




Some ideal convergent multiplier sequence spaces using de la Vallee Poussin mean and Zweier operator

Tanweer Jalal¹  orcid.org/0000-0003-0676-1356

¹National Institute of Technology, Dept. of Mathematics, Srinagar, JK, India.
 tjalal@nitsri.net

Received: June 2018 | Accepted: August 2018

Abstract:

The main objective of this paper is to introduce multiplier type ideal convergent sequence spaces, using Zweier transform and de la Vallee Poussin mean. We study some topological and algebraic properties of these spaces. Further we prove some inclusion relations related to these spaces.

Keywords: Ideal convergence; Zweier transform; Modulus function; De la Vallee Poussin mean.

MSC (2010): 40A05; 40A35; 40C05; 46A45.

Cite this article as (IEEE citation style):

T. Jalal, "Some ideal convergent multiplier sequence spaces using de la Vallee Poussin mean and Zweier operator", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 1, pp. 91-105, Feb. 2020, doi: 10.22199/issn.0717-6279-2020-01-0006. [Accessed dd-mm-yyyy].



Article copyright: © 2020 Tanweer Jalal. This is an open access article distributed under the terms of the Creative Commons Licence, which permits unrestricted use and distribution provided the original author and source are credited.



1. Introduction

Let \mathbf{N} , \mathbf{R} , and \mathbf{C} be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbf{R} \text{ or } \mathbf{C}\}$$

the space of all real or complex sequences. Let ℓ_∞ , c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $\|x\|_\infty = \sup_k |x_k|$. A subspace of ω for example $X, Y \subset \omega$ is called a sequence space. A sequence space X with linear topology is called a K -space provided each of maps $p_i \rightarrow \mathbf{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbf{N}$. A K -space X is called an FK-space provided X is a complete linear metric space. FK-space whose topology is normable is called a BK-space. Let X and Y be two sequence spaces and $A = (a_{nk})$ an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbf{N}$. Then we say that A defines a matrix mapping from X to Y and we denote it by writing $A : X \rightarrow Y$. If for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$ the A -transform of x is in Y , where

$$(1.1) \quad (Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbf{N}).$$

By (X, Y) we denote the class of matrices A such that $A : X \rightarrow Y$. Thus, $A \in (X, Y)$ if and only if series on the right side of (1.1) converges for each $n \in \mathbf{N}$ and every $x \in X$.

The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Basar and Altay [1], Malkowsky [16], Ng and Lee [20] and Wang [33]. Sengonul [27] defined the sequence $y = (y_i)$ which is frequently used as the Z^p -transformation of the sequence $x = (x_i)$ i.e., $(y_i) = px_i + (1-p)x_{i-1}$ where $x_{-1} = 0, p \neq 0, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & (i = k) \\ 1 - p, & (i - 1 = k); (i, k \in \mathbf{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Following Basar and Altay [1], Sengonul [27] introduced the Zweier sequence spaces \mathbf{Z} and \mathbf{Z}_0 as follows:

$$\mathbf{Z} = \{x = (x_k) : Z^p(x) \in c\}$$

$$\mathbf{Z}_0 = \{x = (x_k) : Z^p(x) \in c_0\}$$

Kostyrko et al. [13] introduced the notion of I -convergence based on the structure of admissible ideal I of subset of natural numbers \mathbf{N} . Later on it was studied by Šalat [24], Šalat et. al. [25, 26] and Demirci [2]. Recently it was further studied by Tripathy and Hazarika [29, 30, 31], Mursaleen and Mohiuddine [18], Jalal [6, 7], Khan et. al. [9, 10], Tripathy and Sen [32] and several others.

Let X be a non-empty set. A set $I \subseteq 2^X$ (2^X denoting the power set of X) is said to be an ideal if and only if I is additive i.e., $A < B \in I \Rightarrow A \cup B \in I$ and hereditary i.e., $A \in I, B \subseteq A \Rightarrow B \in I$. A non-empty family of sets $J(I) \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin J(I)$, for $A, B \in J(I)$ we have $A \cap B \in J(I)$ and for each $A \in J(I)$ and $A \subseteq B \Rightarrow B \in J(I)$. An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I , there is a filter $J(I)$ corresponding to I i.e., $J(I) = \{K \subseteq \mathbf{N} : K^c \in I\}$, where $K^c = \mathbf{N} - K$.

