

Nonlinear elliptic equations in dimension two with potentials which can vanish at infinity

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Abstract

We will focus on the existence of nontrivial solutions to the following nonlinear elliptic equation

$$-\Delta u + V(x)u = f(u), \quad x \in \mathbf{R}^2,$$

where V is a nonnegative function which can vanish at infinity or be unbounded from above, and f have exponential growth range. The proof involves a truncation argument combined with Mountain Pass Theorem and a Trudinger-Moser type inequality.

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1. Introduction

The main purpose of this work is to study the existence of solutions for the following nonlinear elliptic equation:

$$(1.1) \quad -\Delta u + V(x)u = f(x, u), \quad x \in \Omega,$$

where $\Omega \subset \mathbf{R}^2$, V is a continuous potential and the nonlinearity f possesses maximal growth range. It is interesting to compare the equation (1.1) with the case where Ω is a subset of \mathbf{R}^N , $N \geq 3$. In this case, the classical Sobolev theorem asserts that the following embedding is continuous: $H_0^1(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq 2^* = 2N/(N-2)$. Thus, using variational methods, the maximal growth of the function f is of type: $f(s) \sim |s|^{2^*-1}$.

In dimension $N = 2$ one has $H_0^1(\Omega) \subset L^q(\Omega)$ for all $q \geq 1$ and $H_0^1(\Omega) \not\subset L^\infty(\Omega)$. In this situation another kind of maximal growth were established independently by Trudinger [20] and Pohožaev [16]. The authors proved that the maximal growth allow us to consider is of type:

$f(s) \sim e^{|s|^2}$. Motivated by this result, it was obtained the following characterization of growth: we say that a function f possesses critical exponential growth, if there exists $\alpha_0 > 0$ such that

$$(1.2) \quad \lim_{|s| \rightarrow \infty} \frac{f(s)}{e^{\alpha|s|^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

The equation (1.1) where f possesses critical exponential growth had been studying in many works (see [5, 4, 10, 11, 9, 12, 3, 8, 7]).

Adimurthi and Yadava [2], Adimurthi et al. [1] and de Figueiredo et al. [5] studied the problem (1.1), where Ω is a bounded smooth domain and the potential V is identically zero.

In [9], do Ó and Ruf studied the equation

$$(1.3) \quad -\Delta u + V(x)u = f(u), \quad x \in \mathbf{R}^2,$$

where V belongs to $\mathcal{C}(\mathbf{R}^2, \mathbf{R})$ and is a 1-periodic function in x_1 and x_2 , and 0 is in a spectral gap of the operator $-\Delta + V$.

In [6], it was found a nontrivial solution of the problem

$$(1.4) \quad -\Delta u + V(x)u = f(x, u), \quad x \in \mathbf{R}^2,$$

where $V : \mathbf{R}^2 \rightarrow \mathbf{R}$ is a continuous function satisfying

$$V(x) \geq V_0 > 0, \quad \text{for all } x \in \mathbf{R}^2,$$

and the potential V is asymptotically periodic at infinity, that is there exists a continuous 1-periodic function $V_1 : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that

$$(i) \quad V_1(x) \geq V(x) > 0, \text{ for all } x \in \mathbf{R}^2.$$

$$(ii) \quad V(x) \rightarrow V_1(x), \text{ as } |x| \rightarrow \infty.$$

In [10], it was considered the equation

$$(1.5) \quad -\Delta u + V(x)u = f(u) + h(x), \quad x \in \mathbf{R}^2,$$

where the potential V satisfy

$$(i) \quad \text{The function } V(x) \geq V_0 > 0, \text{ for all } x \in \mathbf{R}^2$$

$$(ii) \quad V(x) \rightarrow +\infty, \text{ as } |x| \rightarrow \infty.$$

Some extensions of (1.5) can be found in [13, 11, 8].

In [12], the authors studied the existence of nontrivial solutions for the following class of equations

$$(1.6) \quad -\epsilon^2 \Delta u + V(x)u = f(u), \quad x \in \mathbf{R}^2.$$

where ϵ is a small positive parameter and the potential $V : \mathbf{R}^2 \rightarrow \mathbf{R}$ satisfies the following conditions:

- (i) V is locally Hölder continuous in \mathbf{R}^2 and there exists a positive constant V_0 such that

$$V(x) \geq V_0, \quad \text{for all } x \in \mathbf{R}^2.$$

- (ii) There exists a bounded domain $\Omega \subset \mathbf{R}^2$ such that

$$\inf_{\Omega} V(x) < \min_{\partial\Omega} V(x).$$

Motivated by the above mentioned results, we are interested in studying the equation

$$(1.7) \quad -\Delta u + V(x)u = f(u), \quad x \in \mathbf{R}^2,$$

where the nonlinearity f possesses critical exponential growth and the potential V can be vanish at infinity. More specifically V satisfies the following assumptions:

(V₁) $V \in \mathcal{C}(\mathbf{R}^2, \mathbf{R})$ is a radially symmetric nonnegative function.

(V₂) There exist constants a, b, R_0, L_a and L_b , with $0 < a < 2$, $b \leq a$, $R_0 \geq 1$, $L_a \geq R_0^a$ and $L_a R_0^{b-a} \leq L_b$ such that

$$\frac{L_a}{|x|^a} \leq V(x) \leq \frac{L_b}{|x|^b}, \quad \text{for all } |x| \geq R_0.$$

Before starting the assumptions on the nonlinearity f , we define the energy space which will be use to set the variational structure. Following [18], $H_{V,rad}^1(\mathbf{R}^2)$ denote the subspace of the radially symmetric functions in the closure of $\mathcal{C}_0^\infty(\mathbf{R}^2)$ with respect to the norm

$$\|u\| = \|u\|_{H_V^1} := \left(\int_{\mathbf{R}^2} |\nabla u|^2 + V(x)u^2 dx \right)^{1/2}.$$

For $1 \leq p < +\infty$, we consider

$$L_{V,rad}^p(\mathbf{R}^2) := \{u \in \mathcal{M}(\mathbf{R}^2, \mathbf{R}) : u \text{ is radial and } \int_{\mathbf{R}^2} V(x)|u|^p dx < +\infty\},$$

endowed with the norm

$$\|u\|_{L_V^p} = \left(\int_{\mathbf{R}^2} V(x)|u|^p dx \right)^{1/p}.$$

Thus,

$$H_{V,rad}^1(\mathbf{R}^2) = \{u \in L_{V,rad}^2(\mathbf{R}^2) : |\nabla u| \in L^2(\mathbf{R}^2)\}.$$

We note that $H_{V,rad}^1(\mathbf{R}^2)$ is a Hilbert space endowed with inner product

$$\langle u, v \rangle := \int_{\mathbf{R}^2} (\nabla u \nabla v + V(x)uv) dx, \quad u, v \in H_{V,rad}^1(\mathbf{R}^2).$$

Throughout this paper, we denote by E the space $H_{V,rad}^1(\mathbf{R}^2)$ and by E^{-1} the dual space of E with the usual norm.

Now, we state a basic embedding result (see [18, 19], for a proof).

Lemma 1.1. *Suppose V satisfies (V₁) – (V₂). Taking R_0, a and b given by (V₂), consider $a^* = (4 + 2a)/(2 - a)$ and $b^* = 2(2 - 2b + a)/(2 - a)$. Then,*

- (i) *The embedding $H_{V,rad}^1(\mathbf{R}^2) \hookrightarrow L^p(\mathbf{R}^2)$ is continuous for $a^* \leq p < +\infty$ and compact for $a^* < p < +\infty$.*

(ii) The embedding $H_{V,rad}^1(\mathbf{R}^2) \hookrightarrow L_{V,rad}^p(\mathbf{R}^2)$ is continuous for $b^* \leq p < +\infty$ and compact for $b^* < p < +\infty$.

