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Further inequalities for log-convex functions related to Hermite-Hadamard result

S. S. Dragomir

Victoria University, Australia

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Abstract

Some unweighted and weighted inequalities of Hermite-Hadamard type for log-convex functions defined on real intervals are given.

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1. Introduction

A function $f : I \rightarrow (0, \infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$(1.1) \quad f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let us recall the *Hermite-Hadamard inequality*

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2},$$

where $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is a convex function on the interval I , $a, b \in I$ and $a < b$.

For related results, see [1]-[22], [25]-[29], [30]-[40] and [41]-[53].

Note that if we apply the above inequality for the log-convex functions $f : I \rightarrow (0, \infty)$, we have that

$$(1.3) \quad \ln \left[f\left(\frac{a+b}{2}\right) \right] \leq \frac{1}{b-a} \int_a^b \ln f(x) dx \leq \frac{\ln f(a) + \ln f(b)}{2},$$

from which we get

$$(1.4) \quad f\left(\frac{a+b}{2}\right) \leq \exp \left[\frac{1}{b-a} \int_a^b \ln f(x) dx \right] \leq \sqrt{f(a)f(b)}$$

that is an inequality of Hermite-Hadamard's type for log-convex functions.

By using simple properties of log-convex functions Dragomir and Mond proved in 1998 the following result [32].

Theorem 1. *Let $f : I \rightarrow (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$(1.5) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \sqrt{f(x)f(a+b-x)} dx \leq \sqrt{f(a)f(b)}.$$

The inequality between the first and second term in (1.5) may be improved as follows [32]. A different upper bound for the middle term in (1.5) can be also provided.

Theorem 2. Let $f : I \rightarrow (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:

$$(1.6) \quad \begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp \left[\frac{1}{b-a} \int_a^b \ln f(x) dx \right] \\ &\leq \frac{1}{b-a} \int_a^b \sqrt{f(x)f(a+b-x)} dx \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)), \end{aligned}$$

where $L(p, q)$ is the logarithmic mean of the strictly positive real numbers p, q , i.e.,

$$L(p, q) := \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p, p) := p.$$

The last inequality in (1.6) was obtained in a different context in [43]. As shown in [59], the following result also holds:

Theorem 3. Let $f : I \rightarrow (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:

$$(1.7) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{b-a} \int_a^b \sqrt{f(x)} dx \right)^2 \leq \frac{1}{b-a} \int_a^b f(x) dx.$$

We define the p -logarithmic mean as

$$L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, & \text{with } a \neq b \\ a, & \text{if } a = b \end{cases}$$

for $p \neq 0, -1$ and $a, b > 0$.

In the recent work [28] we generalized the inequality (1.6) as follows:

Theorem 4. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function on $[a, b]$. Then for any $p > 0$ we have the inequality

$$\begin{aligned}
 (1.8) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln f(x) dx\right] \\
 &\leq \left(\frac{1}{b-a} \int_a^b f^p(x) f^p(a+b-x) dx\right)^{\frac{1}{2p}} \\
 &\leq \left(\frac{1}{b-a} \int_a^b f^{2p}(x) dx\right)^{\frac{1}{2p}} \\
 &\leq \begin{cases} [L_{2p-1}(f(a), f(b))]^{1-\frac{1}{2p}} [L(f(a), f(b))]^{\frac{1}{2p}}, & p \neq \frac{1}{2}; \\ L(f(a), f(b)), & p = \frac{1}{2}. \end{cases}
 \end{aligned}$$

If $p \in (0, \frac{1}{2})$, then we have

$$\begin{aligned}
 (1.9) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln f(x) dx\right] \\
 &\leq \left(\frac{1}{b-a} \int_a^b f^p(x) f^p(a+b-x) dx\right)^{\frac{1}{2p}} \\
 &\leq \left(\frac{1}{b-a} \int_a^b f^{2p}(x) dx\right)^{\frac{1}{2p}} \leq \frac{1}{b-a} \int_a^b f(x) dx.
 \end{aligned}$$