The idea of modulus function was structured by Nakano in 1953 [19]. A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus function if

- (i) $f(t) = 0$ if and only if $t = 0$,
- (ii) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$
- (iii) f is non-decreasing, and
- (iv) f is continuous from the right at zero.

Ruckle [21, 22, 23] used the idea of a modulus function f to construct the sequence space

$$X(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

This space is an FK-space, and Ruckle [21] proved that the intersection of all such $X(f)$ spaces is ϕ , the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [21, 22, 23] proved that, for any modulus f , $X(f) \subset \ell_1$ and $X(f)^\alpha \subset \ell_\infty$ where

$$X(f)^\alpha = \left\{ y = (y_k) : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty \right\}.$$

The space $X(f)$ is a Banach space with respect to the norm [21]

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch [5]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling [3, 4], Kothe [14] and Ruckle [21, 22, 23]. Later Kolk [11, 12] gave an extension of $X(f)$ by considering a sequence of modulus $F = (f_k)$ and defined the sequence space

$$X(F) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f_k(|x_k|) \in X \right\}.$$

The following well known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = H$, $D = \max \left\{ 1, 2^{H-1} \right\}$ then

$$(1.2) \quad |a_k + b_k|^{p_k} \leq D (|a_k|^{p_k} + |b_k|^{p_k})$$

for all $k \in \mathbf{N}$ and $a_k, b_k \in \mathbf{C}$. Also $|a_k|^{p_k} \leq \max \left\{ 1, |a|^G \right\}$ for all $a \in \mathbf{C}$.

2. Definitions and Preliminaries

Definition 2.1. A sequence space E is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$.

Definition 2.2. A sequence space E is said to be monotone if it contains the canonical pre images of all its step spaces.

Definition 2.3. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 2.4. A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 2.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where $\pi(k)$ is a permutation on \mathbf{N} .

Definition 2.6. Let $K = \{k_1 < k_2 < \dots\} \subset \mathbf{N}$ and let E be a sequence space. A K -step of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_n) \in E\}$.

Definition 2.7. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_n) \in \omega$ defined by

$$(y_n) = \begin{cases} x_n & , \text{ if } n \in K \\ 0 & , \text{ otherwise.} \end{cases}$$

Definition 2.8. A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all the elements in λ_K^E i.e., y is the canonical pre-image λ_K^E if and only if is the canonical pre-image of some $x \in \lambda_K^E$.

Definition 2.9. A sequence $(x_k) \in w$ is said to be I -convergent to a number L if for every $\epsilon > 0$, $\{k \in \mathbf{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I - \lim x_k = L$. The space c^I of all I -convergent sequences to L is given by

$$c^I = \{(x_k) \in w : \{k \in \mathbf{N} : |x_k - L| \geq \epsilon\} \in I, \text{ for some } L \in \mathbf{C}\}.$$

Definition 2.10. A sequence $(x_k) \in w$ is said to be I -null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 2.11. A sequence $(x_k) \in w$ is said to be I -Cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbf{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 2.12. A sequence $(x_k) \in w$ is said to be I -bounded if there exists $M > 0$ such that $\{k \in \mathbf{N} : |x_k| \geq M\} \in I$.

Definition 2.13. A modulus function f is said to satisfy Δ_2 -condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

Definition 2.14. Take for I the class I_f of all finite subsets of \mathbf{N} then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X [13].

Definition 2.15. For $I = I_\delta$ and $A \subset \mathbf{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ -convergence is said to be logarithmic statistical convergence [13].

The following Lemma will be used for establishing some results of this article.

Lemma 2.16. *Let E be a sequence space. If E is solid then E is monotone ([8], Page 53).*

Throughout the paper $\mathbf{Z}^I, \mathbf{Z}_0^I, \mathbf{Z}_\infty^I, m_{\mathbf{Z}}^I$ and $m_{\mathbf{Z}_0}^I$ represent the Zweier I -convergent, Zweier I -null, Zweier bounded, Zweier bounded I -convergent and Zweier bounded I -null sequence spaces, respectively .