(iii) The embedding $H_{V,rad}^1(B_R) \hookrightarrow H^1(B_R)$ is continuous for $R > R_0$.

Remark 1.2. 1. As a consequence of (iii) and Sobolev embedding theorem, the space $H_{V,rad}^1(\mathbf{R}^2)$ is compactly immersed in $L^p(B_R)$ for all $1 \leq p < +\infty$.

We assume the following assumptions on the nonlinearity f :

(H₁) $f \in \mathcal{C}(\mathbf{R})$ and $f(s) = 0$ for all $s \leq 0$.

Taking $b^* \in \mathbf{R}$ as in Lemma 1.1, consider

(H₂) There exists a constant $\mu > b^*$ such that

$$0 < \mu F(s) \leq sf(s), \quad \text{for all } s > 0,$$

where $F(s) = \int_0^s f(t) dt$.

(H₃) There exist constants $s_1 > 0$ and $M > 0$ such that

$$0 < F(s) \leq Mf(s), \quad \text{for all } s > s_1.$$

Setting μ given by (H₂) and a given by (V₂), we suppose:

(H₄) There exists $\theta \geq 4a/(2-a)$ such that $f(s) = O(s^{\mu-1+\theta})$ as $s \rightarrow 0^+$.

(H₅) There exists $\alpha_0 > 0$ such that

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{e^{\alpha|s|^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

(H₆) The following limit hold:

$$\liminf_{s \rightarrow +\infty} \frac{sf(s)}{e^{\alpha_0 s^2}} > \frac{4e}{\alpha_0}.$$

The main result of this paper is stated as follows.

Theorem 1.3. *Suppose that V satisfies $(V_1) - (V_2)$ and f satisfies $(H_1) - (H_6)$. Then, there exists $L^* = L^*(f, \mu, \alpha_0, \theta, a, b) > 0$ such that equation (1.7) possesses a nontrivial weak solution $u \in E$ provided that $L_a \geq L^*$, namely $u \in E$ satisfies*

$$\int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u\phi) dx = \int_{\mathbf{R}^2} f(u)\phi dx, \quad \text{for all } \phi \in E.$$

In [9], under the hypotheses considered in that work, the potential V can not vanish at infinity. Indeed, combining the fact that V is a periodic function with $V(x) \rightarrow 0$, as $|x| \rightarrow \infty$, we obtain that $V \equiv 0$. Thus, 0 is not in the spectral gap of the operator $-\Delta \equiv -\Delta + V$.

We notice that in the equations (1.4), (1.5) and (1.6), it was assumed that the potential is bounded below for a positive constant. Thus, in [5, 10, 13, 11, 8, 12], the authors did not treat the case where V can tend to zero at infinity or be zero somewhere.

In our work, the main difference with the above-mentioned results is that, by assumption (V_2) , the potential can vanish at infinity, and V can be zero in $|x| < R_0$.

In order to find solutions of the equation (1.7), we combine a truncation argument with a finite-dimensional approximation and Mountain Pass Theorem.

The paper is organized as follows: Section 2 contains some preliminaries results. In section 3, we set up an auxiliary functional and show that its energy functional associated has the mountain pass geometry. In section 4, we estimate the Palais-Smale sequences and minimax levels of the auxiliary functional. In section 5, we find a nontrivial critical point of the auxiliary functional. Finally, in sections 6, we present the proof of our main result.

2. Preliminaries

In the first result of this section, we state the following lemma which proof can be found in [17, Lemma 2.1].

Lemma 2.1. *Suppose that (V_1) and (V_2) hold. Then,*

$$|w(x)| \leq \frac{\|w\|}{L_a^{1/4} \pi^{1/2} |x|^{\frac{2-a}{4}}}, \quad \text{for all } |x| \geq R_0,$$

for every $w \in E$.

Let Ω is a bounded domain in \mathbf{R}^2 . A famous result obtained independently by Pohožaev [16] and Trudinger [20] states that $e^{\alpha u^2} \in L^1(\Omega)$ for all $u \in H_0^1(\Omega)$ and $\alpha > 0$. Furthermore, Moser [15] showed that there exists $C = C(\alpha) > 0$ such that

$$(2.1) \quad \sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) dx \leq C|\Omega|.$$

Moreover, inequality (2.1) is sharp, in the sense that for any $\alpha > 4\pi$ the corresponding supremum becomes infinity.

In what follows, we will use the following version of the Trudinger-Moser inequality which is defined on the space E .

Proposition 2.2. (See [17]) Assume V satisfies (V_1) and (V_2) . Then,

$$(2.2) \quad \int_{\mathbf{R}^2} \left(e^{\alpha|u|^2} - \sum_{j=0}^{j_a} \frac{\alpha^j |u|^{2j}}{j!} \right) dx < +\infty, \quad \text{for all } u \in E \text{ and } \alpha > 0,$$

where $j_a = \lfloor 4/(2-a) \rfloor$. Furthermore, if $0 < \alpha < 4\pi$, there exists a positive constant $C = C(\alpha, a, R_0)$ such that

$$(2.3) \quad \sup_{u \in E, \|u\| \leq 1} \int_{\mathbf{R}^2} \left(e^{\alpha u^2} - \sum_{j=0}^{j_a} \frac{\alpha^j |u|^{2j}}{j!} \right) dx \leq C.$$

Lemma 2.3. Let $\alpha > 0$ and $m > 1$. Then, for each $n > m$ there exists a positive constant $C = C(n)$ such that

$$\left(e^{\alpha|t|^2} - \sum_{j=0}^{j_a} \frac{\alpha^j |t|^{2j}}{j!} \right)^m \leq C \left(e^{n\alpha|t|^2} - \sum_{j=0}^{j_a} \frac{n^j \alpha^j |t|^{2j}}{j!} \right), \quad \text{for all } t \in \mathbf{R}.$$

Proof. Since

$$\lim_{|t| \rightarrow 0} \frac{\left(e^{\alpha|t|^2} - \sum_{j=0}^{j_a} \frac{\alpha^j |t|^{2j}}{j!} \right)^m}{\left(e^{n\alpha|t|^2} - \sum_{j=0}^{j_a} \frac{n^j \alpha^j |t|^{2j}}{j!} \right)} = 0 = \lim_{|t| \rightarrow \infty} \frac{\left(e^{\alpha|t|^2} - \sum_{j=0}^{j_a} \frac{\alpha^j |t|^{2j}}{j!} \right)^m}{\left(e^{n\alpha|t|^2} - \sum_{j=0}^{j_a} \frac{n^j \alpha^j |t|^{2j}}{j!} \right)}$$

the conclusion follows. \square

3. The auxiliary functional

Given $R > R_0$, we define a function $\tilde{f} : \mathbf{R}^2 \times [0, +\infty) \rightarrow [0, +\infty)$ by

$$\tilde{f}(x, t) = \begin{cases} f(t), & |x| \leq R, \\ \min\{f(t), V(x)t^{\mu-1}\}, & |x| > R, \end{cases}$$

where $\mu > b^*$ is given by (H_2) . Moreover, we set $\tilde{f}(x, t) = 0$ for $t \leq 0$.

Lemma 3.1. (See [17]) *Suppose that f satisfies (H_1) and (H_2) . Then,*

$$0 < \mu \tilde{F}(x, t) \leq t \tilde{f}(x, t), \quad \text{for all } t > 0,$$

where $\mu > b^*$ is given by (H_2) and $\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) ds$.

