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Total domination and vertex-edge domination in trees

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Abstract

A vertex v of a graph $G = (V, E)$ is said to ve-dominate every edge incident to v, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a vertex-edge dominating set if every edge of E is ve-dominated by at least one vertex of S. The minimum cardinality of a vertex-edge dominating set of G is the vertex-edge domination number $\gamma_{ve}(G)$. In this paper we prove $(\gamma_t(T)-\ell+1)/2 \leq \gamma_{\mathrm{ve}}(\mathrm{T}) \leq (\gamma_{\mathrm{t}}(\mathrm{T})+\ell-1)/2$ and characterize trees attaining each of these bounds.

Keywords: *Vertex-edge dominating set, total dominating set, trees.*

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1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n.

A subset $D \subseteq V(G)$ is a dominating set, abbreviated DS, of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A subset $D \subseteq V(G)$ is a total dominating set, abbreviated TDS, of G if every vertex of $V(G)$ has a neighbor in D. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of G of minimum cardinality is called a $\gamma_t(G)$-set. For more details on total domination, see [2].

An edge $e \in E(G)$ is vertex-edge dominated (ve-dominated) by a vertex $v \in V(G)$ if e is incident to v, or e is adjacent to an edge incident to v. A subset $D \subseteq V(G)$ is a vertex-edge dominating set, abbreviated VEDS, of G if every edge of G is vertex-edge dominated by a vertex of D. The vertex-edge domination number of G, denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex-edge dominating set of G. A vertex-edge dominating set of G of minimum cardinality is called a $\gamma_{ve}(G)$-set. Vertex-edge domination in graphs was introduced in [5], and further studied in [1, 3, 4].

In [4], trees with equal domination number and vertex-edge domination number are characterized. Here, we prove $(\gamma_t(T) - \ell + 1)/2 \leq \gamma_{\mathrm{ve}}(\mathrm{T}) \leq (\gamma_{\mathrm{t}}(\mathrm{T}) + \ell - 1)/2$ and characterize trees attaining each of these bounds.

2. Main Results

The one vertex graph does not have total dominating set and vertex-edge dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following observations:

Observation 1. *Every support vertex of a graph G is in every TDS of graph G.*

Observation 2. *Let G be a graph and $u \in V(G)$. Let the vertex u be adjacent to two paths vw and xy. Let v and x be adjacent to u. Let $H = G - \{x, y\}$. Then $\gamma_t(G) = \gamma_t(H) + 1$ and $\gamma_{ve}(G) = \gamma_{ve}(H)$.*

Proof. Let D be a $\gamma_t(G)$-set. By observation 1, the vertices $v, x \in D$. To dominate the vertices v and x, the vertex $u \in D$. It is easy to see that $D \setminus \{x\}$ is a TDS of the graph H. Thus $\gamma_t(H) \leq \gamma_t(G) - 1$. Let D' be a $\gamma_t(H)$-set. By observation 1, the vertex $v \in D'$. To dominate v, the vertex $u \in D'$. It is clear that $D' \cup \{x\}$ is a TDS of the graph G. Thus $\gamma_t(G) \leq \gamma_t(H) + 1$. We get $\gamma_t(G) = \gamma_t(H) + 1$.

Let D be a $\gamma_{ve}(T)$-set. To dominate the edges xy and vw, the vertex $u \in D$. Obviously D is a VEDS of the graph H. Thus $\gamma_{ve}(H) \leq \gamma_{ve}(G)$. Let D' be a $\gamma_{ve}(H)$-set. To dominate the edge vw, the vertex $u \in D'$. Clearly the vertex u dominates the edges ux and xy in the graph G. The set D' is a VEDS of the graph G. Thus $\gamma_{ve}(G) \leq \gamma_{ve}(H)$. We get $\gamma_{ve}(G) = \gamma_{ve}(H)$. □

Observation 3. *Let H be a graph with a leaf u adjacent to a weak support vertex v. Let G be a graph obtained from H by joining a path $P_4 : xyzw$ to the leaf u. Let u be adjacent to x. Then $\gamma_t(G) = \gamma_t(H) + 2$ and $\gamma_{ve}(G) = \gamma_{ve}(H) + 1$.*

Proof. Let D' be a $\gamma_t(H)$-set. It is obvious that $D' \cup \{z, y\}$ is a TDS of the graph G. Thus $\gamma_t(G) \leq \gamma_t(H) + 2$. Let D be a $\gamma_t(G)$-set. By observation 1, $z \in D$. To dominate z, the vertex $y \in D$. It is clear that $D \setminus \{y, z\}$ is a TDS of the graph H. Thus $\gamma_t(H) \leq \gamma_t(G) - 2$. We get $\gamma_t(G) = \gamma_t(H) + 2$.