3. New Zweier multiplier sequence spaces

Let $\lambda = (\lambda_r)$ be an increasing sequence of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_{r+1}, \lambda_1 = 1$. The generalized de la Vallee Poussin mean is defined by $t_r(x) = \frac{1}{\lambda_r} \sum_{k \in J_r} x_k$ where $J_r = [r - \lambda_r + 1, r]$ for $r = 1, 2, 3, \dots$.

A sequence $x = (x_k)$ is said to be (V, λ) - summable to a number L if $t_r(x) \rightarrow L$ as $r \rightarrow \infty$ [15]. If $\lambda_r = r$ then (V, λ) summability is reduced to Cesáro summability. We denote by \wedge the set of all increasing sequences of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_r + 1, \lambda_1 = 1$.

Let $F = (f_k)$ be a sequence of modulus functions, $v = (v_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers.

In this paper we introduce the following classes of multiplier sequence spaces.

$$\begin{aligned} [\mathbf{Z}^\lambda(F, p, v)]^I &= \left\{ x \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k - L|]^{p_k} \geq \epsilon, \text{ for some } L \in \mathbf{C} \right\} \in I \right\}, \\ [\mathbf{Z}_0^\lambda(F, p, v)]^I &= \left\{ x \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k|]^{p_k} \geq \epsilon \right\} \in I \right\}, \\ [\mathbf{Z}_\infty^\lambda(F, p, v)]^I &= \left\{ x \in \omega : \exists K > 0 \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k|]^{p_k} \geq K \right\} \in I \right\}. \end{aligned}$$

Also

$$[m_{\mathbf{Z}}^\lambda(F, p, v)]^I = [\mathbf{Z}^\lambda(F, p, v)]^I \cap [\mathbf{Z}_\infty^\lambda(F, p, v)]^I$$

and

$$[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I = [\mathbf{Z}_0^\lambda(F, p, v)]^I \cap [\mathbf{Z}_\infty^\lambda(F, p, v)]^I.$$

For further properties of multiplier sequence spaces see [17, 28, 31]

Theorem 3.1. Let $F = (f_k)$ be a sequence of modulus functions, $v = (v_k)$, be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers, then the classes of sequences $[\mathbf{Z}^\lambda(F, p, v)]^I$, $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ and $[\mathbf{Z}_\infty^\lambda(F, p, v)]^I$ are linear spaces over the complex field \mathbf{C} .

Proof. We shall prove the theorem for the space $[\mathbf{Z}^\lambda(F, p, v)]^I$. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $[\mathbf{Z}^\lambda(F, p, v)]^I$ and let α, β be two scalars in \mathbf{R} . Then

$$\left\{ x \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k - L_1|]^{p_k} \geq \epsilon \right\} \in I \right\} \text{ for some } L_1 \in \mathbf{C}$$

and

$$\left\{ y \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zy)_k - L_1|]^{p_k} \geq \epsilon \right\} \in I \right\}, \text{ for some } L_2 \in \mathbf{C}$$

Let

$$A_1 = \left\{ x \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k - L_1|]^{p_k} \geq \frac{\epsilon}{2} \right\} \in I \right\}$$

and

$$A_2 = \left\{ y \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zy)_k - L_2|]^{p_k} \geq \frac{\epsilon}{2} \right\} \in I \right\}$$

Since $F = (f_k)$ is a sequence of modulus functions, from inequality (1.2), we have

$$\begin{aligned} & \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [(\alpha v_k(Zx)_k + \beta v_k(Zy)_k) - (\alpha L_1 + \beta L_2)]^{p_k} \\ & \leq D(T_\alpha^H) \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k - L_1|]^{p_k} + D(T_\beta^H) \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zy)_k - L_2|]^{p_k} \end{aligned}$$

where T_α and T_β are positive integers such that $|\alpha| \leq T_\alpha$ and $|\beta| \leq T_\beta$. On the other hand from the above inequality we get

$$\begin{aligned} & \left\{ x \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|(\alpha v_k(Zx)_k + \beta v_k(Zy)_k) - (\alpha L_1 + \beta L_2)| |^{p_k} \geq \epsilon \right\} \right\} \\ & \subseteq \left\{ x \in \omega : \left\{ r \in \mathbf{N} : D(T_\alpha^H) \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k - L_1| |^{p_k} \geq \epsilon \right\} \right\} \\ & \cup \left\{ y \in \omega : \left\{ r \in \mathbf{N} : D(T_\beta^H) \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zy)_k - L_2| |^{p_k} \geq \epsilon \right\} \right\}. \end{aligned}$$

