Proyecciones Journal of Mathematics
Vol. 38, N^o 2, pp. 191-202, June 2019.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172019000200191

Further results on 3-product cordial labeling

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Received : November 2015. Accepted : November 2018

Abstract

A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with i, $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i(mod\ 3)$. A graph with 3-product cordial labeing is called 3-product cordial graph. In this paper we establish that switching of an apex vertex in closed helm, double fan, book graph $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, I)$ are 3-product cordial graphs.

Key Words. *cordial labeling, product cordial labeling, 3-product cordial labeling, 3-product cordial graph.*

AMS Subject Classification (2010) *: 05C78*

1. Introduction

All graphs considered here are simple, finite, connected and undirected. For basic notations and terminology, we follow [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a complete survey of graph labeling is available in [2]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in [1]. Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)-f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Sundaram et al. introduced the concept of product cordial labeling in [10]. Let f be a function from $V(G)$ to $\{0,1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i=0,1)$. The same authors have introduced the concept of EP-cordial labeling in [11]. A vertex labeling $f: V(G) \to \{-1,0,1\}$ is said to be an EP-cordial labeling if it induces the edge labeling f^* defined by $f^*(uv) = f(u)f(v)$ for each $uv \in E(G)$ and if $|v_f(i)-v_f(j)| \leq 1$ and $|e_f(i)-e_f(j)| \leq 1$ for any $i \neq j$ $i,j \in \{-1,0,1\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and edges of G having the label $x \in \{-1,0,1\}$. In [11] it is remarked that any EP-cordial labeling is a 3-product cordial labeling. A mapping $f: V(G) \to \{0,1,2\}$ is called 3-product cordial labeling if $|v_f(i)-v_f(j)| \leq 1$ and $|e_f(i)-e_f(j)| \leq 1$ for any $i,j \in \{0,1,2\}$, where $v_f(i)$ denotes the number of edges xy with $f(x)f(y) = i(mod\ 3)$. A graph with 3-product cordial labeling is called a 3-product cordial graph. Jeyanthi and Maheswari [4]-[8] proved that the graphs $\langle B_{n,n} : w \rangle, C_n \cup P_n, C_m \circ \overline{K_n}$ if $m \geq 3$ and $n \geq 1$, $P_m \circ \overline{K_n}$ if $m,n \geq 1$, duplicating arbitrary vertex in cycle C_n, duplicating arbitrary edge in cycle C_n, duplicating arbitrary vertex in wheel W_n, middle graph of P_n, the splitting graph of P_n, the total graph of $P_n, P_n[P_2], p_n^2, K_{2,n}$, vertex switching of C_n, ladder L_n, triangular ladder TL_n, the graph $\left\langle w_n^{(1)} : w_n^{(2)} : \ldots : w_n^{(k)} \right\rangle$, the splitting graphs $S'(K_{1,n}), S'(B_{n,n})$, the shadow graph $D_2(B_{n,n})$, the square graph $B_{n,n}^2$, triangular snake, double alternate triangular snake and alternate triangular snake graphs are 3-product cordial graphs. Also they proved that a complete graph K_n is a 3-product cordial graph if and only if $n \leq 2$.

In addition, they proved that if $G(p,q)$ is a 3-product cordial graph (i)

$p \equiv 1(mod\ 3)$ then $q \leq \dfrac{p^2 - 2p + 7}{3}$. (ii) $p \equiv 2(mod\ 3)$ then $q \leq \dfrac{p^2 - p + 4}{3}$ (iii) $p \equiv 0(mod\ 3)$ then $q \leq \dfrac{p^2 - 3p + 6}{3}$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graph. In this paper we establish that switching of an apex vertex in closed helm, double fan, $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, I)$ are 3-product cordial graphs. We use the following definitions in the subsequent section.

Definition 1.1. *The vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, by removing all the edges incident with v and joining the vertex v to every vertex which is not adjacent to v in G.*

Definition 1.2. *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.3. *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to from a cycle.*

Definition 1.4. *The graph $P_n + 2K_1$ is called a double fan DF_n.*

Definition 1.5. *For any permutation f on $1, 2, 3, \ldots, n$, the f-permutation graph on a graph G, $P(G, f)$ consists of two disjoint copies of G, say G_1 and G_2, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with n edges obtained by joining each v_i in G_1 to $v_{f(i)}$ in G_2. We denote the identity permutation by I.*

For any real number n, $\lceil n \rceil$ denotes the smallest integer $\geq n$ and $\lfloor n \rfloor$ denotes the greatest integer $\leq n$.