Using the function \tilde{f} , we consider the following auxiliary functional $\tilde{J} : E \rightarrow \mathbf{R}$ defined by

$$\tilde{J}(u) = \int_{\mathbf{R}^2} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbf{R}^2} \tilde{F}(x, u) dx, \quad \text{for all } u \in E.$$

Fix $1 \leq p < +\infty$. We consider the subspace $\Xi^p = H_{V, rad}^1(\mathbf{R}^2) \cap L_V^p(\mathbf{R}^2)$ endowed with the norm

$$\|u\|_{\Xi^p} := \|u\|_{H_V^1(\mathbf{R}^2)} + \|u\|_{L_V^p(\mathbf{R}^2)}.$$

Lemma 3.2. (See [17]) *If $u_n \rightarrow u$ in Ξ^p , then there exist a subsequence (w_n) of (u_n) and g in $L_V^p(\mathbf{R}^2)$ such that, almost everywhere in \mathbf{R}^2 , $w_n(x) \rightarrow u(x)$ and*

$$|u(x)|, |w_n(x)| \leq g(x).$$

Lemma 3.3. *The functional \tilde{J} is well defined. Moreover, \tilde{J} belongs to $\mathcal{C}^1(E, \mathbf{R})$ and*

$$(3.1) \quad \tilde{J}'(u)\phi = \int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u\phi) dx - \int_{\mathbf{R}^2} \tilde{f}(x, u)\phi dx,$$

for all $u, \phi \in E$.

Proof. Consider $\tilde{J}_1 : E \rightarrow \mathbf{R}$ given by

$$\tilde{J}_1(u) = \int_{\mathbf{R}^2} (|\nabla u|^2 + V(x)u^2) dx.$$

By definition of the space E , we have that the functional \tilde{J}_1 is well defined. Moreover, let $B : E \times E \rightarrow \mathbf{R}$ given by

$$B(u, v) = \int_{\mathbf{R}^2} (\nabla u \nabla v + V(x)uv) dx, \quad \text{for all } (u, v) \in E \times E.$$

Then, B is a bilinear function. From Hölder's inequality, we have

$$\begin{aligned} |B(u, v)| &\leq \left(\int_{\mathbf{R}^2} |\nabla u|^2 dx \right)^{1/2} \left(\int_{\mathbf{R}^2} |\nabla v|^2 dx \right)^{1/2} \\ &\quad + \left(\int_{\mathbf{R}^2} V(x)u^2 dx \right)^{1/2} \left(\int_{\mathbf{R}^2} V(x)v^2 dx \right)^{1/2} \\ &\leq 2\|u\|\|v\|. \end{aligned}$$

Thus, B is a continuous bilinear function and $B \in \mathcal{C}^\infty(E \times E, \mathbf{R})$. Moreover, since $\tilde{J}_1(u) = B(u, u)$, we conclude that $\tilde{J}_1 \in \mathcal{C}^\infty(E, \mathbf{R})$ and

$$(3.2) \quad J'_1(u)\phi = B(u, \phi) = \int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u\phi) dx, \quad \text{for all } u, \phi \in E.$$

On the other hand, setting $\tilde{J}_{F_1} : H^1(B_R) \rightarrow \mathbf{R}$ and $\tilde{J}_{F_2} : \Xi^\mu \rightarrow \mathbf{R}$ defined by

$$\tilde{J}_{F_1}(u) = \int_{\mathbf{R}^2} \tilde{F}(x, u) \chi_{B_R}(x) dx \quad \text{and} \quad \tilde{J}_{F_2}(u) = \int_{\mathbf{R}^2} \tilde{F}(x, u) (1 - \chi_{B_R}(x)) dx.$$

We recall the existence of an extension operator $P : H^1(B_R) \rightarrow H^1(\mathbf{R}^2)$ such that $Pu|_{B_R} = u$. Using the Trudinger-Moser's inequality in the whole space (see [4, Lemma 2.1]), for all $u \in H^1(B_R)$ and $\alpha > 0$, we have

$$\int_{B_R} (e^{\alpha|u|^2} - 1) dx = \int_{B_R} (e^{\alpha|Pu|^2} - 1) dx \leq \int_{\mathbf{R}^2} (e^{\alpha|Pu|^2} - 1) dx < +\infty,$$

which implies

$$(3.3) \quad \int_{B_R} e^{\alpha|u|^2} dx < +\infty, \quad \text{for all } u \in H^1(B_R), \alpha > 0.$$

We observe that

$$\tilde{J}_{F_1}(u) = \int_{B_R} \tilde{F}(x, u) dx, \quad \text{for all } u \in H^1(B_R).$$

From (H_1) and (H_5) , for $\alpha > \alpha_0$ there exists $c > 0$ such that

$$(3.4) \quad f(s) \leq ce^{\alpha|s|^2}, \quad \text{for all } s \in \mathbf{R}.$$

Thus, for $|x| \leq R$, we have

$$(3.5) \quad |\tilde{F}(x, t)| = \left| \int_0^t \tilde{f}(x, s) ds \right| \leq \int_0^t f(s) ds \leq c \int_0^{|t|} e^{\alpha|s|^2} ds \leq \frac{c}{2}(e^{2\alpha|t|^2} + |t|^2).$$

Using (3.3), (3.5) and the embedding of $H^1(B_R)$ in $L^2(B_R)$, we obtain

$$\int_{B_R} \tilde{F}(x, u) dx < +\infty, \quad \text{for all } u \in H^1(B_R).$$

Thus, \tilde{J}_{F_1} is well defined. Now, set $u, v \in H^1(B_R)$ and $0 < |t| < 1$. By the mean value theorem, there exists $\theta(x, t) \in (0, 1)$ such that

$$(3.6) \quad \frac{\tilde{F}(x, u + tv) - \tilde{F}(x, u)}{t} = \tilde{f}(x, u + \theta(x, t)tv)v.$$

Since the function $\tilde{f}(x, t)$ is continuous in the second variable, it follows that

$$\lim_{t \rightarrow 0} \frac{\tilde{F}(x, u + tv) - \tilde{F}(x, u)}{t} = \tilde{f}(x, u)v.$$

Moreover, using (3.4) in (3.6) and the fact that $\tilde{f}(x, t) \leq f(t)$, we get

$$\left| \frac{\tilde{F}(x, u + tv) - \tilde{F}(x, u)}{t} \right| \leq ce^{\alpha(|u|+|v|)^2}|v| \leq \frac{c}{2}(e^{2\alpha(|u|+|v|)^2} + |v|^2) \in L^1(B_R).$$

From Dominated convergence theorem, we find

$$\begin{aligned} \tilde{J}'_{F_1}(u)v &= \lim_{t \rightarrow 0} \frac{\tilde{J}_{F_1}(u + tv) - \tilde{J}_{F_1}(u)}{t} \\ &= \lim_{t \rightarrow 0} \int_{\Omega} \frac{\tilde{F}(x, u + tv) - \tilde{F}(x, u)}{t} dx \\ &= \int_{B_R} \lim_{t \rightarrow 0} \frac{\tilde{F}(x, u + tv) - \tilde{F}(x, u)}{t} dx \\ &= \int_{B_R} \tilde{f}(x, u)v dx. \end{aligned}$$