Let D' be a $\gamma_{ve}(H)$-set. It is clear that $D' \cup \{y\}$ is a VEDS of the graph G. Thus $\gamma_{ve}(G) \leq \gamma_{ve}(H) + 1$. Let D be a $\gamma_{ve}(G)$-set. To dominate the edge zw, the vertex $y \in D$. It is obvious that $D \setminus \{y\}$ is a VEDS of the graph H. Thus $\gamma_{ve}(H) \leq \gamma_{ve}(G) - 1$. We get $\gamma_{ve}(H) = \gamma_{ve}(G) - 1$. □

First we show that if T is a nontrivial tree of order n with ℓ leaves, then $\gamma_{ve}(T)$ is bounded below by $(\gamma_t(T) - \ell + 1)/2$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 = P_5$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a path P_2 by joining one of its vertices to a vertex of T_k adjacent to a path P_2.

- Operation $\mathcal{O}_2$: Attach a path P_4 by joining one of its leaves to a leaf of T_k whose support vertex is weak.

Lemma 4. *If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (\gamma_t(T) - \ell + 1)/2$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T_1 = P_5$, then $\gamma_{ve}(T_1) = 1$ and $\gamma_t(T_1) = 3$. It can be verified that $\gamma_{ve}(T_1) = (\gamma_t(T_1) - \ell + 1)/2$ is satisfied. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let ℓ' be the number of leaves of the tree T'. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let x be a vertex to which a path $P_2 = yz$ is attached. Let x be adjacent to y. Let uv be a path different from yz attached at x. Let u be adjacent to x. By observation 2, $\gamma_t(T) = \gamma_t(T') + 1$ and $\gamma_{ve}(T) = \gamma_{ve}(T')$. It is easy to see $\ell' = \ell - 1$. We get $(\gamma_t(T) - \ell + 1)/2 = (\gamma_{\mathrm{t}}(\mathrm{T}') + 1 - (\ell' + 1) + 1)/2 = \gamma_{\mathrm{ve}}(\mathrm{T}') = \gamma_{\mathrm{ve}}(\mathrm{T})$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. Let x be a leaf to which a path $P_4 : uvwz$ is attached. Let x be adjacent to u. By observation 3, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{ve}(T') = \gamma_{ve}(T) - 1$. It is easy to see $\ell' = \ell$. We get $(\gamma_t(T) - \ell + 1)/2 = (\gamma_{\mathrm{t}}(\mathrm{T}') + 2 - \ell' + 1)/2 = (\gamma_{\mathrm{t}}(\mathrm{T}') - \ell' + 1)/2 + 1 = \gamma_{\mathrm{ve}}(\mathrm{T}') + 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$. $\square$

Theorem 5. *If T is a tree with ℓ leaves then $\gamma_{ve}(T) \geq (\gamma_t(T) - \ell + 1)/2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $diam(T) = 1$, then $T = P_2$. We have $(\gamma_t(T) - \ell + 1)/2 = (2 - 2 + 1)/2 < 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$. If $diam(T) = 2$, then T is a star $K_{1,n-1}$. We have $(\gamma_t(T) - \ell + 1)/2 = (2 - (\mathrm{n} - 1) + 1)/2 = (4 - \mathrm{n})/2 < 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$ as $n \geq 3$. Now assume that $diam(T) \geq 3$. Thus the order n of the tree is at least four. We prove the result by induction on n. Assume that the theorem is true for every tree T' of order $n' < n$ with ℓ' leaves.