2. Main Results

Theorem 2.1. *The graph obtained by switching of an apex vertex in closed helm CH_n admits 3-product cordial labeling if and only if $n \equiv 2(mod\ 3)$.*

Proof. Let v be the apex vertex $v_1, v_2, v_3, \ldots, v_n$ be the vertices of inner cycle and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of outer cycle CH_n. Let G_v denotes graph obtained by switching of an apex vertex v of $G = CH_n$. Then $|V(G_v)| = 2n + 1$ and $|E(G_v)| = 4n$. We define $f : V(G_v) \to \{0, 1, 2\}$ as follows:

$f(v)=2$, For $1 \le i \le \left\lfloor \dfrac{2n+1}{3} \right\rfloor$, $f(v_i)=0$.
For $n \equiv 0,2,3(mod\ 4), 1 \le i \le \left\lceil \dfrac{n}{3} \right\rceil$,

$$f\left(v_{\lfloor \frac{2n+1}{3} \rfloor + i}\right) = \begin{cases} 1 & \text{for } i \equiv 1,2(mod\ 4) \\ 2 & \text{for } i \equiv 0,3(mod\ 4) \end{cases}$$

For $n \equiv 1(mod\ 4), n > 5, 1 \le i \le \left\lceil \dfrac{n}{3} \right\rceil - 2$,

$$f\left(v_{\lfloor \frac{2n+1}{3} \rfloor + i}\right) = \begin{cases} 1 & \text{for } i \equiv 1,2(mod\ 4) \\ 2 & \text{for } i \equiv 0,3(mod\ 4) \end{cases}$$

$f(v_{n-1}) = 1$ and $f(v_n) = 2$.
For $n = 5$, $f(v_4) = 1$ and $f(v_5) = 2$.

For $n \equiv 0,1,3(mod\ 4), 1 \le i \le n, f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1,2(mod\ 4) \\ 2 & \text{if } i \equiv 0,3(mod\ 4) \end{cases}$

For $n \equiv 2(mod\ 4), 1 \le i \le n-2, f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1,2(mod\ 4) \\ 2 & \text{if } i \equiv 0,3(mod\ 4) \end{cases}$

$f(u_{n-1}) = 1$ and $f(u_n) = 2$.

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{2n+1}{3} \right\rceil$, $e_f(0) = e_f(1) = e_f(2)+1 = \left\lceil \dfrac{4n}{3} \right\rceil$ if $n \equiv 0,1(mod\ 4)$ and $e_f(0) = e_f(1)+1 = e_f(2) = \left\lceil \dfrac{4n}{3} \right\rceil$ if $n \equiv 2,3(mod\ 4)$.

Thus we have $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ for all $i, j = 0,1,2$. Hence f is a 3-product cordial labeling of G_v if $n \equiv 2(mod\ 3)$.

Conversely, we assume that $n \equiv 0(mod\ 3)$ and take $n = 3k$. Then $|V(G_v)| = 6k+1$ and $|E(G_v)| = 12k$.

Let f be a 3-product cordial labeling of G_v. Hence we have $v_f(0) = v_f(1) - 1 = v_f(2) = 2k$ or $v_f(0) = v_f(1) = v_f(2) - 1 = 2k$ and $e_f(0) = e_f(1) = e_f(2) = 4k$. If $f(u_i) = 0$ if $1 \le i \le 2k$, then $e_f(0) = 6k+1$. If $f(v_i) = 0$ if $1 \le i \le 2k$, then $e_f(0) = 4k+1$. If $f(u_i) = 0$ if $1 \le i \le 2k-1$ and $f(v) = 0$, then $e_f(0) = 7k-1$. If $f(v_i) = 0$ if $1 \le i \le 2k-1$ and $f(v) = 0$, then $e_f(0) = 7k-1$. Thus, none of $f(u_i), f(v_i)$ and $f(v)$ is 0. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence G_v is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$.

We assume that $n \equiv 1(mod\ 3)$ and take $n = 3k+1$. Then $|V(G_v)| = 6k+3$ and $|E(G_v)| = 12k+4$.

