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ON SOME NONLINEAR EQUATIONS

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Abstract.

A new method for finding large solutions of quadratic equations is presented.

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Introduction.

Consider the quadratic equation

$$x = y + \lambda B(x,x) \tag{1}$$

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in a Banach space X, where $y \in X$ is fixed, λ is a positive number and B is a bounded symmetric bilinecr operator on X [1], [5], [7].

The above equation has already been studied in [1], [2], [3], [5] and the references there. Briefly, it is known that if

$$4\lambda||B|| \cdot ||y|| < 1 , \qquad (2)$$

then a small solution x of (1) exists (i.e., a solution tending to 0 with y) such that

$$||x|| \leq \frac{1\sqrt{1-4\lambda||B||\cdot||y||}}{2\lambda||B||} . \qquad (3)$$

The problem of finding a not necessarily small solution x of (1) is of great importance. The best known approach to this problem is the application of the Newton's Kantorovich iteration [6]

$$x_{n+1} = x_n - (I - 2\lambda B(x_n))^{-1}(y + \lambda B(x_n,x_n) - x_n), \quad n = 0,1,2,... \qquad (4)$$

for some $x_0 \in X$. The application of the above iteration, however, dces not necessarily guarantee that the obtained solution x is not the small solution.

Here motivated by the solution of the real quadratic equation and the work in [5] and [6], we seek a solution x of (1) expressed as

$$x = \frac{1}{\lambda} v + \sum_{n=0}^{\infty} \lambda^n x_n , \qquad (5)$$

where, v , $x_n \in X$, $n = 0,1,2,\ldots$ are to be specified. Under certain assumptions on v we show that if (2) holds the solution x of (1) given by (5) is such that

$$||x|| \geq \frac{1 + \sqrt{1-4\lambda||B||\cdot||y||}}{2\lambda||B||} \tag{6}$$

We now state the main results.

Theorem. Assume:

(a) there exists $v \in X$ satisfying

$$B(v,v) = v \ , \ v \neq 0 \tag{7}$$

and such that the linear operator $(I - 2B(v))^{-1}$ exists on X.

(b) Let k denote the norm of $(I - 2B(v))^{-1}$ and set

$$\begin{aligned} x_0 &= (I - 2B(v))^{-1}(y) \\ x_n &= \sum_{j=0}^{n-1} (I - 2B(v))^{-1}B(x_j, x_{n-j-1}) \ , \quad n = 1,2,\ldots \end{aligned} \tag{8}$$

with

$$4\lambda k^2||B|| \cdot ||y|| < 1 \quad \text{and} \quad ||B|| \neq 0. \tag{9}$$

Then there exists a solution x of (1) given by (5) and satisfying

$$||x|| \leq \frac{1}{\lambda}\,||v|| + \frac{1-\sqrt{1-4\lambda k^2||B||\cdot||y||}}{2\lambda||B||\cdot k} \tag{10}$$

Proof. As in [5], formal substitution of (5) into (1) and equation of like powers of λ shows that if x is a solution then v, x_n, $n = 0,1,2,...$ must be giben by (7) and (8).

The real series

$$\frac{1}{\lambda}\,||v|| + \sum_{n=0}^{\infty} \lambda^n z_n \,,$$

where

$$z_0 = k||y||$$

$$z_n = \sum_{j=0}^{n-1} k||B||\, z_j z_{n-j-1} \,,$$

obviously dominates the series given by (5). Moreover, by (9), we have

$$\sum_{n=0}^{\infty} \lambda^n z_n = \frac{1-\sqrt{1-4\lambda k^2||B||\cdot||y||}}{2\lambda||B||\cdot k} .$$

Therefore, the series given by (5) converges to a solution x of (1) satisfying (10) and the proof is completed.

We now prove the existence of a not small solution. For simplicity we take $\lambda = 1$.

Proposition. If the hypotheses of the theorem are satisfied, k is such that

$$0 < k \leq 1 \quad \text{and} \quad 1 - 4||B|| \cdot ||y|| > 0 ,$$

then there exists a solution x of (1) given by (5) and satisfying

$$||x|| \geq \frac{1 + \sqrt{1-4||B||\cdot||y||}}{2||B||} . \tag{11}$$

Proof. The solution x of (1) given by (5) is guaranteed by the theorem. Hence, it is enough to show (11). By (5) we have

$$||x|| \geq ||v|| - ||\sum_{n=0}^{\infty} x_n||$$

$$\geq ||v|| - \sum_{n=0}^{\infty} ||x_n||$$

$$\geq ||v|| - \frac{1 - \sqrt{1-4k^2||B||\cdot||y||}}{2||B||k} \tag{12}$$

If v is a nonzero solution of (7), then

$$||v|| = ||B(v,v)|| \leq ||B|| \cdot ||v||^2 .$$

Therefore,

$$||v|| \geq \frac{1}{||B||} . \tag{13}$$

Now, (12), because of (13), becomes

$$||x|| \geq \frac{1}{||B||} - \frac{1 - \sqrt{1-4k^2||B||\cdot||y||}}{2||B||k} . \tag{14}$$

By (14), to show (11), it is enough to show

$$\frac{(2k-1) + \sqrt{1-4k^2||B||\cdot||y||}}{2||B||\cdot k} \geq \frac{1 + \sqrt{1-4||B||\cdot||y||}}{2||B||} \tag{15}$$

After the simplifications showing (15) becomes easily equivalent to showing

$$0 < k \leq 1 ,$$

which is true by hypothesis and the proof is completed.

Note that in the case of the real quadratic equation

$$r = \alpha + \lambda\beta r^2 ,$$

where $\alpha = ||y||$ and $\beta = ||B||$ equality is achieved (3) and (6). The solutions are then given by

•

$$r^- = \frac{1 - \sqrt{1-4\lambda\alpha\beta}}{2\lambda\beta}$$

$$= \sum_{n=0}^{\infty} \lambda^n z_n$$

$$= \sum_{n=0}^{\infty} 2^n \lambda^n \alpha^{n+1} \beta^n \frac{1\ 3\ \ldots\ (2n-1)}{1\ 2\ \ldots\ (n+1)} .$$

and

$$r^+ = \frac{1}{\lambda\beta} - r^- .$$

Finally, note that for $v = 0$ in theorem 2, we obtain the result in [5].

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