Remark 1. If we take in (1.8) $p = 1$, then we get

$$\begin{aligned}
 (1.10) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln f(x) dx\right] \\
 &\leq \left(\frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx\right)^{\frac{1}{2}} \\
 &\leq \left(\frac{1}{b-a} \int_a^b f^2(x) dx\right)^{\frac{1}{2}} \\
 &\leq [A(f(a), f(b))]^{\frac{1}{2}} [L(f(a), f(b))]^{\frac{1}{2}}.
 \end{aligned}$$

If we take $p = \frac{1}{4}$ in (1.9), then we get

$$\begin{aligned}
 (1.11) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln f(x) dx\right] \\
 &\leq \left(\frac{1}{b-a} \int_a^b \sqrt[4]{f(x)f(a+b-x)} dx\right)^2 \\
 &\leq \left(\frac{1}{b-a} \int_a^b \sqrt{f(x)} dx\right)^2 \leq \frac{1}{b-a} \int_a^b f(x) dx.
 \end{aligned}$$

This improves the inequality (1.7).

Motivated by the above results, we establish in this paper some new un-weighted and weighted inequalities for log-convex functions, some of them improving earlier results. Applications for special means are also provided.

2. New Results

The following result holds.

Theorem 1. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. Then for every $t \in [0, 1]$ we have

$$\begin{aligned}
 (2.1) \quad \int_a^b f(x) dx &\geq \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t dx \\
 &\geq \begin{cases} \frac{1}{1-2t} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) du, & \text{if } t \neq \frac{1}{2} \\ (b-a) f\left(\frac{a+b}{2}\right), & \text{if } t = \frac{1}{2}. \end{cases}
 \end{aligned}$$

Proof. The cases $t = 0, \frac{1}{2}, 1$ are obvious.

Assume that $t \in (0, 1) \setminus \left\{\frac{1}{2}\right\}$. By the log-convexity of f we have

$$\begin{aligned}
 (2.2) \quad [f(x)]^{1-t} [f(a+b-x)]^t &\geq f((1-t)x + t(a+b-x)) \\
 &= f((1-2t)x + t(a+b))
 \end{aligned}$$

for any $x \in [a, b]$.

Integrating the inequality (2.2) over x on $[a, b]$ we have

$$\int_a^b [f(x)]^{1-t} [f(a+b-x)]^t dx \geq \int_a^b f((1-2t)x + t(a+b)) dx.$$

Since $t \neq \frac{1}{2}$, then $u := (1-2t)x + t(a+b)$ is a change of variable with $du = (1-2t)dx$.

For $x = a$ we get $u = (1-t)a + tb$ and for $x = b$ we get $u = ta + (1-t)b$. Therefore

$$\int_a^b f((1-2t)x + t(a+b)) dx = \frac{1}{1-2t} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) du$$

and the second inequality in (2.1) is proved.

By the Hölder integral inequality for $p = \frac{1}{1-t}$, $q = \frac{1}{t}$ we have

$$\begin{aligned} & \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t dx \\ & \leq \left(\int_a^b ([f(x)]^{1-t})^{\frac{1}{1-t}} dx \right)^{1-t} \left(\int_a^b ([f(a+b-x)]^t)^{\frac{1}{t}} dx \right)^t \\ & = \left(\int_a^b f(x) dx \right)^{1-t} \left(\int_a^b f(a+b-x) dx \right)^t \\ & = \left(\int_a^b f(x) dx \right)^{1-t} \left(\int_a^b f(x) dx \right)^t = \int_a^b f(x) dx, \end{aligned}$$

that proves the first inequality in (2.1). $\square$

Corollary 1. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. Then for every $t \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}$ we have for $p > 0$ that

$$(2.3) \quad \left(\frac{1}{b-a} \int_a^b f^{2p}(x) dx \right)^{\frac{1}{2p}}$$

$$\begin{aligned}
&\geq \left(\frac{1}{b-a} \int_a^b [f^{2p}(x)]^{1-t} [f^{2p}(a+b-x)]^t dx \right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u) du \right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^p(u) f^p(a+b-u) dx \right)^{\frac{1}{2p}} \\
&\geq \exp \left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u) du \right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Proof. Follows from Theorem 5 applied for the log-convex function f^{2p} with $p > 0$ and by Theorem 4 applied for the interval $[(1-t)a+tb, ta+(1-t)b]$ when $t \in (0, \frac{1}{2})$ or $[ta+(1-t)b, (1-t)a+tb]$ when $t \in (\frac{1}{2}, 1)$. $\square$