The last two sets on the right hand side belongs to I and this completes the proof. \square

Theorem 3.2. *Let (f_k) and (g_k) be sequences of modulus functions for some fixed k and satisfy the Δ_2 -condition. If X is any of the spaces $[\mathbf{Z}^\lambda(F, p, v)]^I$, $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ and $[\mathbf{Z}_\infty^\lambda(F, p, v)]^I$, then*

- (i) $X(g_k) \subseteq X(f_k \circ g_k)$
- (ii) $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$.

Proof. (i) Let $x = (x_k) \in [\mathbf{Z}^\lambda(G, p, v)]^I$. Then, we have

$$(3.1) \quad \left\{ y \in \omega : \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} g_k [|v_k(Zx)_k - L| |^{p_k} \geq \epsilon \right\} \in I \right\}.$$

Let $\epsilon > 0$ and hence choose $0 < \delta < 1$ such that $f_k(t) \leq \epsilon$ for $0 \leq t \leq \delta$. We write $y_k = g_k [|v_k(Zx)_k - L|]$ and consider

$$(3.2) \quad \frac{1}{\lambda_r} \sum_{k \in J_r} [f_k(y_k)]^{p_k} = \frac{1}{\lambda_r} \sum_{k \in J_r, y_k \leq \delta} [f_k(y_k)]^{p_k} + \frac{1}{\lambda_r} \sum_{k \in J_r, y_k > \delta} [f_k(y_k)]^{p_k}$$

Since $F = (f_k)$ is continuous, we have

$$(3.3) \quad \sum_{k \in J_r, y_k \leq \delta} [f_k(y_k)]^{p_k} \leq [f_k(2)]^G + \sum_{k \in J_r, y_k \leq \delta} [(y_k)]^{p_k}, \quad G = \sup_k p_k.$$

For second summation (i.e $y_k > \delta$), we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since $F = (f_k)$ is non-decreasing, it follows that

$$f_k(y_k) < f_k\left(1 + \frac{y_k}{\delta}\right) \leq \frac{1}{2}f_k(2) + \frac{1}{2}f_k\left(2\frac{y_k}{\delta}\right).$$

Again, since $F = (f_k)$ satisfies Δ_2 -condition, we can write

$$f_k(y_k) < \frac{1}{2}K\left(\frac{y_k}{\delta}\right)f_k(2) + \frac{1}{2}K\left(\frac{y_k}{\delta}\right)f_k(2) = K\left(\frac{y_k}{\delta}\right)f_k(2)$$

Hence, we have

$$(3.4) \quad \sum_{k \in J_r, y_k > \delta} [f_k(y_k)]^{p_k} \leq \max\left\{1, \left(k\delta^{-1}f_k(2)\right)^H\right\} + \frac{1}{\lambda_r} \sum_{k \in J_r} [f_k(y_k)]^{p_k}.$$

From equations (3.1), (3.2), (3.3), (3.4), it follows that

$$\left\{x \in \omega : \left\{r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} g_k(f_k[|v_k(Zx)_k - L|]^{p_k}) \geq \epsilon\right\} \in I\right\}.$$

Hence $X(g_k) \subseteq X(f_k \circ g_k)$.