In order to prove the continuity of \tilde{J}_{F_1} , let (u_n) be a sequence in $H^1(B_R)$ such that $u_n \rightarrow u$ in $H^1(B_R)$. Arguing similarly as Proposition 2.7 in [10], we can assume that $u_n \rightarrow u$ almost everywhere in B_R and there exists $v \in H^1(B_R)$ such that $|u_n(x)| \leq v(x)$ almost everywhere in B_R . Consequently,

$$|\tilde{f}(x, u_n) - \tilde{f}(x, u)|^2 \leq 2c(e^{2\alpha|v|^2} + e^{2\alpha|u|^2}) \in L^1(B_R),$$

by the continuity of \tilde{f} almost everywhere in B_R , we get

$$|\tilde{f}(x, u_n) - \tilde{f}(x, u)|^2 \rightarrow 0, \quad \text{almost everywhere in } B_R.$$

By Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \|\tilde{J}'_{F_1}(u_n) - \tilde{J}'_{F_1}(u)\| &= \sup_{\|v\|_{H^1(B_R)} \leq 1} |\langle \tilde{J}'_{F_1}(u_n) - \tilde{J}'_{F_1}(u), v \rangle| \\ &= \sup_{\|v\|_{H^1(B_R)} \leq 1} \left| \int_{B_R} (\tilde{f}(x, u_n) - \tilde{f}(x, u)) v \, dx \right| \\ &\leq \sup_{\|v\|_{H^1(B_R)} \leq 1} \|\tilde{f}(x, u_n) - \tilde{f}(x, u)\|_{L^2(B_R)} \|v\|_{L^2(B_R)} \\ &= o_n(1). \end{aligned}$$

Thus, $\tilde{J}_{F_1} \in \mathcal{C}^1(H^1(B_R), \mathbf{R})$. Since $E \hookrightarrow H^1(B_R)$ continuously, it follows that $\tilde{J}_{F_1} \in \mathcal{C}^1(E, \mathbf{R})$ and

$$(3.7) \quad \tilde{J}'_{F_1}(u)\phi = \int_{B_R} \tilde{f}(x, u)\phi \, dx, \quad \text{for all } u, \phi \in E.$$

Notice that $\tilde{F}(x, s)(1 - \chi_{B_R}(x))$ is a Carathéodory function in $(x, s) \in \mathbf{R}^2 \times \mathbf{R}$ and

$$|\tilde{f}(x, s)(1 - \chi_{B_R}(x))| \leq V(x)|s|^{\mu-1}, \quad \text{for all } (x, s) \in \mathbf{R}^2 \times \mathbf{R}.$$

Using Lemma 3.2 and arguing similarly as Lemma 17.1 in [14], we have $\tilde{J}_{F_2} \in \mathcal{C}^1(\Xi^\mu, \mathbf{R})$, by Lemma 1.1, the embedding $E \hookrightarrow \Xi^\mu$ is continuous, we have $\tilde{J}_{F_2} \in \mathcal{C}^1(E, \mathbf{R})$ and

$$(3.8) \quad \tilde{J}'_{F_2}(u)\phi = \int_{\mathbf{R}^2 \setminus B_R} \tilde{f}(x, u)\phi \, dx, \quad \text{for all } u, \phi \in E.$$

Finally, note that $\tilde{J} = \tilde{J}_1 - \tilde{J}_{F_1} - \tilde{J}_{F_2}$. Thus, \tilde{J} is well defined and using (3.2), (3.7) and (3.8), we obtain (3.1). \square

3.1. The geometry of Mountain Pass

This section is devoted to set the geometry of the Mountain Pass Theorem of the auxiliary functional.

Lemma 3.4. *Suppose $(V_1), (V_2), (H_1), (H_4)$ and (H_5) hold. Then, there exist $\sigma > 0$ and $\rho > 0$ such that*

$$\tilde{J}(u) \geq \sigma, \quad \text{for all } u \in E, \quad \|u\| = \rho.$$

Proof. From (H_4) , we have $f(s) = o(s^{a^*-1})$. Thus, there exists $\delta_0 > 0$ such that

$$|f(s)| \leq |s|^{a^*-1}, \quad \text{for all } |s| < \delta_0.$$

By (H_5) , we can find constants $c > 0$, $\delta_1 > \delta_0$ and $q \geq a^*$ such that

$$|f(s)| \leq c|s|^{q-1} \left(e^{2\alpha_0|s|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |s|^{2j}}{j!} \right), \quad \text{for all } |s| \geq \delta_1.$$

Note also that for all $\delta_0 \leq |s| \leq \delta_1$, we have

$$|f(s)| \leq \frac{|s|^{q-1} \left(e^{2\alpha_0|s|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |s|^{2j}}{j!} \right)}{|\delta_0|^{q-1} \left(e^{2\alpha_0|\delta_0|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |\delta_0|^{2j}}{j!} \right)} \max_{\delta_0 \leq |s| \leq \delta_1} |f(s)|.$$

From these estimates, we get a constant $c > 0$ such that

$$|f(s)| \leq |s|^{a^*-1} + c|s|^{q-1} \left(e^{2\alpha_0|s|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |s|^{2j}}{j!} \right), \quad \text{for all } s \in \mathbf{R}.$$

Then,

$$|\tilde{F}(x, s)| \leq |F(s)| \leq |s|^{a^*} + c|s|^q \left(e^{2\alpha_0|s|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |s|^{2j}}{j!} \right), \quad \text{for all } s \in \mathbf{R}$$

By Lemma 2.3 and Proposition 2.2, we obtain

$$\begin{aligned} \int_{\mathbf{R}^2} |u|^q \left(e^{2\alpha_0|u|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |u|^{2j}}{j!} \right) dx \\ \leq \|u\|_{2q}^q \left(\int_{\mathbf{R}^2} \left(e^{2\alpha_0|u|^2} - \sum_{j=0}^{j_a} \frac{2^j \alpha_0^j |u|^{2j}}{j!} \right)^2 dx \right)^{1/2} \\ \leq c \|u\|_{2q}^q \left(\int_{\mathbf{R}^2} \left(e^{6\alpha_0|u|^2} - \sum_{j=0}^{j_a} \frac{6^j \alpha_0^j |u|^{2j}}{j!} \right) dx \right)^{1/2} \\ \leq c \|u\|_{2q}^q, \end{aligned}$$

provided that $\|u\| \leq \rho_1$ for some $\rho_1 > 0$ such that $6\alpha_0\rho_1^2 < 4\pi$.

Thus,

$$\int_{\mathbf{R}^2} \tilde{F}(x, u) dx \leq c\|u\|_{a^*}^{a^*} + c\|u\|_{2q}^q.$$

By Lemma 1.1, we obtain

$$\tilde{J}(u) \geq \|u\|^2 - \int_{\mathbf{R}^2} \tilde{F}(x, u) dx \geq \|u\|^2 - c\|u\|^{a^*} - c\|u\|^q.$$

Therefore, we can find $\rho > 0$ and $\sigma > 0$ with ρ sufficiently small such that $\tilde{J}(u) \geq \sigma$, for all $u \in E$ with $\|u\| = \rho$. \square

Lemma 3.5. *Suppose that $(V_1) - (V_2)$ and $(H_1) - (H_2)$ hold. Then, there exists $e \in E$ such that*

$$\tilde{J}(e) < \rho \quad \text{and} \quad \|e\| > \rho.$$

where $\rho > 0$ is given by Lemma 3.4.