If we take $p = 1$ in (2.3), then we get

$$\left(\frac{1}{b-a} \int_a^b f^2(x) dx \right)^{\frac{1}{2}} \geq \left(\frac{1}{b-a} \int_a^b [f^2(x)]^{1-t} [f^2(a+b-x)]^t dx \right)^{\frac{1}{2}}$$

(2.4)

$$\begin{aligned}
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^2(u) du \right)^{\frac{1}{2}} \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) f(a+b-u) dx \right)^{\frac{1}{2}} \\
&\geq \exp \left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u) du \right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

If we take $p = \frac{1}{2}$ in (2.3), then we get

$$\frac{1}{b-a} \int_a^b f(x) dx$$

(2.5)

$$\begin{aligned}
&\geq \frac{1}{b-a} \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t dx \\
&\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) du \\
&\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u) f(a+b-u)} du \\
&\geq \exp \left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u) du \right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Corollary 2. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. Then for every $t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$ we have for $p \in \left(0, \frac{1}{2}\right)$ that

$$\begin{aligned}
(2.6) \quad &\frac{1}{b-a} \int_a^b f(x) dx \\
&\geq \frac{1}{b-a} \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t dx \\
&\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) du \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u) du \right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f^p(u) f^p(a+b-u) du \right)^{\frac{1}{2p}} \\
&\geq \exp \left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u) du \right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Follows by Theorem 5 and Theorem 4 for $p \in \left(0, \frac{1}{2}\right)$.

If we take $p = \frac{1}{4}$ in (2.6), then we get

$$(2.7) \quad \frac{1}{b-a} \int_a^b f(x) dx$$

$$\begin{aligned}
&\geq \frac{1}{b-a} \int_a^b [f(x)]^{1-t} [f(a+b-x)]^t dx \\
&\geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) du \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u)} du \right)^2 \\
&\geq \left(\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt[4]{f(u)f(a+b-u)} du \right)^2 \\
&\geq \exp \left[\frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u) du \right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

If we use the inequality

$$(2.8) \quad \frac{1}{b-a} \int_a^b f(x) dx \geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} f(u) du,$$

where $t \neq \frac{1}{2}$, for the log-convex function $f : [a, b] \rightarrow (0, \infty)$, $f(x) = \frac{1}{x}$, then we get

$$\frac{\ln b - \ln a}{b-a} \geq \frac{\ln(ta + (1-t)b) - \ln((1-t)a + tb)}{(ta + (1-t)b) - ((1-t)a + tb)}$$

which, in terms of logarithmic mean, is

$$(2.9) \quad L(ta + (1-t)b, (1-t)a + tb) \geq L(a, b)$$

for any $t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$.

When $t = \frac{1}{2}$ the inequality (2.9) becomes $A(a, b) \geq L(a, b)$ that is also true.

For $q \neq 0, -1$ define the L_q -logarithmic mean as

$$L_q(a, b) := \begin{cases} \left(\frac{b^{q+1} - a^{q+1}}{(q+1)(b-a)} \right)^{\frac{1}{q}} & \text{if } b \neq a, \\ a & \text{if } b = a. \end{cases}$$

If we use the inequality (2.8) for the log-convex function $f : [a, b] \rightarrow (0, \infty)$, $f(x) = \frac{1}{x^p}$, with $p > 0$, $p \neq 1$, then we have for any $t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$ that

$$\frac{1}{b-a} \int_a^b x^{-p} dx \geq \frac{1}{(1-2t)(b-a)} \int_{(1-t)a+tb}^{ta+(1-t)b} u^{-p} du,$$

i.e.