First assume that some support vertex of T, say x, is strong. Let y and z be two leaves adjacent to x. Let $T' = T - y$. Let D' be a $\gamma_t(T')$-set. By observation 1, $x \in D'$. To dominate x, a vertex in $N_G(x)$ is in D'. Clearly the vertex x dominates y in the tree T. The set D' is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T')$. It is clear that $\ell' = \ell - 1$. Let D be a $\gamma_{ve}(T)$-set. It is obvious that D is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. We now get $(\gamma_t(T) - \ell + 1)/2 \le (\gamma_{\rm t}({\rm T}') - \ell' - 1 + 1)/2 = \gamma_{\rm ve}({\rm T}') - 1/2 < \gamma_{\rm ve}({\rm T})$. Henceforth, every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and let u be the parent of v in the rooted tree. If $diam(T) \ge 4$, then let w be the parent of u. If $diam(T) \ge 5$, then let d be the parent of w. If $diam(T) \ge 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree.

From the last but one paragraph, we get $d_T(v) = 2$. Assume that among the children of u there is a support vertex, say x, other than v. Let y be the leaf adjacent to x. Let $T' = T - T_x$. We have $\ell' = \ell - 1$. Let D' be a $\gamma_t(T')$-set. By observation 1, $v \in D'$. To dominate v, the vertex $u \in D'$. Clearly $D' \cup \{x\}$ is a TDS of the tree T. Thus $\gamma_t(T) \le \gamma_t(T') + 1$. Let D be a $\gamma_{ve}(T)$-set. To dominate the edge vt and xy, the vertex $u \in D$. Clearly D is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \le \gamma_{ve}(T)$. We now get $(\gamma_t(T) - \ell + 1)/2 \le (\gamma_{\rm t}({\rm T}') + 1 - \ell' - 1 + 1)/2 = \gamma_{\rm ve}({\rm T}') \le \gamma_{\rm ve}({\rm T})$. This implies that $(\gamma_t(T') - \ell' + 1)/2 = \gamma_{\rm ve}({\rm T}')$. By the induction hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Assume that among the children of u, other than v, there is a leaf x. Let $T' = T - x$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$-set. By observation 1, $v \in D'$. To dominate v, the vertex $u \in D'$. Clearly D' is a TDS of the tree T. Thus $\gamma_t(T) \le \gamma_t(T')$. Let D be a $\gamma_{ve}(T)$-set. To dominate the edge vt, the vertex $u \in D$. It is easy to see that D is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \le \gamma_{ve}(T)$. We now get $(\gamma_t(T) - \ell + 1)/2 \le (\gamma_{\rm t}({\rm T}') - \ell' - 1 + 1)/2 = \gamma_{\rm ve}({\rm T}') - 1/2 < \gamma_{\rm ve}({\rm T})$.

We assume that $d_T(u) = 2$. Now assume that among the children of w, other than u, there is a vertex x such that the distance of w to the most distant vertex of T_x is three. It suffices to consider that w is adjacent to a path $P_3 : xyz$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \le \gamma_t(T') + 2$. Let D be a $\gamma_{ve}(T)$-set. To dominate the edges vt and yz, the vertices $u, x \in D$. It is easy to observe that $D \setminus \{u\}$ is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \le \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \le (\gamma_{\rm t}({\rm T}') + 2 - \ell' - 1 + 1)/2 = \gamma_{\rm ve}({\rm T}') + 1/2 < \gamma_{\rm ve}({\rm T})$.

Assume that among the children of w, other than u, there is a vertex x such that the distance of w to the most distant vertex of T_x is two. It suffices to consider that w is adjacent to a path $P_2 : xy$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$-set. It is easy to observe that $D' \cup \{u, v\}$

is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Let D be a $\gamma_{ve}(T)$-set. To dominate the edges vt and yx, the vertices $u, w \in D$. It is clear that $D \setminus \{u\}$ is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_{\rm t}({\rm T}') + 2 - \ell' - 1 + 1)/2 = \gamma_{\rm ve}({\rm T}') + 1/2 < \gamma_{\rm ve}({\rm T})$.

Assume that among the children of w, other than u, there is a leaf x. Let $T' = T - T_w$. We have $\ell = \ell' + 2$. Let D' be a $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v, w\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 3$. Let D be a $\gamma_{ve}(T)$-set. To dominate the edges vt, uv, uw, wx and wd, the vertex $u \in D$. It is clear that $D \setminus \{u\}$ is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_{\rm t}({\rm T}') + 3 - \ell' - 2 + 1)/2 = \gamma_{\rm ve}({\rm T}') + 1/2 < \gamma_{\rm ve}({\rm T})$.