Proof. It follows from Lemma 3.1 the existence of $c > 0$ and $\vartheta > 2$ such that

$$\tilde{F}(x, s) \geq c|s|^\vartheta - s^2, \quad \text{for all } (x, s) \in \overline{B_1}(0) \times [0, +\infty).$$

Let $0 \neq e_{k_0} \in E$ fixed. Then,

$$\begin{aligned} \tilde{J}(te_{k_0}) &= t^2\|e_{k_0}\|^2 - \int_{\mathbf{R}^2} \tilde{F}(x, te_{k_0}) dx \\ &\leq t^2\|e_{k_0}\|^2 + \int_{B_1} \left(|te_{k_0}|^2 - c|te_{k_0}|^\vartheta \right) dx \\ &\leq t^2\|e_{k_0}\|^2 + t^2\|e_{k_0}\|_2^2 - ct^\vartheta\|e_{k_0}\|_\vartheta^\vartheta \end{aligned}$$

Since, $\vartheta > 2$. Then, $\tilde{J}(te_{k_0}) \rightarrow -\infty$. Thus, we can take $e = t_0e_{k_0}$ with $t_0 > 0$ sufficiently large such that $\tilde{J}(e) < 0$ and $\|e\| > \rho$.

\square

By Lemmas 3.4 and 3.5 in Mountain Pass Theorem (see [21, Theorem 1.15]) and Ekeland's variational principle (see [21, Theorem 2.4]), there exists a Palais-Smale sequence at level $d \geq \sigma$, where σ is given by Lemma 3.4, that is, there exists a sequence $(u_n) \subset E$ such that

$$(3.9) \quad \tilde{J}(u_n) \rightarrow d \quad \text{and} \quad \|\tilde{J}'(u_n)\|_{E^-} \rightarrow 0,$$

and $d > 0$ can be characterized as

$$(3.10) \quad d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{J}(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.6. *Let $(u_n) \subset E$ be a Palais-Smale sequence satisfying (3.9). Then, $\|u_n\| \leq c$, for every $n \in \mathbf{N}$ and for some positive constant c .*

Proof. From Lemma 3.1, we obtain

$$\begin{aligned} J(u_n) - \frac{1}{\mu} \tilde{J}'(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 - \frac{1}{\mu} \int_{\mathbf{R}^2} (\mu \tilde{F}(x, u_n) - \tilde{f}(x, u_n)u_n) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2. \end{aligned}$$

Using (3.9), for n sufficiently large, we have

$$\tilde{J}(u_n) \leq d + 1 \quad \text{and} \quad \|\tilde{J}'(u_n)\|_{E^-} \leq \mu,$$

Thus, for n sufficiently large, we get

$$\left| \tilde{J}(u_n) - \frac{1}{\mu} \tilde{J}'(u_n)u_n \right| \leq d + 1 + \|u_n\|.$$

Then, for n sufficiently large, we obtain

$$\left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 \leq d + 1 + \|u_n\|,$$

which implies that the sequence (u_n) is bounded. \square

Lemma 3.7. (See [5, Lemma 2.1]) *Let Ω be a bounded subset in \mathbf{R}^N , $f : \overline{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$ a continuous function and (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u(x))$ and $f(x, u_n(x))$ are also $L^1(\Omega)$ functions. If*

$$\int_{\Omega} |f(x, u_n)u_n| dx \leq C,$$

then, $f(x, u_n)$ converges in $L^1(\Omega)$ to $f(x, u)$.

Lemma 3.8. *Let (u_n) be a Palais-Smale sequence satisfying (3.9) and suppose that $u_n \rightharpoonup u$ in E . Then, there exists a subsequence still denoted by (u_n) such that*

$$\tilde{f}(x, u_n) \rightarrow \tilde{f}(x, u) \quad \text{in} \quad L^1(B_{R_1}),$$

where $R_1 > R_0$ and

$$\tilde{F}(x, u_n) \rightarrow \tilde{F}(x, u) \quad \text{in} \quad L^1(\mathbf{R}^2).$$

Proof. According to Remark 1.2, we can assume that $u_n \rightarrow u$ in $L^1(B_{R_1})$. Moreover, by the exponential growth of f and Proposition 2.2, we have that $\tilde{f}(x, u_n) \in L^1(B_{R_1})$. By Lemma 3.6, the sequence $(\|u_n\|)$ is bounded and since $\|\tilde{J}'(u_n)\|_{E^-} \rightarrow 0$, we obtain

$$|\tilde{J}'(u_n)u_n| \leq \|\tilde{J}'(u_n)\|_{E^-} \|u_n\| \rightarrow 0.$$

Thus,

$$\tilde{J}'(u_n)u_n = \|u_n\|^2 - \int_{\mathbf{R}^2} \tilde{f}(x, u_n)u_n \, dx \rightarrow 0.$$

Then, there exists $c > 0$ such that

$$\int_{\mathbf{R}^2} \tilde{f}(x, u_n)u_n \, dx \leq c.$$

Using Lemma 3.7, we conclude that $\tilde{f}(x, u_n) \rightarrow \tilde{f}(x, u)$ in $L^1(B_{R_1})$.

On the other hand, by the first part, given $R_1 \geq R$, where R is given by the definition of \tilde{f} , we obtain

$$\int_{B_{R_1}} \tilde{f}(x, u_n) \, dx \rightarrow \int_{B_{R_1}} \tilde{f}(x, u) \, dx.$$

Thus, there exists $p \in L^1(B_{R_1})$ such that

$$(3.11) \quad f(u_n) \leq p(x), \quad \text{almost everywhere in } B_{R_1}.$$

From (H_1) and (H_3) , we obtain

$$(3.12) \quad F(t) \leq \max_{t \in [0, s_0]} F(t) + Mf(t), \quad \text{for all } t \in \mathbf{R}.$$

Using (3.11) and (3.12), we have

$$\tilde{F}(x, u_n) \leq F(u_n) \leq \max_{t \in [0, s_0]} F(t) + Mp(x), \quad \text{almost everywhere in } B_{R_1}.$$

By Lebesgue's dominated convergence theorem, we obtain

$$\int_{B_{R_1}} \tilde{F}(x, u_n) \, dx \rightarrow \int_{B_{R_1}} \tilde{F}(x, u) \, dx.$$

Consequently, to prove that

$$\int_{\mathbf{R}^2} \tilde{F}(x, u_n) \, dx \rightarrow \int_{\mathbf{R}^2} \tilde{F}(x, u) \, dx,$$

it is sufficient to show that given $\delta > 0$, there exists $R_1 > 0$ such that

$$\int_{\mathbf{R}^2 \setminus B_{R_1}} \tilde{F}(x, u_n) dx < \delta \quad \text{and} \quad \int_{\mathbf{R}^2 \setminus B_{R_1}} \tilde{F}(x, u) dx < \delta.$$

Using definition of the function \tilde{f} , we have

$$\tilde{f}(x, t) \leq V(x)t^{\mu-1}, \quad \text{for all } x \in \mathbf{R}^2 \setminus B_{R_1}.$$

Then, for all $x \in \mathbf{R}^2 \setminus B_{R_1}$ and $t > 0$, we have

$$\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) ds \leq \int_0^t V(x)s^{\mu-1} ds = \frac{1}{\mu} V(x)t^\mu.$$

Thus,

$$\tilde{F}(x, u_n) \leq \frac{1}{\mu} V(x)|u_n|^\mu, \quad \text{for all } x \in \mathbf{R}^2 \setminus B_{R_1}.$$

Hence,

$$\begin{aligned} \int_{\mathbf{R}^2 \setminus B_{R_1}} \tilde{F}(x, u_n) dx &\leq \frac{1}{\mu} \int_{\mathbf{R}^2 \setminus B_{R_1}} V(x)|u_n|^\mu dx \\ &\leq \frac{2^{\mu-1}}{\mu} \left(\int_{\mathbf{R}^2 \setminus B_{R_1}} V(x)|u_n - u|^\mu dx + \int_{\mathbf{R}^2 \setminus B_{R_1}} V(x)|u|^\mu dx \right). \end{aligned}$$