$$\frac{b^{-p+1} - a^{-p+1}}{(1-p)(b-a)} \geq \frac{(ta + (1-t)b)^{-p+1} - ((1-t)a + tb)^{-p+1}}{((1-p))[(ta + (1-t)b) - ((1-t)a + tb)]},$$

which can be written as

$$L_{-p}^{-p}(a, b) \geq L_{-p}^{-p}(ta + (1-t)b, (1-t)a + tb).$$

Therefore we have

$$(2.10) \quad L_{-p}(ta + (1-t)b, (1-t)a + tb) \geq L_{-p}(a, b)$$

for any $p > 0$, $p \neq 1$ and for any $t \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$. The case $t = \frac{1}{2}$ reduces to the known inequality $A(a, b) \geq L_{-p}(a, b)$ for any $p > 0$.

3. Weighted Inequalities

We have the following generalized weighted version of the inequality (1.5).

Theorem 1. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. If $w : [a, b] \rightarrow [0, \infty)$ is integrable and $\int_a^b w(x) dx > 0$, then

$$(3.1) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)}$$

for any $p > 0$.

In particular, we have

$$(3.2) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x) dx}{\int_a^b w(x) dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}.$$

Proof. We know that, see [32] or [35, p. 198], if g is log-convex, then

$$(3.3) \quad g\left(\frac{a+b}{2}\right) \leq \sqrt{g(x)g(a+b-x)} \leq \sqrt{g(a)g(b)}$$

for any $x \in [a, b]$.

For any $p > 0$ the function f^{2p} is log-convex and by (3.3) we have

$$(3.4) \quad f^{2p}\left(\frac{a+b}{2}\right) \leq f^p(x) f^p(a+b-x) \leq f^p(a) f^p(b)$$

for any $x \in [a, b]$.

If we multiply (3.4) by $w(x) \geq 0$ and integrate, then we get

$$\begin{aligned} f^{2p}\left(\frac{a+b}{2}\right) \int_a^b w(x) dx &\leq \int_a^b w(x) f^p(x) f^p(a+b-x) dx \\ &\leq f^p(a) f^p(b) \int_a^b w(x) dx \end{aligned}$$

namely

$$(3.5) \quad f^{2p}\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx} \leq f^p(a) f^p(b).$$

Taking the power $\frac{1}{2p}$ in (3.5) we get the desired result (3.1). $\square$

We also have the inequality

$$(3.6) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(x) \sqrt{f(x)f(a+b-x)} dx}{\int_a^b w(x) dx} \leq \sqrt{f(a)f(b)},$$

that is a weighted version of (1.5).

If we take $p = \frac{1}{4}$ in (3.1), then we get

$$(3.7) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) \sqrt[4]{f(x)f(a+b-x)} dx}{\int_a^b w(x) dx} \right)^2 \leq \sqrt{f(a)f(b)}.$$

Using Jensen's inequality for the power $p \geq 1$ ($p \in (0, 1)$), namely

$$\left(\frac{\int_a^b w(x) g(x) dx}{\int_a^b w(x) dx} \right)^p \leq (\geq) \frac{\int_a^b w(x) g^p(x) dx}{\int_a^b w(x) dx},$$

we can start the following more precise result:

Corollary 3. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function and $w : [a, b] \rightarrow [0, \infty)$ be integrable and $\int_a^b w(x) dx > 0$.

If $p \geq 1$, then

$$(3.8) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x) dx}{\int_a^b w(x) dx}\right)^{\frac{1}{2}} \\ \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)}.$$

If $p \in (0, 1)$, then

$$(3.9) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx}\right)^{\frac{1}{2p}} \\ \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x) dx}{\int_a^b w(x) dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}.$$

Remark 2. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. We observe that if we take in (3.1) $w(x) = f^{-p}(a+b-x)$, $p > 0$, then we get

$$(3.10) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f^p(x) dx}{\int_a^b f^{-p}(x) dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)}$$

for any $p > 0$.