Assume $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$-set. It is clear that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Let D be a $\gamma_{ve}(T)$-set. To dominate the edges vt, uv, uw and wd, the vertex $u \in D$. It is clear that $D \setminus \{u\}$ is a VEDS of the tree T'. Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_{\rm t}({\rm T}') + 2 - \ell' - 1 + 1)/2 = \gamma_{\rm ve}({\rm T}') + 1/2 < \gamma_{\rm ve}({\rm T})$.

Suppose $d = r$, we have $T = P_5 = rwuvt$. Then $(\gamma_t(T) - \ell + 1)/2 = (3 - 2 + 1)/2 = 1 = \gamma_{\rm ve}({\rm T})$. Thus $T \in \mathcal{T}$. Now assume that $d_T(d) = 2$. Let $T' = T - T_w$. We have $\ell = \ell'$. Placing the arguments as in the previous case, we get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_{\rm t}({\rm T}') + 2 - \ell' + 1)/2 \leq \gamma_{\rm ve}({\rm T}') + 1 \leq \gamma_{\rm ve}({\rm T})$. This implies that $(\gamma_t(T') - \ell' + 1)/2 = \gamma_{\rm ve}({\rm T}')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. $\square$

We now show that if T is a nontrivial tree of order n with ℓ leaves, then $\gamma_{ve}(T)$ is bounded above by $(\gamma_t(T) + \ell - 2)/2$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3, P_4\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by operation $\mathcal{O}_2$.

It is easy to see that $\mathcal{F}$ consists of paths P_n where $n \neq 4k+1$ for positive integer k.

Lemma 6. *If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (\gamma_t(T) + \ell - 2)/2$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T_1 \in \{P_2, P_3, P_4\}$, then $\gamma_{ve}(T_1) = 1$ and $\gamma_t(T_1) = 2$. It can be verified that $\gamma_{ve}(T_1) = (\gamma_t(T_1) + \ell - 2)/2$ is satisfied. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let ℓ' be the number of leaves of the tree T'. Let $T = T_{k+1}$ be a tree of the family $\mathcal{F}$ constructed by k operations.

Assume that T is obtained from T' by operation $\mathcal{O}_2$. Let x be a leaf to which a path $P_4 : uvwz$ is attached. Let x be adjacent to u. By observation 3, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{ve}(T') = \gamma_{ve}(T) - 1$. It is easy to see $\ell' = \ell$. We get $(\gamma_t(T) + \ell - 2)/2 = (\gamma_t(\mathrm{T}') + 2 + \ell' - 2)/2 = (\gamma_t(\mathrm{T}') + \ell' - 2)/2 + 1 = \gamma_{\mathrm{ve}}(\mathrm{T}') + 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$. $\square$

Theorem 7. *If T is a tree with ℓ leaves then $\gamma_{ve}(T) \leq (\gamma_t(T) + \ell - 2)/2$ with equality if and only if $T \in \mathcal{F}$.*

Proof. If $diam(T) = 1$, then $T = P_2$. We have $(\gamma_t(T) + \ell - 2)/2 = (2 + 2 - 2)/2 = 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$. Thus $T \in \mathcal{F}$. If $diam(T) = 2$. If T is a path P_3, we have $(\gamma_t(T) + \ell - 2)/2 = (2 + 2 - 2)/2 = 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$. Thus $T \in \mathcal{F}$. T is a star $K_{1,n-1}$ other than P_3. We have $(\gamma_t(T) + \ell - 2)/2 = (2 + (\mathrm{n} - 1) - 2)/2 = (\mathrm{n} - 1)/2 > 1 = \gamma_{\mathrm{ve}}(\mathrm{T})$ as $n > 3$. Now assume that $diam(T) \geq 3$. Thus the order n of the tree is at least four. We prove the result by induction on n. Assume that the theorem is true for every tree T' of order $n' < n$ with ℓ' leaves.

