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Classification of Osborn loops of order $4n$

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Abstract

The smallest non-associative Osborn loop is of order 16*. Attempts in the past to construct higher orders have been very difficult. In this paper, some examples of finite Osborn loops of order* $4n$*,* $n = 4$*,* 6*,* 8*,* 9*,* 12*,* 16 *and* 18 *were presented. The orders of certain elements of the examples were considered. The nuclei of two of the examples were also obtained and these were used to establish the classification of these Osborn loops up to isomorphism. Moreover, the central properties of these examples were examined and were all found to be having a trivial center and no non-trivial normal subloop. Therefore, these examples of Osborn loops are simple Osborn loops.*

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1. Introduction

Osborn loop is more or less recent and only few examples are available. Whenever there are examples of a structure then it is always of interest to researchers to know how many of such examples are distinct or non-isomorphic. Therefore, classification of algebraic examples, up to isomorphism, is imperative. The Bol and Moufang loops have been classified extensively, but Osborn loops are yet to attain that status-compare ([30]). This work is aimed at classifying up to isomorphism some examples of finite Osborn loops of order $4n$. By a loop $G(\cdot)$ we shall mean a non-empty set G together with a binary operation $(\cdot)$ such that the following properties hold: (i) given $a, b \in G$ the equations $a \cdot x = b, y \cdot a = b$ have unique solutions x, y respectively, in G; (ii) $G(\cdot)$ possesses an identity element, i.e. there exists $e \in G$ such that $e \cdot x = x \cdot e = x$ for all $x \in G$ ([32]). An overview of loop theory can be found in Jaiyéọlá [18].

Definition 1.1. *A loop is called an Osborn loop [31, 4] if it obeys any of the following:*

$$(x^\lambda \backslash y) \cdot zx = x(yz \cdot x) \tag{1.1}$$

or

$$x(yz \cdot x) = (x \cdot yE_x) \cdot zx \ \forall \ x, y, z \in G \tag{1.2}$$

where $E_x = R_x R_{x^\rho} = (L_x L_x^\lambda)^{-1} = R_x L_x R_x^{-1} L_x^{-1}$

Among the class of Bol-Moufang type of loops is the Bol loop.

Definition 1.2. *A loop L is called a Bol loop if:*

$$(xy \cdot z)y = x(yz \cdot y) \qquad \text{for all } x, y, z \in L \tag{1.3}$$

Strictly speaking, (1.3) defines a right Bol loop. A left Bol-loop $(L, \cdot)$ is defined as:

Definition 1.3.

$$x(y \cdot xz) = (x \cdot yx)z \qquad \text{for all } x, y, z \in L \tag{1.4}$$

A Bol loop commonly refers to a left or right Bol loop. The loop that satisfies both (1.3) and (1.4) is called a Moufang loop [7, 8]. Therefore, the necessary and sufficient condition for a loop to be a Moufang loop is that the loop is both a left Bol loop and right Bol loop ([9]). The smallest order for which a non-associative finite Bol loop exist is 8. There are exactly six Bol loops of order 8 that are not associative ([34]). These loops were classified by Burn [6]. Solarin and Sharma [35] determined and classified all Bol loops of order 12 that are not associative. Purtill [33] has shown that Moufang loops of orders pqr and p^2q where p, q and r are distinct odd primes with $p \leq q \leq r$ are groups (See [2]). It is to be noted that a Moufang loop is a variety of Osborn loops. Some of the earliest examples of Osborn loops were constructed by Huthnance [10] in 1968. Other examples are presented in Kinyon [10] and Isere *et al.* [12, 13]. Some recent studies on this class of loops are by Isere [11], Adeniran and Isere [3], Isere *et al.* [14, 15, 16], Jaiyéọlá [17, 20, 22, 23], Jaiyéọlá and Adéníran [24, 25, 26, 28], Jaiyéọlá *et al.* [29, 23]. The application of some identities in universal Osborn loops to cryptography were reported in Jaiyéọlá [19, 21], Jaiyéọlá and Adéníran [27].

Bruck [5] defined the following as important subloops of a loop $(L, \cdot)$.