In particular, we have the inequalities

$$(3.11) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f(x) dx}{\int_a^b \frac{1}{f(x)} dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)},$$

and

$$(3.12) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b \sqrt{f(x)} dx}{\int_a^b \frac{1}{\sqrt{f(x)}} dx} \leq \sqrt{f(a) f(b)}.$$

If we take in (3.10) $f(x) = x^{-1}$, $x \in [a, b] \subset (0, \infty)$, which is log-convex, then we have

$$(3.13) \quad \left(\frac{a+b}{2}\right)^{-1} \leq \left(\frac{\int_a^b x^{-p} dx}{\int_a^b x^p dx}\right)^{\frac{1}{2p}} \leq \left(\sqrt{f(a)f(b)}\right)^{-1}$$

Observe that

$$\begin{aligned} \frac{\int_a^b x^{-p} dx}{\int_a^b x^p dx} &= \frac{\frac{1}{b-a} \int_a^b x^{-p} dx}{\frac{1}{b-a} \int_a^b x^p dx} \\ &= \frac{[L_{-p}(a,b)]^{-p}}{[L_p(a,b)]^p} \end{aligned}$$

and by (3.13) we have

$$\left(\frac{a+b}{2}\right)^{-1} \leq \left(\frac{1}{[L_p(a,b)][L_{-p}(a,b)]}\right)^{\frac{1}{2}} \leq \left(\sqrt{f(a)f(b)}\right)^{-1}$$

that is equivalent to

$$G(a,b) \leq \sqrt{[L_p(a,b)][L_{-p}(a,b)]} \leq A(a,b).$$

This can be also written as

$$(3.14) \quad G(a,b) \leq G(L_p(a,b), L_{-p}(a,b)) \leq A(a,b).$$

If we take in the first inequality (3.14) $p = 1$, then we get

$$(3.15) \quad G^2(a,b) \leq A(a,b)L(a,b).$$

We have the following weighted version of (1.4).

Theorem 2. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. If $w : [a, b] \rightarrow [0, \infty)$ is integrable and $\int_a^b w(x) dx > 0$, then

$$\begin{aligned} (3.16) \quad & f\left(\frac{\int_a^b w(x) x dx}{\int_a^b w(x) dx}\right) \\ & \leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) \\ & \leq [f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx} - a\right)} [f(a)]^{\frac{1}{b-a}\left(b - \frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx}\right)} \\ & \leq \frac{1}{b-a} \left[\left(\frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx} - a\right) f(b) + \left(b - \frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx}\right) f(a) \right]. \end{aligned}$$

Proof. Since $\ln f$ is convex, then by Jensen's inequality we have

$$(3.17) \quad \ln f \left(\frac{\int_a^b w(x) x dx}{\int_a^b w(x) dx} \right) \leq \frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}.$$

Taking the exponential in (3.17) we get the first inequality in (3.16). Since $\ln f$ is convex, then

$$(3.18) \quad \ln f(x) = \ln f \left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a \right) \leq \frac{x-a}{b-a} \ln f(b) + \frac{b-x}{b-a} \ln f(a)$$

for any $x \in [a, b]$.

By taking the weighted integral mean in (3.18) we get

$$(3.19) \quad \frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx} \leq \frac{1}{b-a} \left[\left(\frac{\int_a^b xw(x) dx}{\int_a^b w(x) dx} - a \right) \ln f(b) + \left(b - \frac{\int_a^b xw(x) dx}{\int_a^b w(x) dx} \right) \ln f(a) \right] \\ = \ln \left([f(b)]^{\frac{1}{b-a} \left(\frac{\int_a^b xw(x) dx}{\int_a^b w(x) dx} - a \right)} [f(a)]^{\frac{1}{b-a} \left(b - \frac{\int_a^b xw(x) dx}{\int_a^b w(x) dx} \right)} \right).$$

By taking the exponential in (3.19), we get the second inequality in (3.16).