Using the compactness of the embedding $E \hookrightarrow L_{V, rad}^\mu(\mathbf{R}^2)$ and the weak convergence $u_n \rightharpoonup u$ in E , we can choose $R_1 > 0$ sufficiently large such that

$$\int_{\mathbf{R}^2 \setminus B_{R_1}} \tilde{F}(x, u_n) dx < \delta.$$

Since $\tilde{F}(\cdot, u) \in L^1(\mathbf{R}^2)$, we may assume that

$$\int_{\mathbf{R}^2 \setminus B_{R_1}} \tilde{F}(x, u) dx < \delta.$$

Combining all the above estimates, since $\delta > 0$ is arbitrary, we have

$$\int_{\mathbf{R}^2} \tilde{F}(x, u_n) dx \rightarrow \int_{\mathbf{R}^2} \tilde{F}(x, u) dx.$$

□

4. Estimates

In this section, we establish the estimates for the auxiliary functional that are used to prove Proposition 5.1. We start with the definition of Moser type functions. Consider $k \in \mathbf{N}$. Let $\delta_k > 0$ be a sequence which will be fixed such that $\delta_k \rightarrow 0$, as $k \rightarrow +\infty$. The Moser type functions are defined by

$$e_k = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\ln k} (1 - \delta_k)^{1/2}, & |x| \leq \frac{1}{k}, \\ \ln \left(\frac{1}{|x|} \right) \frac{(1 - \delta_k)^{1/2}}{\sqrt{\ln k}}, & \frac{1}{k} < |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Therefore,

$$\|\nabla e_k\|_2^2 = 1 - \delta_k$$

and

$$\int_{\mathbf{R}^2} V(x) e_k^2 dx \leq (1 - \delta_k) \left(\frac{\ln k}{k^2} + \frac{1}{4 \ln k} \right).$$

Then, we may choose δ_k , depending on k such that

$$\|e_k\| = 1, \quad \text{for all } k \geq 1.$$

Furthermore, we can see that

$$\delta_k \leq (1 - \delta_k) \left(\frac{\ln k}{k^2} + \frac{1}{4 \ln k} \right) \leq \left(\frac{\ln k}{k^2} + \frac{1}{4 \ln k} \right).$$

Thus,

$$(4.1) \quad \delta_k \ln k \leq \frac{1}{2}, \quad \text{for } k \text{ sufficiently large.}$$

Proposition 4.1. *Suppose that $(H_1) - (H_6)$ hold. Then, there exists $k_0 \in \mathbf{N}$ such that*

$$\max \{ \tilde{J}(te_{k_0}) : t \geq 0 \} < \frac{2\pi}{\alpha_0}.$$

Proof. Suppose, by contradiction, that for all $k \in \mathbf{N}$

$$\max \{ \tilde{J}(te_k) : t \geq 0 \} \geq \frac{2\pi}{\alpha_0}.$$

Thus, for all fixed $k \geq 1$, there exists $t_k > 0$ such that

$$\tilde{J}(t_k e_k) = \max \{ \tilde{J}(te_k) : t \geq 0 \} \geq \frac{2\pi}{\alpha_0}.$$

Then,

$$\tilde{J}(t_k e_k) = \frac{t_k^2 \|e_k\|^2}{2} - \int_{\mathbf{R}^2} \tilde{F}(x, t_k e_k) dx \geq \frac{2\pi}{\alpha_0}.$$

and

$$\frac{d}{dt} \tilde{J}(t e_k) = 0, \quad \text{in } t = t_k.$$

That is,

$$t_k \|e_k\|^2 - \int_{\mathbf{R}^2} \tilde{f}(x, t_k e_k) e_k dx = 0.$$

Using last equations and the fact that $\|e_k\| = 1$, we have

$$(4.2) \quad t_k^2 \geq \frac{4\pi}{\alpha_0}$$

and

$$t_k^2 = \int_{\mathbf{R}^2} \tilde{f}(x, t_k e_k) t_k e_k dx.$$

Set $l > 0$ such that

$$(4.3) \quad \liminf_{t \rightarrow +\infty} \frac{tf(t)}{e^{\alpha_0 t^2}} > l > \frac{4e}{\alpha_0}.$$

Thus, given $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$(4.4) \quad tf(t) \geq (l - \epsilon) e^{\alpha_0 t^2}, \quad \text{for all } t \geq R_\epsilon.$$

Using the fact that (t_k) is bounded below, there exists $k_0 > 0$ such that

$$(1 - \delta_k)^{1/2} t_k \frac{\sqrt{\ln k}}{2\pi} \geq R_\epsilon, \quad \text{for all } k \geq k_0.$$

Since,

$$e_k(x) = (1 - \delta_k)^{1/2} \sqrt{\frac{\ln k}{2\pi}}, \quad \text{for all } x \in B_{1/k},$$

we get

$$\begin{aligned} t_k^2 &= \int_{\mathbf{R}^2} \tilde{f}(x, t_k e_k) t_k e_k dx \\ &\geq \int_{B_{1/k}} \tilde{f}(x, t_k e_k) t_k e_k dx \\ &= \int_{B_{1/k}} f(t_k e_k) t_k e_k dx \\ &\geq (l - \epsilon) \int_{B_{1/k}} e^{\alpha_0 (1 - \delta_k) \frac{\ln k}{2\pi} t_k^2} dx, \end{aligned}$$

for every $k \geq k_0$. Define $s_k := t_k^2 - \frac{4\pi}{\alpha_0}$. Then,

$$\begin{aligned} \frac{4\pi}{\alpha_0} + s_k &\geq (l - \epsilon) \int_{B_{1/k}} e^{\alpha_0(1-\delta_k)\frac{\ln k}{2\pi}(\frac{4\pi}{\alpha_0}+s_k)} dx \\ &= (l - \epsilon) e^{\alpha_0\frac{\ln k}{2\pi}(\frac{4\pi}{\alpha_0}+s_k)} e^{-\frac{\alpha_0}{2\pi}(\frac{4\pi}{\alpha_0}+s_k)\delta_k \ln k} \int_{1/k}^1 1 dx \\ &= (l - \epsilon) e^{2 \ln k} e^{\alpha_0\frac{\ln k}{2\pi}s_k} e^{-\frac{\alpha_0}{2\pi}(\frac{4\pi}{\alpha_0}+s_k)\delta_k \ln k} \frac{\pi}{k^2} \\ &= (l - \epsilon) \pi e^{\alpha_0\frac{\ln k}{2\pi}s_k} e^{-\frac{\alpha_0}{2\pi}(\frac{4\pi}{\alpha_0}+s_k)\delta_k \ln k} \end{aligned}$$

for every $k \geq k_0$. Using (4.1), we obtain

$$\frac{4\pi}{\alpha_0} + s_k \geq (l - \epsilon) \pi e^{\alpha_0\frac{\ln k}{2\pi}s_k} e^{-\frac{\alpha_0}{4\pi}(\frac{4\pi}{\alpha_0}+s_k)}.$$

Thus,

$$(4.5) \quad \frac{4\pi}{\alpha_0} + s_k \geq (l - \epsilon) \pi e^{\frac{\alpha_0 s_k}{2\pi}(\ln k - 1/2)} e^{-1}.$$