(ii) Let $x = (x_k) \in [\mathbf{Z}^\lambda(F, p, v)]^I \cap [\mathbf{Z}^\lambda(G, p, v)]^I$. Then we have

$$\left\{x \in \omega : \left\{r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k[|v_k(Zx)_k - L|]^{p_k} \geq \epsilon\right\} \in I\right\}$$

and

$$\left\{x \in \omega : \left\{r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} g_k[|v_k(Zx)_k - L|]^{p_k} \geq \epsilon\right\} \in I\right\}$$

The rest of the proof follows from the following relation;

$$\begin{aligned} & \left\{r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} [f_k + g_k][|v_k(Zx)_k - L|]^{p_k} \geq \epsilon\right\} \\ & \subseteq \left\{r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k[|v_k(Zx)_k - L|]^{p_k} \geq \epsilon\right\} \cup \quad \square \\ & \left\{x \in \omega : r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} g_k[|v_k(Zx)_k - L|]^{p_k} \geq \epsilon\right\} \end{aligned}$$

Corollary 3.3. $X \subseteq X(F)$, for $X = [\mathbf{Z}^\lambda(p, v)]^I, [\mathbf{Z}_0^\lambda(p, v)]^I, [m_{\mathbf{Z}}^\lambda(p, v)]^I$ and $[m_{\mathbf{Z}_0}^\lambda(p, v)]^I$.

Theorem 3.4. The spaces $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ and $[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I$ are solid and monotone.

Proof. We shall prove the result for the space $[\mathbf{Z}_0^\lambda(F, p, v)]^I$, the result for $[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I$ can be proved similarly. Let $x = (x_k) \in [\mathbf{Z}_0^\lambda(F, p, v)]^I$, then

$$(3.5) \quad \left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k|]^{p_k} \geq \epsilon \right\} \in I$$

Let α_k be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$, then the result follows from (3.5) and the following inequality

$$f_k [|\alpha_k v_k(Zx)_k|]^{p_k} \leq |\alpha_k| f_k [|v_k(Zx)_k|]^{p_k} \leq f_k [|v_k(Zx)_k|]^{p_k}.$$

The space $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ is monotone follows from Lemma 2.16. \square

Theorem 3.5. The spaces $[\mathbf{Z}^\lambda(F, p, v)]^I$ and $[m_{\mathbf{Z}}^\lambda(F, p, v)]^I$ are neither solid nor monotone in general.

Proof. The proof of this result follows from the following example.

Let $I = I_f$, $f_k(x) = x$, for $x \in [0, \infty)$, $p = (p_k) = 1$ and $v = (v_k) = 1$. Consider the K -step space T_k of T defined as follows:

Let $(x_k) \in T_k$ and $(y_k) \in T_k$ be such that

$$y_k = \begin{cases} x_k & : \text{ if } k \text{ is odd;} \\ 0 & : \text{ otherwise.} \end{cases}$$

Consider the sequence (x_k) defined as $x_k = \frac{1}{2}$ for all $k \in \mathbf{N}$, then $(x_k) \in [\mathbf{Z}^\lambda(F, p, v)]^I$ but its K -step space preimage does not belong to $[\mathbf{Z}^\lambda(F, p, v)]^I$. Thus $[\mathbf{Z}^\lambda(F, p, v)]^I$ is not monotone. Hence $[\mathbf{Z}^\lambda(F, p, v)]^I$ is not solid by Lemma 2.16. \square

Theorem 3.6. The spaces $[\mathbf{Z}^\lambda(F, p, v)]^I$ and $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ are sequence algebras.

Proof. We prove that $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ is sequence algebra. For $[\mathbf{Z}^\lambda(F, p, v)]^I$ the result can be proved similarly. Let $x = (x_k), y = (y_k) \in [\mathbf{Z}_0^\lambda(F, p, v)]^I$. Then

$$\left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k|]^{p_k} \geq \epsilon \right\} \in I$$

and

$$\left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zy)_k|]^{p_k} \geq \epsilon \right\} \in I.$$

Therefore

$$\left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k(Zy)_k|]^{p_k} \geq \epsilon \right\} \in I$$

Thus $(x_k y_k) \in [\mathbf{Z}_0^\lambda(F, p, v)]^I$. Hence $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ is a sequence algebra. \square

Theorem 3.7. Let $F = (f_k)$ be a sequence of modulus functions. Then

$$[\mathbf{Z}_0^\lambda(F, p, v)]^I \subset [\mathbf{Z}^\lambda(F, p, v)]^I \subset [\mathbf{Z}_\infty^\lambda(F, p, v)]^I$$

and the inclusions are proper.

