\|_{q(x),w}^{q^+}).$$

By the Poincaré inequality, we have

$$\|u\|_{p(x),w} \leq C \|\nabla u\|_{p(x),w} = C \|Tv\|_{1,p(x),w}.$$

From the continuous embedding $L^{p(x)}(\Omega, w) \hookrightarrow L^{q(x)}(\Omega, w)$ and the Poincaré inequality and, we have

$$\begin{aligned} \|u\|_{q(x),w}^{q^+} &\leq C \|\nabla u\|_{q(x),w}^{q^+} = C \|Tv\|_{1,q(x),w}^{q^+}, \\ \|u\|_{q(x),w}^{q^-} &\leq C \|\nabla u\|_{q(x),w}^{q^-} = C \|Tv\|_{1,q(x),w}^{q^-} \leq C \|Tv\|_{1,q(x),w}^{q^+}, \\ \|\nabla u\|_{q(x),w}^{q^+} &= \|Tv\|_{1,q(x),w}^{q^+}. \end{aligned}$$

We can then deduct the estimate

$$\|Tv\|_{1,p(x),w}^{p^-} \leq \text{const}(\|Tv\|_{1,p(x),w} + \|Tv\|_{1,p(x),w}^{q^+}).$$

It follows that $\{Tv|v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (3.9) that the set B is bounded in $W^{-1,p'(x)}(\Omega, w^*)$.

Consequently, there exists $R > 0$ such that

$$\|v\|_{-1,p'(x),w^*} < R \text{ for all } v \in B.$$

This says that

$$v + tSoTv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0, 1].$$

From Lemma 3.4 it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = AoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Consider a homotopy $H : [0, 1] \times \overline{B_R(0)} \rightarrow W^{-1,p'(x)}(\Omega, w^*)$ given by

$$H(t, v) := v + tSoTv \text{ for } (t, v) \in [0, 1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem 2.7, we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

Finally, we conclude that $u = Tv$ is a weak solution of (1.1).

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