Inequality (4.5) implies that (s_k) is bounded for each $k \geq k_0$. Therefore, there exists $s \in \mathbf{R}$ such that $\limsup_{k \rightarrow \infty} s_k = s$. By (4.2), $s \geq 0$. Using the last limit in (4.5) and taking $k \rightarrow +\infty$, we see that necessarily $s = 0$. Then, $\lim_{k \rightarrow \infty} s_k = 0$. Using this in (4.5), yields

$$\frac{4\pi}{\alpha_0} \geq (l - \epsilon) \pi e^{-1}.$$

This contradicts (4.3) because $\epsilon > 0$ is arbitrary. \square

Remark 4.2. Taking e_{k_0} given by Proposition 4.1 in Lemma 3.5. Thus, $e = t_0 e_{k_0}$. Define $\gamma_0(t) = t t_0 e_{k_0}$. Then, $\gamma_0 \in \Gamma = \{\gamma \in \mathcal{C}([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$. By Proposition 4.1 and (3.10), we obtain

$$d = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \tilde{J}(\gamma(t)) \leq \max_{t \in [0, 1]} \tilde{J}(\gamma_0(t)) \leq \max_{t \in [0, 1]} \tilde{J}(t t_0 e_{k_0}) \leq \max_{t \geq 0} \tilde{J}(t e_{k_0}) < \frac{2\pi}{\alpha_0}.$$

5. Existence of critical point of the auxiliary functional

Proposition 5.1. Suppose that V satisfies $(V_1) - (V_2)$ and f satisfies $(H_1) - (H_6)$. Then, \tilde{J} possesses a nontrivial critical point.

Proof. Let $(u_n) \subset E$ be a sequence satisfying (3.9). Then,

$$(5.1) \quad \tilde{J}(u_n)\phi = \int_{\mathbf{R}^2} (\nabla u_n \nabla \phi + V(x)u_n \phi) dx - \int_{\mathbf{R}^2} \tilde{f}(x, u_n)\phi dx = o_n(1),$$

for all $\phi \in \mathcal{C}_{0,r}^\infty(\mathbf{R}^2)$.

By Lemma 3.6, the sequence (u_n) is bounded in E . Thus, we can assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ in E , using this together with Lemma 5.2 in (5.1), we obtain passing to limit

$$\int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u \phi) dx - \int_{\mathbf{R}^2} \tilde{f}(x, u)\phi dx = 0, \quad \text{for all } \phi \in \mathcal{C}_{0,r}^\infty(\mathbf{R}^2).$$

Using the fact that $\mathcal{C}_{0,rad}^\infty(\mathbf{R}^2)$ is dense in E , yields

$$\int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u \phi) dx = \int_{\mathbf{R}^2} \tilde{f}(x, u)\phi dx, \quad \text{for all } \phi \in E.$$

Thus, $u \in E$ is a critical point of \tilde{J} . To conclude the proof, it only remains to prove that u is nontrivial. Suppose, by contradiction, that $u \equiv 0$. From Lemma 1.1, we can assume that

$$(5.2) \quad u_n \rightarrow 0 \quad \text{in } L_{V,rad}^r(\mathbf{R}^2), \quad \text{for all } r > b^*.$$

Using the fact that $\tilde{J}(u_n) \rightarrow d$, we have

$$(5.3) \quad \tilde{J}(u_n) = \frac{\|u_n\|^2}{2} - \int_{\mathbf{R}^2} \tilde{F}(x, u_n) dx = d + o_n(1).$$

Since, we suppose that $u_n \rightharpoonup 0$, by Lemma 3.8, we obtain

$$\int_{\mathbf{R}^2} \tilde{F}(x, u_n) dx \rightarrow \int_{\mathbf{R}^2} \tilde{F}(x, 0) dx = 0.$$

Replacing in (5.3), we have

$$(5.4) \quad \frac{\|u_n\|^2}{2} = d + o_n(1).$$

By Remark 4.2, we get

$$\|u_n\|^2 = 2d + o_n(1) < \frac{4\pi}{\alpha_0} + o_n(1).$$

Thus, we can assume that there exists $\delta > 0$ sufficiently small such that

$$\|u_n\|^2 \leq \frac{4\pi}{\alpha_0} - \delta, \quad \text{for all } n \text{ sufficiently large.}$$

Taking $p > 1$ sufficiently close to 1 and $\epsilon > 0$ sufficiently small such that

$$(5.5) \quad p(\alpha_0 + \epsilon) \left(\frac{4\pi}{\alpha_0} - \delta \right) < 4\pi.$$

From the continuity and the critical growth of f there exists a positive constant C such that

$$(5.6) \quad |f(s)| \leq |s|^{a^*-1} + C \left(e^{(\alpha_0 + \epsilon)|s|^2} - \sum_{j=0}^{j_a} \frac{(\alpha_0 + \epsilon)^j |s|^{2j}}{j!} \right), \quad \text{for all } s \in \mathbf{R}.$$

By Hölder's inequality and (5.6), we have

$$(5.7) \quad \begin{aligned} \int_{B_R} \tilde{f}(x, u_n) u_n \, dx &= \int_{B_R} f(u_n) u_n \, dx \\ &\leq \|u_n\|_{a^*}^{a^*} + C \int_{B_R} \left(e^{(\alpha_0 + \epsilon)|u_n|^2} - \sum_{j=0}^{j_a} \frac{(\alpha_0 + \epsilon)^j |u_n|^{2j}}{j!} \right) |u_n| \, dx \\ &\leq \|u_n\|_{a^*}^{a^*} + C \|u_n\|_{p'} \left(\int_{B_R} \left(e^{(\alpha_0 + \epsilon)|u_n|^2} - \sum_{j=0}^{j_a} \frac{(\alpha_0 + \epsilon)^j |u_n|^{2j}}{j!} \right)^p \, dx \right)^{1/p}. \end{aligned}$$

From (5.5), we can find for some $p_0 > p$ such that

$$(5.8) \quad p_0(\alpha_0 + \epsilon) \left(\frac{4\pi}{\alpha_0} - \delta \right) < 4\pi.$$

Using Lemma 2.3 in the last integral of (5.7) there exists $C_0 > 0$ such that

$$\begin{aligned} &\int_{B_R} \left(e^{(\alpha_0 + \epsilon)|u_n|^2} - \sum_{j=0}^{j_a} \frac{(\alpha_0 + \epsilon)^j |u_n|^{2j}}{j!} \right)^p \, dx \\ &\leq C_0 \int_{B_R} \left(e^{p_0(\alpha_0 + \epsilon)|u_n|^2} - \sum_{j=0}^{j_a} \frac{p_0^j (\alpha_0 + \epsilon)^j |u_n|^{2j}}{j!} \right) \, dx \\ &\leq C_0 \int_{B_R} \left(e^{p_0(\alpha_0 + \epsilon) \left(\frac{4\pi}{\alpha_0} - \delta \right) \frac{|u_n|^2}{\|u_n\|^2}} - \sum_{j=0}^{j_a} \frac{(p_0(\alpha_0 + \epsilon) \left(\frac{4\pi}{\alpha_0} - \delta \right))^j |u_n|^{2j}}{j! \|u_n\|^{2j}} \right) \, dx. \end{aligned}$$

Using (5.8) and Theorem 2.1 in the last inequality, we can find some $C > 0$ such that

$$(5.9) \quad \int_{B_R} \left(e^{(\alpha_0 + \epsilon)|u_n|^2} - \sum_{j=0}^{j_a} \frac{(\alpha_0 + \epsilon)^j |u_n|^{2j}}{j!} \right)^p dx \leq C.$$

Replacing (5.9) in (5.7), we obtain

$$\int_{B_R} \tilde{f}(x, u_n) u_n dx \leq \|u_n\|_{a^*}^{a^*} + C \|u_n\|_{p'}.$$

By (5.2), we get

$$(5.10) \quad \int_{B_R} \tilde{f}(x, u_n) u_n dx \rightarrow 0.$$

On the other hand,

$$\int_{\mathbf{R}^2 \setminus B_R} \tilde{f}(x, u_n) u_n dx \leq \int_{\mathbf{R}^2 \setminus B_R} V(x) u_n^\mu dx \leq \int_{\mathbf{R}^2} V(x) u_n^\mu dx$$