Let f be a 3-product cordial labeling of G_v. Hence we have $v_f(0) = v_f(1) = v_f(2) = 2k+1$ and $e_f(0) = 4k+1$ or $4k+2$. If $v_f(0) = 2k+1$, we assign 0 to $2k+1$ vertices of degree 3. We get $e_f(0) = 4k+3$. we assign

0 to $2k+1$ vertices of degree 4. We get $e_f(0) = 6k+4$. If $f(u_i) = 0$ if $1 \leq i \leq 2k$ and $f(v) = 0$, then $e_f(0) = 7k+2$. If $f(v_i) = 0$ if $1 \leq i \leq 2k$ and $f(v) = 0$, then $e_f(0) = 7k+2$. Thus, none of $f(u_i), f(v_i)$ and $f(v)$ is 0. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence G_v is not a 3-product cordial graph if $n \equiv 1(mod\, 3)$. □ An example for the 3-product cordial labeling of a closed helm CH_8 by switching of an apex vertex is shown in Figure 1.

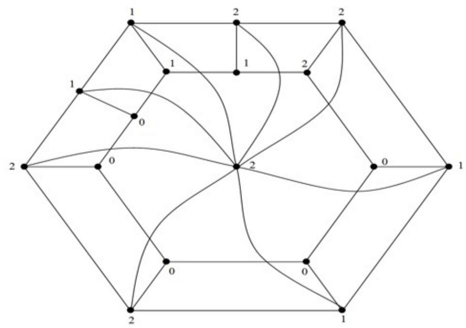

Figure 1

Theorem 2.2. *The double fan graph DF_n is a 3-product cordial graph if and only if $n \equiv 0(mod\ 3)$.*

Proof. Let DF_n be the double fan with apex vertices u, v and $v_1, v_2, \ldots, v_n$ be the vertices of common path. Then $|V(DF_n)| = n+2$ and $|E(DF_n)| = 3n-1$. To define $f : V(DF_n) \to \{0,1,2\}$ as follows:

Let $f(u) = 1, f(v) = 2, f(v_i) = 0$ if $1 \leq i \leq \dfrac{n}{3}$.

For n is even, $i = \dfrac{n}{3}+j, 1 \leq j \leq \dfrac{2n}{3}, f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,2(mod\ 4) \\ 2 & \text{if } j = 0,3(mod\ 4) \end{cases}$

For n is odd, $i = \dfrac{n}{3}+j, 1 \leq j \leq \dfrac{2n}{3}-2, f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,2(mod\ 4) \\ 2 & \text{if } j = 0,3(mod\ 4) \end{cases}$

$f(v_{n-1}) = 1, f(v_n) = 2$.

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \dfrac{n}{3}+1$ and $e_f(0) = e_f(1) = e_f(2)+1 = n$ if n is even, $e_f(0) = e_f(1)+1 = e_f(2) = n$ if n if n is odd.

Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0,1,2$. Thus, f is a 3-product cordial labeling. Therefore, DF_n is a 3-product cordial graph if $n \equiv 0(mod\ 3)$.

Conversely, we assume that $n \equiv 2(mod\ 3)$ and take $n = 3k+2$. Then $|V(DF_n)| = 3k+4$ and $|E(DF_n)| = 9k+5$.

Let f be a 3-product cordial labeling of DF_n. Hence we have $v_f(0) = v_f(1) - 1 = v_f(2) = k + 1$ or $v_f(0) = v_f(1) = v_f(2) - 1 = k + 1$ and $e_f(0) = 3k + 1$ or $3k + 2$. If $f(u) = 0, f(v) = 0$ and the remaining $k - 1$ vertices are $f(v_i) = 0$ then we get, $e_f(0) > 7k + 3$. If $f(u) = 0$ or $f(v) = 0$ and the remaining k vertices are $f(v_i) = 0$ for $1 \leq i \leq k$ then we get, $e_f(0) = 5k + 2$. If $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then we get $e_f(0) = 3k + 3$. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence DF_n is not a 3-product cordial graph if $n \equiv 2(mod\ 3)$.

We assume that $n \equiv 1(mod\ 3)$ and take $n = 3k + 1$. Then $|V(DF_n)| = 3k + 3$ and $|E(DF_n)| = 9k + 2$.

Let f be a 3-product cordial labeling of DF_n. Hence we have $v_f(0) = v_f(1) = v_f(2) = k + 1$ and $e_f(0) = 3k$ or $3k + 1$. If $f(u) = 0, f(v) = 0$ and the remaining $k - 1$ vertices are $f(v_i) = 0$ then we get, $e_f(0) > 7k + 1$. If $f(u) = 0$ or $f(v) = 0$ and the remaining k vertices are $f(v_i) = 0$ for $1 \leq i \leq k$ then we get, $e_f(0) = 5k + 1$. If $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then we get $e_f(0) = 3k + 3$. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence DF_n is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$. □

An example for the 3-product cordial labeling for the graph DF_6 is shown in Figure 2.