Proof. Let $x = (x_k) \in [\mathbf{Z}^\lambda(F, p, v)]^I$. Then there exists $L \in \mathbf{C}$ such that

$$\left\{ r \in \mathbf{N} : \frac{1}{\lambda_r} \sum_{k \in J_r} f_k [|v_k(Zx)_k - L|]^{p_k} \geq \epsilon \right\} \in I.$$

We have

$$f_k [|v_k(Zx)_k - L|]^{p_k} \leq \frac{1}{2} f_k [|v_k(Zx)_k - L|]^{p_k} + \frac{1}{2} f_k [|L|]^{p_k}.$$

Taking the supremum over k on both sides we get $x = (x_k) \in [\mathbf{Z}_\infty^\lambda(F, p, v)]^I$. The inclusion $[\mathbf{Z}_0^\lambda(F, p, v)]^I \subset [\mathbf{Z}^\lambda(F, p, v)]^I$ is obvious. The inclusion is proper follows from the following example.

Let $I = I_\delta, f_k(x) = x^2$, for $x \in [0, \infty)$, $v = (v_k) = 1$, $p = (p_k) = 1$ for all $k \in \mathbf{N}$.

(i) Consider the sequence (x_k) defined by $x_k = 1$ for all $k \in \mathbf{N}$. Then

$(x_k) \in [\mathbf{Z}^\lambda(F, p, v)]^I$ but $(x_k) \notin [\mathbf{Z}_0^\lambda(F, p, v)]^I$.

(ii) Consider the sequence (y_k) defined as

$$y_k = \begin{cases} 2 & , \text{ if } k \text{ is even;} \\ 0 & , \text{ otherwise.} \end{cases}$$

Then $(y_k) \in [\mathbf{Z}_\infty^\lambda(F, p, v)]^I$ but $(y_k) \notin [\mathbf{Z}^\lambda(F, p, v)]^I$. \square

Theorem 3.8. *The spaces $[\mathbf{Z}^\lambda(F, p, v)]^I$ and $[\mathbf{Z}_0^\lambda(F, p, v)]^I$ are not convergence free in general.*

Proof. The proof of this theorem follows from the following example.

Let $I = I_f, f_k(x) = x^2$, for $x \in [0, \infty)$, $v = (v_k) = 1$, $p = (p_k) = 1$ for all $k \in \mathbf{N}$.

Consider the sequence (x_k) and (y_k) defined by $x_k = \frac{1}{k^2}$ and $y_k = k^2$ for all $k \in \mathbf{N}$. Then $(x_k) \in [\mathbf{Z}^\lambda(F, p, v)]^I$ and $[\mathbf{Z}_0^\lambda(F, p, v)]^I$, but (y_k) does not belong to both $[\mathbf{Z}^\lambda(F, p, v)]^I$ and $[\mathbf{Z}_0^\lambda(F, p, v)]^I$.

Hence the spaces are not convergence free in general. \square

Theorem 3.9. *The spaces $[m_{\mathbf{Z}}^\lambda(F, p, v)]^I$ and $[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I$ are not separable.*

Proof. We shall prove the result for the space $[m_{\mathbf{Z}}^\lambda(F, p, v)]^I$. Let A be an infinite subset of \mathbf{N} of increasing natural numbers such that $A \in I$. Let

$$p_k = \begin{cases} 1 & , \text{ if } k \in A; \\ 2 & , \text{ otherwise} \end{cases}$$

and $v = (v_k) = 1$ for all $k \in \mathbf{N}$. Let

$$P_0 = (x_n) = \begin{cases} x_n = 1 & , \quad n \in A; \\ x_n = 0 & , \quad \text{otherwise.} \end{cases}$$

Since A is infinite, so P_0 is uncountable. Consider the class of open balls

$B_1 = \left\{ B\left(z, \frac{1}{2}\right) : z \in P_0 \right\}$. Let C_1 be an open cover of $[m_{\mathbf{Z}}^\lambda(F, p, v)]^I$ and $[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I$ containing B_1 . Since B_1 is uncountable, so is C_1 can not be reduced to a countable subcover for $[m_{\mathbf{Z}}^\lambda(F, p, v)]^I$ as well as $[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I$. Thus $[m_{\mathbf{Z}}^\lambda(F, p, v)]^I$ and $[m_{\mathbf{Z}_0}^\lambda(F, p, v)]^I$ are not separable. \square

Acknowledgement: The author is thankful to the reviewers for their valuable suggestions and comments which improved the presentation of the paper.