Using Lemma 1.1 and the fact that $u_n \rightharpoonup 0$ in E , we can suppose up to a subsequence that

$$\int_{\mathbf{R}^2} V(x) u_n^\mu dx = \|u_n\|_{L_V^\mu(\mathbf{R}^2)}^\mu \rightarrow 0.$$

Thus,

$$(5.11) \quad \int_{\mathbf{R}^2 \setminus B_R} \tilde{f}(x, u_n) u_n dx \rightarrow 0.$$

Combining (5.10) with (5.11), we get

$$(5.12) \quad \int_{\mathbf{R}^2} \tilde{f}(x, u_n) u_n dx \rightarrow 0.$$

Using the fact that $(\|u_n\|)$ is bounded and $\|\tilde{J}'(u_n)\|_{E^-} \rightarrow 0$, we obtain

$$(5.13) \quad |\tilde{J}'(u_n) u_n| \leq \|\tilde{J}'(u_n)\|_{E^-} \|u_n\| \rightarrow 0.$$

Since,

$$\tilde{J}'(u_n) u_n = \|u_n\|^2 - \int_{\mathbf{R}^2} \tilde{f}(x, u_n) u_n dx.$$

By (5.12) and (5.13), we have

$$\|u_n\|^2 = \tilde{J}'(u_n) u_n + \int_{\mathbf{R}^2} \tilde{f}(x, u_n) u_n dx \rightarrow 0.$$

From (5.4), we have $\|u_n\|^2 \rightarrow 2d$. Then, $d = 0$ which is a contradiction. Thus, u is a nontrivial critical point of \tilde{J} . \square

Lemma 5.2. *Let u be the critical point of \tilde{J} given by Proposition 5.1. Then,*

$$\|u\| \leq \sqrt{\frac{4\pi}{\alpha_0}}.$$

Proof. Since $\tilde{J}(u_n) \rightarrow d$, we have

$$\tilde{J}(u_n) = \frac{\|u_n\|^2}{2} - \int_{\mathbf{R}^2} \tilde{F}(x, u_n) dx = d + o_n(1).$$

Using the fact that $\tilde{F}(x, s) \geq 0$ for all $(x, s) \in \mathbf{R}^2$ and Remark 4.2, we obtain

$$\|u_n\|^2 \leq 2d + o_n(1) < \frac{4\pi}{\alpha_0} + o_n(1).$$

Since, $u_n \rightharpoonup u$ in E , we get

$$\|u\|^2 \leq \liminf_{n \rightarrow \infty} \|u_n\|^2 \leq \frac{4\pi}{\alpha_0},$$

this complete the proof. \square

6. Proof of Theorem 1.3

Let $0 \neq u \in E$ given by Proposition 5.1. We start showing that

$$(6.1) \quad \tilde{f}(x, u(x)) = f(u(x)), \quad \text{for all } x \in \mathbf{R}^2.$$

Notice that, by definition $\tilde{f}(x, u(x)) = f(u(x))$ for all $|x| \leq R_0$. Moreover, if $u(x) \leq 0$ then, $\tilde{f}(x, u(x)) = 0 = f(u(x))$. Thus, we can assume that $u(x) > 0$ for all $|x| > R_0$. From (H_4) , there exist constants $C_1 = C_1(f, \mu, \theta) > 0$ and $s_1 > 0$ such that

$$(6.2) \quad \frac{f(s)}{s^{\mu-1}} \leq C_1 s^\theta \leq C_1 s^\theta e^{(\alpha_0+1)|s|^2}, \quad \text{for all } 0 < s < s_1.$$

From (H_5) , we have

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{\mu-1} s^\theta e^{(\alpha_0+1)|s|^2}} \rightarrow 0.$$

Thus, there exist constants $C_2 = C_2(f, \mu, \alpha_0, \theta) > 0$ and $s_2 > 0$ such that

$$(6.3) \quad \frac{f(s)}{s^{\mu-1}} \leq C_2 s^\theta e^{(\alpha_0+1)|s|^2}, \quad \text{for all } s > s_2.$$

From the continuity of f , there exists a constant $C_3 = C_3(f, \mu, \alpha_0, \theta) > 0$ such that

$$(6.4) \quad \frac{f(s)}{s^{\mu-1} s^\theta e^{(\alpha_0+1)|s|^2}} \leq C_3, \quad \text{for all } s_1 \leq s \leq s_2.$$

Combining the estimates (6.2), (6.3) with (6.4), there exists a positive constant $C = C(f, \mu, \alpha_0, \theta)$ such that

$$\frac{f(s)}{s^{\mu-1}} \leq C s^\theta e^{(\alpha_0+1)|s|^2}, \quad \text{for all } s > 0.$$

Since, we suppose that $u(x) > 0$ for all $|x| > R_0$, we have

$$\frac{f(u)}{u(x)^{\mu-1}} \leq C u(x)^\theta e^{(\alpha_0+1)|u|^2}, \quad \text{for all } |x| > R_0.$$

By Lemma 2.1 and the fact that $L_a \geq 1$, we have

$$\frac{f(u)}{u(x)^{\mu-1}} \leq C \frac{\|u\|^\theta e^{\frac{(\alpha_0+1)\|u\|^2}{\frac{2-a}{2}}}}{|x|^{(\frac{2-a}{4})\theta}}, \quad \text{for all } |x| > R_0.$$

Using Lemma 5.2 and the fact that $R_0 \geq 1$, we get

$$\frac{f(u)}{u(x)^{\mu-1}} \leq \frac{C(4\pi)^{\theta/2} e^{\frac{(\alpha_0+1)4\pi}{\alpha_0}}}{\alpha_0^{\theta/2} |x|^{(\frac{2-a}{4})\theta}}, \quad \text{for all } |x| > R_0.$$

Set

$$L^* := \frac{C(4\pi)^{\theta/2} e^{\frac{(\alpha_0+1)4\pi}{\alpha_0}}}{\alpha_0^{\theta/2}}$$

Since $\theta \geq \frac{4a}{2-a}$, we get

$$\frac{f(u)}{u(x)^{\mu-1}} \leq \frac{L^*}{|x|^{(\frac{2-a}{4})\theta}} \leq \frac{L^*}{|x|^a}, \quad \text{for all } |x| > R_0.$$

Moreover, for $L_a \geq L^*$, we obtain

$$\frac{f(u)}{u(x)^{\mu-1}} \leq \frac{L_a}{|x|^a}, \quad \text{for all } |x| > R_0.$$

From (V_2) , we obtain

$$\frac{f(u)}{u(x)^{\mu-1}} \leq V(x), \quad \text{for all } |x| > R_0.$$

Thus,

$$\tilde{f}(x, u(x)) = \min\{f(u(x)), V(x)u(x)^{\mu-1}\} = f(u(x)), \quad \text{for all } |x| > R_0.$$

Hence, (6.1) follows.

Since u is a nontrivial critical point of \tilde{J} , we have

$$\int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u\phi) \, dx = \int_{\mathbf{R}^2} \tilde{f}(x, u)\phi \, dx, \quad \text{for all } \phi \in E.$$

Using (6.1), we obtain

$$\int_{\mathbf{R}^2} (\nabla u \nabla \phi + V(x)u\phi) \, dx = \int_{\mathbf{R}^2} f(u)\phi \, dx, \quad \text{for all } \phi \in E.$$

That is, equation (1.7) possesses a nontrivial weak solution.

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