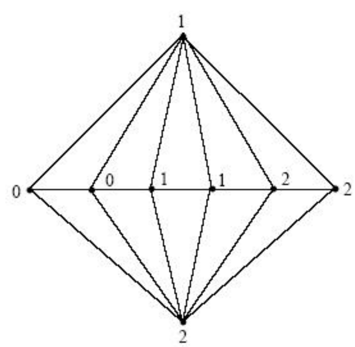

Figure 2.

Theorem 2.3. *The book graph $K_{1,n} \times K_2$ is a 3-product cordial graph.*

Proof. Let the vertices of $K_{1,n} \times K_2$ be $\{u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edges are $\{uv\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{uu_i/1 \leq i \leq n\}$. Clearly $K_{1,n} \times K_2$ has $2n + 2$ vertices and $3n + 1$ edges. Define $f : V(K_{1,n} \times K_2) \rightarrow \{0, 1, 2\}$ by the following cases.

$f(u) = 1, f(v) = 2$.

Case (i): $n \equiv 0(mod\ 3), n = 3k, k > 1.$

For k is even, $f(u_i) = \begin{cases} 0 & \text{if } 1 \le i \le k \\ 2 & \text{if } k+1 \le i \le \dfrac{3k}{2} \\ 1 & \text{if } \dfrac{3k}{2}+1 \le i \le 3k \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{if } 1 \le i \le k \\ 1 & \text{if } k+1 \le i \le \dfrac{3k}{2} \\ 2 & \text{if } \dfrac{3k}{2}+1 \le i \le 3k \end{cases}$

For k is odd, $f(u_i) = \begin{cases} 0 & \text{if } 1 \le i \le k \\ 2 & \text{if } k+1 \le i < \left\lceil \dfrac{3k}{2} \right\rceil \\ 1 & \text{if } \left\lceil \dfrac{3k}{2} \right\rceil \le i \le 3k \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{if } 1 \le i \le k \\ 1 & \text{if } k+1 \le i < \left\lceil \dfrac{3k}{2} \right\rceil \\ 2 & \text{if } \left\lceil \dfrac{3k}{2} \right\rceil \le i \le 3k \end{cases}$

For $k = 1, f(u_1) = f(v_1) = 0,\ f(u_2) = f(u_3) = 1,\ f(v_2) = f(v_3) = 2.$

From the above labeling we have, $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{6k+2}{3} \right\rceil$, $e_f(0) = e_f(1) = e_f(2) - 1 = 3k$ if k is even and $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{6k+2}{3} \right\rceil, e_f(0) = e_f(1) - 1 = e_f(2) = 3k$ if k is odd.

Case (ii): $n \equiv 1(mod\ 3), n = 3k+1, k > 1.$

For k is even, $f(u_i) = \begin{cases} 0 & \text{if } 1 \le i \le k+1 \\ 2 & \text{if } k+2 \le i \le \left\lceil \dfrac{3k+1}{2} \right\rceil \\ 1 & \text{if } \left\lceil \dfrac{3k+1}{2} \right\rceil + 1 \le i \le 3k+1 \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{if } 1 \le i \le k \\ 1 & \text{if } k+2 \le i \le \left\lceil \dfrac{3k+1}{2} \right\rceil \\ 2 & \text{if } i = k+1 \text{ and } \left\lceil \dfrac{3k+1}{2} \right\rceil + 1 \le i \le 3k+1 \end{cases}$

For k is odd, $f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 2 & \text{if } k+2 \leq i \leq \left\lceil \dfrac{3k+1}{2} \right\rceil \\ 1 & \text{if } \left\lceil \dfrac{3k+1}{2} \right\rceil + 1 \leq i \leq 3k+1 \end{cases}$

and $\mathrm{f}(\mathrm{v}_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i \leq \left\lceil \dfrac{3k+1}{2} \right\rceil \\ 2 & \text{if } \left\lceil \dfrac{3k+1}{2} \right\rceil + 1 \leq i \leq 3k+1 \end{cases}$

For $k = 1, f(u_1) = f(u_2) = f(v_1) = 0$, $f(v_2) = f(u_4) = 1$, $f(u_3) = f(v_3) = f(v_4) = 2$.