References

- [1] F. Başar and B. Altay, "On the spaces of sequences of p -bounded variation and related matrix mappings", *Ukrainian mathematical journal*, vol. 55, pp. 136-147, Jan. 2003, doi: 10.1023/A:1025080820961.
- [2] K. Demirci, "I-limit superior and limit inferior", *Mathematical communications*, vol. 6, no. 2, pp. 165-172, 2001. [On line]. Available: <https://bit.ly/38HWRNc>
- [3] D. J. H. Garling, "On symmetric sequence spaces", *Proceedings of the London mathematical society*, vol. s3-16, no. 1, pp. 85-106, Jan. 1966, doi: 10.1112/plms/s3-16.1.85.
- [4] D. J. H. Garling, "Symmetric bases of locally convex spaces", *Studia mathematica*, vol. 30, pp. 163-181, 1968, doi: 10.4064/sm-30-2-163-181.
- [5] B. Gramsch, "Die Klasse metrischer linearer Räume \mathcal{L}_p ", *Mathematische Annalen*, vol. 171, pp. 61-78, Mar. 1967, doi: 10.1007/BF01433094.
- [6] T. Jalal, "Some new I-lacunary generalised difference sequence space in n -normed space", in *Modern mathematical methods and high performance computing in science and technology*, vol. 171, V. Singh, H. Srivastava, E. Venturino, M. Resch, and V. Gupta, Eds. Singapore: Springer, 2016, pp. 249-258, doi: 10.1007/978-981-10-1454-3_21.
- [7] T. Jalal, "On generalized A-difference strongly summable sequence space defined by ideal convergence on a real n -normed space", *Bulletin of the Calcutta mathematical society*, vol. 106, no. 6, pp. 415-426, 2014.
- [8] P. K. Kamthan and M. Gupta, *Sequence spaces and series*. New York, NY: Dekker, 1981.
- [9] V. A. Khan, and K. Ebadullah, "On some I-convergent sequence spaces defined by a modulus function", *Theory and applications of mathematics & computer science*, vol. 1, no. 2, pp. 22-30, Nov. 2011. [On line]. Available: <https://bit.ly/2RrCGxc>
- [10] V. A. Khan, K. Ebadullah, and Yasmeen, "On Zweier I-convergent sequence spaces", *Proyecciones (Antofagasta, On line)*, vol. 33, no. 3, pp. 259- 276, Sep. 2014, doi: 10.4067/S0716-09172014000300003.