The last part of (3.16) follows by the weighted geometric mean-arithmetic mean inequality. $\square$

Remark 3. If we take $w(x) = 1$, $x \in [a, b]$ in the first two inequalities (3.16), we recapture (1.4).

If we take $w(x) = \frac{1}{x}$, $x \in [a, b] \subset (0, \infty)$ in (3.16), then we get

$$(3.20) \quad f(L(a, b)) \leq \exp \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(x)}{x} dx \right) \\ \leq [f(b)]^{\frac{L(a,b)-a}{b-a}} [f(a)]^{\frac{b-L(a,b)}{b-a}} \\ \leq \frac{(L(a,b)-a)f(b) + (b-L(a,b))f(a)}{b-a},$$

where $L(a, b)$ is the logarithmic mean, i.e.

$$L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ a & \text{if } a = b. \end{cases}$$

If we take $w(x) = \frac{1}{x^2}$, $x \in [a, b] \subset (0, \infty)$ in (3.16), then we get

$$\begin{aligned} (3.21) \quad f\left(\frac{G^2(a, b)}{L(a, b)}\right) &\leq \exp\left(\frac{ab}{b-a} \int_a^b \frac{\ln f(x)}{x^2} dx\right) \\ &\leq [f(b)]^{\frac{1}{b-a}\left(\frac{G^2(a, b)}{L(a, b)} - a\right)} [f(a)]^{\frac{1}{b-a}\left(b - \frac{G^2(a, b)}{L(a, b)}\right)} \\ &\leq \frac{1}{b-a} \left[\left(\frac{G^2(a, b)}{L(a, b)} - a\right) f(b) + \left(b - \frac{G^2(a, b)}{L(a, b)}\right) f(a) \right]. \end{aligned}$$

We also have the alternative result:

Theorem 3. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function. If $w : [a, b] \rightarrow [0, \infty)$ is integrable and $\int_a^b w(x) dx > 0$, then

$$\begin{aligned} (3.22) \quad f\left(\frac{\int_a^b w(x) x dx}{\int_a^b w(x) dx}\right) &\leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) \\ &\leq \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx} \leq \left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}} \frac{\int_a^b w(x) \left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}} dx}{\int_a^b w(x) dx} \\ &\leq \frac{1}{b-a} \left[\left(\frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx} - a\right) f(b) + \left(b - \frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx}\right) f(a) \right]. \end{aligned}$$

Proof. Using Jensen's inequality for the exponential function we have

$$\begin{aligned} \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) &\leq \frac{\int_a^b w(x) \exp(\ln f(x)) dx}{\int_a^b w(x) dx} \\ &= \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx} \end{aligned}$$

and the second inequality in (3.22) is proved.

From (3.18) and the arithmetic mean - geometric mean inequality we have

$$\begin{aligned}
 (3.23) \quad f(x) &\leq [f(b)]^{\frac{x-a}{b-a}} [f(a)]^{\frac{b-x}{b-a}} = \left(\frac{[f(a)]^b}{[f(b)]^a} \right)^{\frac{1}{b-a}} \left(\frac{f(b)}{f(a)} \right)^{\frac{x}{b-a}} \\
 &\leq \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a)
 \end{aligned}$$

for any $x \in [a, b]$.

By taking the weighted integral mean in (3.23) we get

$$\begin{aligned}
 &\frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx} \\
 &\leq \left(\frac{[f(a)]^b}{[f(b)]^a} \right)^{\frac{1}{b-a}} \frac{\int_a^b w(x) \left(\frac{f(b)}{f(a)} \right)^{\frac{x}{b-a}} dx}{\int_a^b w(x) dx} \\
 &\leq \frac{1}{b-a} \left[\left(\frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx} - a \right) f(b) + \left(b - \frac{\int_a^b x w(x) dx}{\int_a^b w(x) dx} \right) f(a) \right]
 \end{aligned}$$

and the last part of (3.22) is proved. $\square$

Remark 4. If we take $w(x) = 1$, $x \in [a, b]$ in (3.22), then we have

$$\begin{aligned}
 (3.24) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{1}{b-a} \int_a^b \ln f(x) dx\right) \\
 &\leq \frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)) \leq \frac{f(a)+f(b)}{2}.
 \end{aligned}$$