From the above labeling we have, $v_f(0) + 1 = v_f(1) = v_f(2) + 1 = \left\lceil \dfrac{6k+4}{3} \right\rceil, e_f(0) - 1 = e_f(1) = e_f(2) = 3k+1$ if k is odd and $v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \left\lceil \dfrac{6k+4}{3} \right\rceil, e_f(0) - 1 = e_f(1) = e_f(2) = 3k+1$ if k is even and $k = 1$.

Case (iii): $n \equiv 2(mod\ 3), n = 3k+2$ and k is even.

$\mathrm{f}(\mathrm{u}_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 2 & \text{if } k+2 \leq i \leq \dfrac{3k+2}{2} \\ 1 & \text{if } \dfrac{3k+2}{2} + 1 \leq i \leq 3k+2 \end{cases}$

and $f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 1 & \text{if } k+2 \leq i \leq \dfrac{3k+2}{2} \\ 2 & \text{if } \dfrac{3k+2}{2} + 1 \leq i \leq 3k+2. \end{cases}$

From the above labeling we have, $v_f(0) = v_f(1) = v_f(2) = 2k+2, e_f(0) - 1 = e_f(1) = e_f(2) = 3k+2$. Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Thus, f is a 3-product cordial labeling. Therefore, $K_{1,n} \times K_2$ is a 3-product cordial graph. □

An example for the 3-product cordial labeling for the book graph $K_{1,7} \times K_2$ is shown in Figure 3.

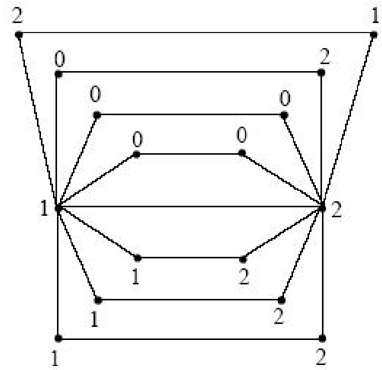

Figure 3.

Theorem 2.4. *The graph $P(K_2+mK_1,I)$ is a 3-product cordial graph if and only if $m \equiv 2(mod\ 3)$.*

Proof. Let $V(P(K_2+mK_1,I)) = \{u,u',v,v',u_i,v_i/1 \leq i \leq m\}$ and $E(P(K_2+mK_1,I)) = \{uu_i,u'u_i,vv_i,v'v_i,uu',vv',u'v,uv',u_iv_i/1 \leq i \leq m\}$. Then $|V(P(K_2+mK_1,I))| = 2m+4$ and $|E(P(K_2+mK_1,I))| = 5m+4$.

Define a vertex labeling $f : V(P(K_2+mK_1,I)) \rightarrow \{0,1,2\}$ by $f(u) = 1, f(u') = 2, f(v) = 2, f(v') = 1$.

$$\mathrm{f(u}_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \left\lceil \dfrac{m}{3} \right\rceil \\ 1 & \text{for } i = \left\lceil \dfrac{m}{3} \right\rceil + j \text{ if } j \equiv 1(mod\ 2) \\ 2 & \text{for } i = \left\lceil \dfrac{m}{3} \right\rceil + j \text{ if } j \equiv 0(mod\ 2) \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \left\lceil \dfrac{m}{3} \right\rceil \\ 1 & \text{for } i = \left\lceil \dfrac{m}{3} \right\rceil + j \text{ if } j \equiv 0,3(mod\ 4) \\ 2 & \text{for } i = \left\lceil \dfrac{m}{3} \right\rceil + j \text{ if } j \equiv 1,2(mod\ 4). \end{cases}$$

In view of the above labeling pattern, we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{2m+4}{3} \right\rceil, e_f(0) = e_f(1)+1 = e_f(2) = \left\lceil \dfrac{5m+4}{3} \right\rceil$ if m is even and $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{2m+4}{3} \right\rceil, e_f(0) = e_f(1) = e_f(2)+1 = \left\lceil \dfrac{5m+4}{3} \right\rceil$ if m is odd. Thus, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i,j = 0,1,2$. Hence, f is a 3-product cordial labeling of $P(K_2+mK_1,I)$ if $m \equiv 2(mod\ 3)$.