Definition 1.4. *Let $(L, \cdot)$ be a loop. The left nucleus of L is defined as*

$$N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall \ x, y \in L\}.$$

The right nucleus of L is defined as

$$N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall \ x, y \in L\}.$$

The middle nucleus of L is defined as

$$N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall \ x, y \in L\}.$$

The nucleus of L is defined as

$$N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot).$$

Artzy [1] has proved that $N_\mu = N_\rho$ holds more generally in any right inverse property loop.

Definition 1.5. *The centrum of L is defined as*

$$C(L, \cdot) = \{a \in L : ax = xa \ \forall \ x \in L\}.$$

The center of L is defined as

$$Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot).$$

Remark 1.1. *It is to be noted that the centrum of a loop is not necessarily a subloop of the loop. However, some authors have found special Bol loops in which their centrum (commutant) are subloops.*

Definition 1.6. *A subloop N of a loop $(L,\cdot)$ is said to be normal in $(L,\cdot)$ if*

$$xN = Nx, \qquad x(yN) = (xyN), \qquad N(xy) = (Nx)y,\ \forall\ x,y \in (L,\cdot)$$

Definition 1.7. *A loop Q that has only the trivial subloops as the only normal subloops of Q is called a simple loop.*

In the next section, some examples of finite Osborn loops of order $4n$, $n = 4, 6, 8, 9, 12, 16$ and 18, as presented in Isere *et al.* [12, 13] will be revisited with the intention of examining their central properties and to show whether or not they are distinct, non-isomorphic Osborn loops. The orders of certain elements of the examples will also be considered. Whenever there is a tie, the nuclei of the examples will be obtained and these will be used to establish the classification of these Osborn loops up to isomorphism.

2. Main Results

2.1. Osborn Loops of order 4n

Example 2.1. *Let $I(\cdot) = C_{2n} \times C_2,\ I = \{(x^\alpha, y^\beta), 0 \le \alpha \le 2n-1, 0 \le \beta \le 1\}$ such that the binary operation $(\cdot)$ is defined as follows:*

$$(x^a, e)\cdot(x^b, y^\beta) = (x^{a+b}, y^\beta) \tag{2.1}$$

$$(x^a, y^\alpha)\cdot(x^b, e) = (x^{a+b}, y^\alpha) \tag{2.2}$$

$$(x^a, y^\alpha)\cdot(x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta}) \text{ if } a \equiv 0 (\text{mod } 2), b \equiv 0 (\text{mod } 2) \tag{2.3}$$

$$= (x^{a+b+ab^2}, y^{\alpha+\beta}) \text{ if } a \equiv 0 (\text{mod } 2), b \equiv 1 (\text{mod } 2) \tag{2.4}$$

$$(x^{b+c}, y^\delta)\cdot(x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta}) \text{ if } a \equiv 0 (\text{mod } 2), b \equiv 0 (\text{mod } 2) \tag{2.5}$$

$(x^{b+c}, y^{\delta}) \cdot (x^a, y^{\alpha}) = (x^{a+b+c+ab^2}, y^{\alpha+\delta})$ *if* $a \equiv 0(\text{mod } 2), b \equiv 1(\text{mod } 2)$
(2.6)

Then $I(\cdot)$ *is an Osborn loop of order* $4n$*, where* $n = 4, 6$ *and* 12*.*

Remark 2.1. *These are non-associative Osborn loops of orders* 16*,* 24 *and* 48*. This is interesting since up to now the smallest Osborn loop constructed is of order* 16*.*

Example 2.2. *Let* $I(\cdot) = C_{2n} \times C_2,\ I = \{(x^{\alpha}, y^{\beta}), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ *such that the binary operation* $(\cdot)$ *is defined as follows:*

$$(x^a, e) \cdot (x^b, y^{\beta}) = (x^{a+b}, y^{\beta}) \tag{2.7}$$

$$(x^a, y^{\alpha}) \cdot (x^b, e) = (x^{a+b}, y^{\alpha}) \tag{2.8}$$

$(x^a, y^{\alpha}) \cdot (x^b, y^{\beta}) = (x^{a+b}, y^{\alpha+\beta})$ if $a \equiv 0(\bmod 2), b \equiv 0(\bmod 2)$
(2.9)

$$= (x^{a+3b}, y^{\alpha+\beta})\ if\ a \equiv 0(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{2.10}$$