- [11] E. Kolk, "On strong boundedness and summability with respect to a sequence of moduli", *Acta et commentationes Universitatis Tartuensis*, vol. 960, no. 1, pp. 41-50, 1993. [On line]. Available: <https://bit.ly/2RsPN0R>
- [12] E. Kolk, "Inclusion theorems for some sequence spaces defined by a sequence of moduli", *Acta et commentationes Universitatis Tartuensis*, vol. 970, no. 1, pp. 65-72, 1994. [On line]. Available: <https://bit.ly/30XF01O>
- [13] P. Kostyrko, T. Sălăt, and W. Wilczyński, "I-convergence", *Real analysis exchange*, vol. 26, no. 2, pp. 669-686, 2000. [On line]. Available: <https://bit.ly/3aMh8CU>
- [14] G. Köthe, *Topological vector spaces I*, Berlin: Springer, 1970.
- [15] L. Leindler, "Über die de la Vallée-Poussinsche summierbarkeit allgenmeiner Orthogonalreihen", *Acta mathematica academiae scientiarum hungarica*, vol. 16, no. 3-4, pp. 375- 387, Sep. 1965, doi: 10.1007/BF01904844.
- [16] E. Malkowsky, "Recent results in the theory of matrix transformation in sequence spaces", *Matematički vesnik*, vol. 49, no. 3-4, pp. 187-196, 1997. [On line]. Available: <https://bit.ly/2uy8hV0>
- [17] M. Et, P. Y. Lee and B. C. Tripathy, "Strongly almost $(V, \lambda)(\Delta r)$ - summable sequences defined by Orlicz function", *Hokkaido mathematical journal*, vol. 35, no. 1, pp. 197-213, Feb. 2006, doi:10.14492/hokmj/1285766306.
- [18] M. Mursaleen and S. A. Mohiuddin, "On ideal convergence in probabilistic normed space", *Mathematica slovacica*, vol. 62, no. 1, pp. 49-62, 2012, doi: 10.2478/s12175-011-0071-9.
- [19] H. Nakano, "Concave modular", *Journal of the mathematical society of Japan*, vol. 5, no. 1, pp. 29-49, 1953, doi:10.2969/jmsj/00510029.
- [20] P. N. Ng, and P.Y. Lee, "Cesàro sequence spaces of non-absolute type", *Commentationes mathematicae*, vol. 20, no. 2, pp. 429-433, 1978. [On line]. Available: <https://bit.ly/30W7AAK>
- [21] W. H. Ruckle, "FK-spaces in which the sequence of coordinate vectors is bounded", *Canadian journal of mathematics*, vol. 25, no. 5, pp. 973-975, Oct. 1973, doi: 10.4153/CJM-1973-102-9.
- [22] W. H. Ruckle, "On perfect symmetric BK-spaces", *Mathematische annalen*, vol. 175, no. 2, pp. 121-126, Jun. 1968, doi: 10.1007/BF01418767.
- [23] W. H. Ruckle, "Symmetric coordinate spaces and symmetric bases", *Canadian journal of mathematics*, vol. 19, pp. 828-838, 1967, doi: 10.4153/CJM-1967-077-9.
- [24] T. Šalăt, "On statistically convergent sequences of real numbers", *Mathematica slovacica*, vol. 30, no. 2, pp. 139-150, 1980. [On line]. Available: <https://bit.ly/2vw9Vad>

- [25] T. Šalát, B. C. Tripathy, and M. Ziman, "On some properties of I-convergence", *Tatra mountains mathematical publications*, vol. 28, pp. 279-286, 2004. [On line]. Available: <https://bit.ly/2O8ODpw>
- [26] T. Šalát, B. C. Tripathy and M. Ziman, "On I-convergence field", *Italian journal of pure and applied mathematics*, vol. 17, pp. 45-54, 2005. [On line]. Available: <https://bit.ly/3aOasnY>
- [27] M. Şengönül, "On the Zweier sequence space", *Demonstratio mathematica*, vol. 40, no. 1, pp. 181-196, 2007, doi: 10.1515/dema-2007-0119.
- [28] B. C. Tripathy and P. Chandra, "On some generalied difference paranormed sequence spaces associated with multiplier sequences defined by modulus functions", *Analysis in theory and applications*, vol. 27, no. 1, pp. 21-27, Mar. 2011, doi: 10.1007/s10496-011-0021-y.
- [29] B. C. Tripathy, and B. Hazarika, "Paranorm I-convergent sequence spaces", *Mathematica slovac*, vol. 59, no. 4, pp. 485-494, Jul. 2009, doi: 10.2478/s12175-009-0141-4.
- [30] B. C. Tripathy and B. Hazarika, "I-convergent sequence spaces associated with multiplier sequence spaces", *Mathematical inequalities & applications*, vol. 11, no. 3, pp. 543-548, 2008, doi: 10.7153/mia-11-43.
- [31] B. C. Tripathy and S. Mahanta, "On a class of vector valued sequences associated with multiplier sequences", *Acta mathematicae applicatae sinica*, vol. 20, no. 3, pp. 487-494, Sep. 2004, doi: 10.1007/s10255-004-0186-7.
- [32] B. C. Tripathy and M. Sen, "On fuzzy I-convergent difference sequence spaces", *Journal of intelligent & fuzzy systems*, vol. 25, no. 3, pp. 643-647, 2013, doi: 10.3233/IFS-120671.
- [33] C. S. Wang, "On Nörlund sequence spaces", *Tamkang journal of mathematics*, vol. 9, no. 2, pp. 269-274, 1978.