Conversely, we assume that $m \equiv 0(mod\ 3)$ and take $m = 3k$. Then $|V(P(K_2 + mK_1, I))| = 6k + 4$ and $|E(P(K_2 + mK_1, I))| = 15k + 4$. Let f be a vertex 3-product cordial labeling of $P(K_2 + mK_1, I)$. Then $v_f(0) - v_f(1) - 1 = v_f(2) = 2k + 1$ or $v_f(0) = v_f(1) = v_f(2) - 1 = 2k + 1$ and $e_f(0) = 5k+1$ or $5k+2$. If $f(u), f(u'), f(v), f(v')$ are zero and the remaining $2k - 3$ vertices are either $f(u_i) = 0$ for $1 \leq i \leq k - 2$ $k > 2$ and $f(v_i) = 0$ for $1 \leq i \leq k - 1, k > 2$ or $f(u_i) = 0$ for $1 \leq i \leq k - 1, k > 2$ and $f(v_i) = 0$ for $1 \leq i \leq k - 2, k > 2$ then $e_f(0) = 13k + 3$. If one of $f(u)$ or $f(u')$ and $f(v)$ or $f(v')$ is zero and the remaining $2k - 1$ vertices are either $f(u_i) = 0$ for $1 \leq i \leq k - 1, k > 1$ and $f(v_i) = 0$ for $1 \leq i \leq k, k > 1$ or $f(u_i) = 0$ for $1 \leq i \leq k, k > 1$ and $f(v_i) = 0$ for $1 \leq i \leq k - 1, k > 1$ then $e_f(0) = 7k + 3$ or $7k + 4$. If all the $2k + 1$ vertices are either $f(u_i) = 0$ for $1 \leq i \leq k + 1$ and $f(v_i) = 0$ for $1 \leq i \leq k$ or $f(u_i) = 0$ for $1 \leq i \leq k$ and $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then $e_f(0) = 5k + 3$. Hence none of $f(u), f(u'), f(v)$ and $f(v')$ is zero. From the above arguments, we get a contradiction to, f is a 3-product cordial labeling. Hence, $P(K_2 + mK_1, I)$ is not a 3-product cordial graph if $m \equiv 0(mod\ 3)$.

We assume that $m \equiv 1(mod\ 3)$ and take $m = 3k + 1$. Then $|V(P(K_2 + mK_1, I))| = 6k+6$ and $|E(P(K_2+mK_1, I))| = 15k+9$. Let f be a vertex 3-product cordial labeling of $P(K_2+mK_1, I)$. Hence we have $v_f(0) = v_f(1) = v_f(2) = 2k+2$ and $e_f(0) = e_f(1) = e_f(2) = 5k+3$. If $f(u), f(u'), f(v), f(v')$ are zero and the remaining $2k - 2$ vertices are $f(u_i) = f(v_i) = 0$ for $1 \leq i \leq k - 1$, $k > 1$ then $e_f(0) = 13k + 7$. If one of $f(u)$ or $f(u')$ and $f(v)$ or $f(v')$ is zero and the remaining $2k$ vertices are $f(u_i) = f(v_i) = 0$ for $1 \leq i \leq k$ then $e_f(0) = 9k + 6$ or $9k + 5$. If all the $2k + 2$ vertices are $f(u_i) = f(v_i) = 0$ for $1 \leq i \leq k + 1$ then $e_f(0) = 5k + 5$. Hence none of $f(u), f(u'), f(v)$ and $f(v')$ is zero. From the above arguments, we get a contradiction to, f is a 3-product cordial labeling. Hence, $P(K_2 + mK_1, I)$ is not a 3-product cordial graph if $m \equiv 1(mod\ 3)$. □

An example for the 3-product cordial labeling for the graph $P(K_2 + 5K_1, I)$ is shown in Figure 4.

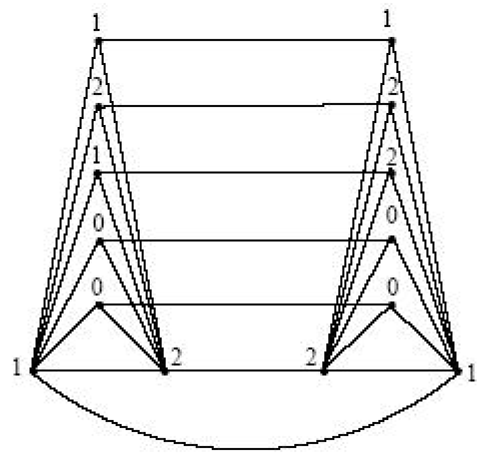

Figure 4.

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