$(x^a, y^{\alpha}) \cdot (x^b, y^{\beta}) = (x^{a+3b}, y^{\alpha+3\beta})$ if $a \equiv 1(\bmod 2), b \equiv 1(\bmod 2)$
(2.11)

$(x^{b+c}, y^{\delta}) \cdot (x^a, y^{\alpha}) = (x^{a+b+c}, y^{\alpha+\delta})$ if $a \equiv 0(\text{mod } 2), b \equiv 0(\text{mod } 2)$
(2.12)

$(x^{b+c}, y^{\delta}) \cdot (x^a, y^{\alpha}) = (x^{a+3b+c}, y^{\alpha+\delta})$ if $a \equiv 0(\bmod 2), b \equiv 1(\bmod 2)$
(2.13)

$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^{\alpha}) = (x^{3a+3b+c}, y^{\alpha+3\beta+\gamma})$ if $a \equiv 1(\bmod 2), b \equiv 1(\bmod 2)$
(2.14)

Then $I(\cdot)$ *is an Osborn loop of order* $4n$*, where* $n = 6, 9,$ *and* 18*.*

Remark 2.2. *These examples of Osborn loops are of orders* 24, 36 *and* 72. *We have a peak order of* 72.

Example 2.3. *Let* $I(\cdot) = C_{2n} \times C_2,\ I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ *such that the binary operation* $(\cdot)$ *is defined as follows:*

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta) \tag{2.15}$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha) \tag{2.16}$$

$$(x^a, y^\alpha)\cdot(x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta})\ if\ a \equiv 0(\bmod 2), b \equiv 0(\bmod 2) \tag{2.17}$$

$$= (x^{a-b}, y^{\alpha+\beta})\ if\ a \equiv 0(\bmod 2), b \equiv 1(\bmod 2) \tag{2.18}$$

$$(x^a, y^\alpha)\cdot(x^b, y^\beta) = (x^{a-b}, y^{\alpha-\beta})\ if\ a \equiv 1(\bmod 2), b \equiv 1(\bmod 2) \tag{2.19}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta})\ if\ a \equiv 0(\bmod 2), b \equiv 0(\bmod 2) \tag{2.20}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a-b+c}, y^{\alpha+\delta})\ if\ a \equiv 0(\bmod 2), b \equiv 1(\bmod 2) \tag{2.21}$$

$$(x^{b+c}, y^{\beta+\gamma})\cdot(x^a, y^\alpha) = (x^{c-a-b}, y^{\alpha-\beta+\gamma})\ if\ a \equiv 1(\bmod 2), b \equiv 1(\bmod 2) \tag{2.22}$$

$$(x^{b+c}, y^{\beta+\gamma})\cdot(x^a, y^\alpha) = (x^{b+c-a}, y^{\beta+\gamma-\alpha})\ if\ a \equiv 1(\bmod 2), b \equiv 0(\bmod 2) \tag{2.23}$$

Then $I(\cdot)$ *is an Osborn loop of order* $4n$, *where* $n = 6$, 9 *and* 18.

Remark 2.3. *These examples of Osborn loops are of orders* 24, 36 *and* 72.

Example 2.4. *Let $I(\cdot) = C_{2n} \times C_2,\ I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ such that the binary operation $(\cdot)$ is defined as follows:*

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta) \tag{2.24}$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha) \tag{2.25}$$

$$(x^a, y^\alpha)\cdot(x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta})\ if\ a \equiv 0(\bmod 2), b \equiv 0(\bmod 2) \tag{2.26}$$

$$= (x^a, y^{\alpha+\beta})\ if\ a \equiv 0(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{2.27}$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^a, y^\alpha)\ if\ a \equiv 1(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{2.28}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta})\ if\ a \equiv 0(\text{mod } 2), b \equiv 0(\text{mod } 2) \tag{2.29}$$

$$(x^{b+c}, y^\delta)\cdot(x^a, y^\alpha) = (x^{a+c}, y^{\alpha+\delta})\ if\ a \equiv 0(\bmod 2), b \equiv 1(\bmod 2) \tag{2.30}$$

$$(x^{b+c}, y^{\beta+\gamma})\cdot(x^a, y^\alpha) = (x^c, y^{\alpha+\gamma})\ if\ a \equiv 1(\bmod 2), b \equiv 1(\bmod 2) \tag{2.31}$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{b+c}, y^{\beta+\gamma})\ if\ a \equiv 1(\text{mod } 2), b \equiv 0(\text{mod } 2) \tag{2.32}$$

Then $I(\cdot)$ is an Osborn loop of order $4n$, where $n = 4, 8$ and 16.