If we take $w(x) = \frac{1}{x}$, $x \in [a, b] \subset (0, \infty)$ in (3.22), then we get

$$\begin{aligned}
 (3.25) \quad f(L(a, b)) &\leq \exp\left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(x)}{x} dx\right) \\
 &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \\
 &\leq \left(\frac{[f(a)]^b}{[f(b)]^a} \right)^{\frac{1}{b-a}} \frac{\int_a^b \frac{1}{x} \left(\frac{f(b)}{f(a)} \right)^{\frac{x}{b-a}} dx}{\ln b - \ln a} \\
 &\leq \frac{(L(a, b) - a)f(b) + (b - L(a, b))f(a)}{b - a}.
 \end{aligned}$$

4. Inequalities for Symmetric Weights

We say that the weight $w : [a, b] \rightarrow [0, \infty)$ is *symmetric* on $[a, b]$ if

$$w(a + b - x) = w(x) \text{ for all } x \in [a, b].$$

It is well known that if $f : [a, b] \rightarrow \mathbf{R}$ is convex and $w : [a, b] \rightarrow [0, \infty)$ is integrable and symmetric on $[a, b]$, then the *Fejér inequality* holds

$$(4.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx} \leq \frac{f(a) + f(b)}{2}.$$

If $f : [a, b] \rightarrow (0, \infty)$ is a log-convex function on $[a, b]$ and $w : [a, b] \rightarrow [0, \infty)$ is integrable and symmetric on $[a, b]$, then by (4.1) we have

$$\ln f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx} \leq \frac{\ln f(a) + \ln f(b)}{2},$$

which is equivalent to

$$(4.2) \quad f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) \leq \sqrt{f(a) f(b)}.$$

Theorem 1. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function on $[a, b]$ and $w : [a, b] \rightarrow [0, \infty)$ be integrable and symmetric on $[a, b]$. Then

$$(4.3) \quad f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) \\ \leq \frac{\int_a^b w(x) \sqrt{f(x)f(a+b-x)} dx}{\int_a^b w(x) dx} \leq \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx}.$$

Proof. By Jensen's integral inequality for the exponential we have

$$(4.4) \quad \exp\left(\frac{\int_a^b w(x) \ln \sqrt{f(x)f(a+b-x)} dx}{\int_a^b w(x) dx}\right) \\ \leq \frac{\int_a^b w(x) \exp(\ln \sqrt{f(x)f(a+b-x)}) dx}{\int_a^b w(x) dx} \\ = \frac{\int_a^b w(x) \sqrt{f(x)f(a+b-x)} dx}{\int_a^b w(x) dx}.$$

Observe, by the symmetry of w , that

$$\begin{aligned}
& \int_a^b w(x) \ln \sqrt{f(x) f(a+b-x)} dx \\
&= \frac{1}{2} \left[\int_a^b w(x) \ln f(x) dx + \int_a^b w(x) \ln f(a+b-x) dx \right] \\
&= \frac{1}{2} \left[\int_a^b w(x) \ln f(x) dx + \int_a^b w(a+b-x) \ln f(a+b-x) dx \right] \\
&= \int_a^b w(x) \ln f(x) dx
\end{aligned}$$

since, obviously

$$\int_a^b w(a+b-x) \ln f(a+b-x) dx = \int_a^b w(x) \ln f(x) dx.$$

By (4.4) we then get the second inequality in (4.3).

By Cauchy-Buniakovski-Schwarz integral inequality we also have

$$\begin{aligned}
& \int_a^b w(x) \sqrt{f(x) f(a+b-x)} dx \\
&\leq \sqrt{\int_a^b w(x) f(x) dx} \sqrt{\int_a^b w(x) f(a+b-x) dx} \\
&= \sqrt{\int_a^b w(x) f(x) dx} \sqrt{\int_a^b w(x) f(x) dx} \\
&= \int_a^b w(x) f(x) dx,
\end{aligned}$$

which proves the third inequality in (4.3). $\square$

The above inequality (4.2) may be generalized as follows by replacing f with f^{2p} for $p > 0$.