First assume that some support vertex of T, say x, is strong. Let y and z be two leaves adjacent to x. Let $T' = T - y$. We have $\ell' = \ell - 1$. Let D' be a $\gamma_{ve}(T')$-set. The vertex which dominates the edge xz in the tree T' dominates the edge xy in the tree T. It is obvious that D' is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let D be a $\gamma_t(T)$-set. By observation 1, $x \in D$. To dominate x, a vertex in $N_G(x)$ is in D. If $y \in D$ then $(D \setminus \{y\}) \cup \{z\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T)$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq (\gamma_t(T') + \ell' - 2)/2 \leq (\gamma_{\mathrm{t}}(\mathrm{T}) + \ell - 1 - 2)/2 < (\gamma_{\mathrm{t}}(\mathrm{T}) + \ell - 2)/2$. Henceforth, every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and let u be the parent of v in the rooted tree. If $diam(T) \geq 4$, then let w be the parent of u. If $diam(T) \geq 5$, then let d be the parent of w. If $diam(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree.

Assume that among the children of u, other than v, there is a leaf x. Let $T' = T - x$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$-set. To dominate the edge vt, the vertex $u \in D'$. Clearly D' is a VEDS of the tree T as u dominates the edge ux. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let D be a $\gamma_t(T)$-set. By observation 1, the vertices $u, v \in D$. It is obvious that D is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T)$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq (\gamma_t(T') + \ell' - 2)/2 \leq (\gamma_t(\mathrm{T}) + \ell - 1 - 2)/2 < (\gamma_t(\mathrm{T}) + \ell - 2)/2$.

Assume that among the children of u there is a support vertex, say x, other than v. Let y be the leaf adjacent to x. Let $T' = T - T_x$. We have $\ell' = \ell - 1$. Let D' be a $\gamma_{ve}(T')$-set. To dominate the edge vt, the vertex $u \in D'$. The vertex u dominates the edges ux and xy in the tree T. Clearly D' is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let D be a $\gamma_t(T)$-set. By observation 1, the vertices $v, x \in D$. To dominate v and x, the vertex $u \in D$. It is obvious that $D \setminus \{x\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T) - 1$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq (\gamma_t(T') + \ell' - 2)/2 \leq (\gamma_t(\mathrm{T}) - 1 + \ell - 1 - 2)/2 < (\gamma_t(\mathrm{T}) + \ell - 2)/2$.

We assume that $d_T(u) = 2$. Now assume that among the children of w, other than u, there is a vertex x such that the distance of w to the most distant vertex of T_x is three. It suffices to consider that w is adjacent to a path $P_3 : xyz$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$-set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$-set. By observation 1, the vertices $y, v \in D$. To dominate the two vertices y and v the vertices $x, u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(\mathrm{T}) - 2 + \ell - 1 - 2)/2 + 1 < (\gamma_t(\mathrm{T}) + \ell - 2)/2$.

Assume that among the children of w, other than u, there is a vertex x such that the distance of w to the most distant vertex of T_x is two. It suffices to consider that w is adjacent to a path $P_2 : xy$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$-set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$-set. By observation 1, the vertices $x, v \in D$. To dominate x, v the vertices $w, u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(\mathrm{T}) - 2 + \ell - 1 - 2)/2 + 1 < (\gamma_t(\mathrm{T}) + \ell - 2)/2$.

Assume that among the children of w, other than u, there is a leaf x. Let $T' = T - T_w$. We have $\ell = \ell' + 2$. Let D' be a $\gamma_{ve}(T')$-set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$-set. By observation 1, the vertices $v, w \in D$. To dominate v and w, the vertex $u \in D$. It is obvious that $(D \setminus \{w, u, v\}) \cup \{a\}$ where a is a vertex in $N_G(d)$ other than w is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(\mathrm{T}) - 2 + \ell - 2 - 2)/2 + 1 < (\gamma_t(\mathrm{T}) + \ell - 2)/2$.

Assume $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$-set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$-set. By observation 1, $v \in D$. To dominate v, the vertex $u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(\mathrm{T}) - 2 + \ell - 1 - 2)/2 + 1 < (\gamma_t(\mathrm{T}) + \ell - 2)/2$.

Now assume $d_T(d) = 2$. Let $T' = T - T_w$. We have $\ell = \ell'$. Let D' be a $\gamma_{ve}(T')$-set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T. Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$-set. By observation 1, $v \in D$. To dominate v, the vertex $u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(\mathrm{T}) - 2 + \ell - 2)/2 + 1 \leq (\gamma_t(\mathrm{T}) + \ell - 2)/2$. This implies that $\gamma_{ve}(T') = (\gamma_t(T') + \ell' - 2)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. $\square$

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