Corollary 4. *Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function on $[a, b]$ and $w : [a, b] \rightarrow [0, \infty)$ be integrable and symmetric on $[a, b]$. Then for any $p > 0$ we have*

$$(4.5) \quad f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right)$$

$$\begin{aligned}
&\leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2p}} \\
&\leq \left(\frac{\int_a^b w(x) f^{2p}(x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2p}}.
\end{aligned}$$

Remark 5. We observe that for $p \geq 1$ we have

$$\begin{aligned}
(4.6) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) \\
&\leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2p}} \\
&\leq \left(\frac{\int_a^b w(x) f^{2p}(x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2p}}
\end{aligned}$$

and for $p \in (0, 1)$

$$\begin{aligned}
(4.7) \quad f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{\int_a^b w(x) \ln f(x) dx}{\int_a^b w(x) dx}\right) \\
&\leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2p}} \\
&\leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{\int_a^b w(x) f^2(x) dx}{\int_a^b w(x) dx} \right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, we have:

Theorem 2. Let $f : [a, b] \rightarrow (0, \infty)$ be a log-convex function on $[a, b]$ and $w : [a, b] \rightarrow [0, \infty)$ be integrable and symmetric on $[a, b]$. Then for any $p > 0$ we have

$$(4.8) \quad f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f^p(x) w(x) dx}{\int_a^b \frac{w(x) dx}{f^p(x)}}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)}.$$

Proof. From (3.4) we have

$$(4.9) \quad f^{2p}\left(\frac{a+b}{2}\right) \frac{1}{f^p(a+b-x)} \leq f^p(x) \leq f^p(a) f^p(b) \frac{1}{f^p(a+b-x)}$$

for any $x \in [a, b]$.

If we multiply by $w(x) \geq 0$ and integrate on $[a, b]$, then we get

$$\begin{aligned} & f^{2p}\left(\frac{a+b}{2}\right) \int_a^b \frac{w(x) dx}{f^p(a+b-x)} \\ & \leq \int_a^b f^p(x) w(x) dx \\ & \leq f^p(a) f^p(b) \int_a^b \frac{w(x) dx}{f^p(a+b-x)}. \end{aligned}$$

Since, by symmetry of w we have

$$\int_a^b \frac{w(x) dx}{f^p(a+b-x)} = \int_a^b \frac{w(a+b-x) dx}{f^p(a+b-x)} = \int_a^b \frac{w(x) dx}{f^p(x)},$$

which implies that

$$\begin{aligned} & f^{2p}\left(\frac{a+b}{2}\right) \int_a^b \frac{w(x) dx}{f^p(x)} \\ & \leq \int_a^b f^p(x) w(x) dx \\ & \leq f^p(a) f^p(b) \int_a^b \frac{w(x) dx}{f^p(x)}. \end{aligned}$$

and the inequality (4.8) is proved. $\square$

Remark 6. If we write the inequality (4.8) for the log-convex function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$, $f(x) = \frac{1}{x}$, then we have for $p > 0$

$$\left(\frac{a+b}{2}\right)^{-1} \leq \left(\frac{\int_a^b x^{-p} w(x) dx}{\int_a^b x^p w(x) dx}\right)^{\frac{1}{2p}} \leq (\sqrt{ab})^{-1},$$

that is equivalent to

$$(4.10) \quad G(a, b) \leq \left(\frac{\int_a^b x^p w(x) dx}{\int_a^b x^{-p} w(x) dx} \right)^{\frac{1}{2p}} \leq A(a, b),$$

for any symmetric integrable weight $w : [a, b] \rightarrow [0, \infty)$.

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S. S. Dragomir

Mathematics,
College of Engineering & Science
Victoria University,
P. O. Box 14428
Melbourne City, MC 8001,
Australia
e-mail : sever.dragomir@vu.edu.au