Remark 2.4. *These examples of Osborn loops are of orders* 24, 32 *and* 64. *In all, we have Osborn loops of orders* 16, 24, 32, 36, 48, 64 *and* 72, *as against the only Osborn loop of order* 16 *that was constructed prior to our work by [30]. For the proof of all these examples as Osborn loops, see references [12] and [13]. But, the challenge is, are these all distinct examples? In other words, are they non-isomorphic Osborn loops? This is the focus of this paper.*

2.2. Classification up to Isomorphism

Two loops shall be considered non-isomorphic if they contain different number of elements of the same order. Whenever, two loops contain the same number of elements, we shall go further to consider the order of elements in their left nucleus. If these coincide in both cases, we shall consider commutative patterns of both loops.

Theorem 2.1. *The four examples stated above are non-isomorphic Osborn loops.*

Proof:
Let us consider the number of elements of order 2 in each example.
(i) Example 2.1 Elements of order 2 are given by
$(x^n, e)^2 = (e, e)$
$(x^a, y^\alpha)^2 = (e, e)$ if $a \equiv 0 (\text{mod } 2) = (x^{2a+a^3}, y^{2\alpha})$

$$= (e, e) \text{ if } a \equiv 1 (\text{mod } 2) \tag{2.33}$$

The only possible solution to the equation (2.33) is a case of $a = 0$ i.e. (e, y^α). Therefore, (e, y^α) and (x^n, e) are the 2 elements of order 2 in Example 2.1.

(ii) Example 2.2 Elements of order 2 are given by
$(x^n, e)^2 = (e, e)$
$(x^a, y^\alpha)^2 = (e, e)$ if $a \equiv 0 (\text{mod } 2)$

$$= (x^{4a}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1 (\text{mod } 2) \tag{2.34}$$

The only possible solution to the equation (2.34) are $a = 0$ and $a = n$ whether n is even or odd. i.e.$(x^{2n}, e) = (e, e)$ and $(x^{4n}, e) = (x^{2n(2)}, e) = (e, e)$. Therefore, (e, y^α) , (x^n, e) and (x^n, y^α) are the 3 elements of order 2 in Example 2.2.

(iii) Example 2.3 Elements of order 2 are given by
$(x^n, e)^2 = (e, e)$
$(x^a, y^\alpha)^2 = (e, e)$ if $a \equiv 0 (\text{mod } 2)$

$$= (x^{a-a}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1 (\text{mod } 2) \tag{2.35}$$

The only possible solution to the equation (2.35) are $a = 0$, $a = n$ and (x^a, y^α) $\forall$ $a \equiv 1 (\text{mod } 2)$, hence, there are $n + 3$ elements of order

2 in Example 2.3, which are (e, y^α) , (x^n, e) , (x^a, y^α) and n number of (x^a, y^α) $\forall$ $a \equiv 1(\text{mod } 2)$.

(iv) Example 2.4 Elements of order 2 are given by

$$(\mathrm{x}^n, e)^2 = (e, e)$$

$$(x^a, y^\alpha)^2 = (e, e) \text{ if } a \equiv 0(\text{mod } 2)$$

$$= (x^a, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1(\text{mod } 2) \tag{2.36}$$

The only possible solution to the equation (2.36) is $a = 0$ i.e. (e, y^α). Therefore, (e, y^α) and (x^n, e) are the two elements of order 2 in Example 2.4. From the above, isomorphism is only possible in the set $\{(i), (iv)\}$.

Now let us consider elements of order 4. For Example 2.1 and Example 2.4, we obtain:

(1) $(x^{n/2}, e)^4 = (x^{2n}, e) = (e, e)$

(2) $(x^{n/2}, y^\alpha)^4 = (x^{2n}, e) = (e, e)$

(3) $(e, y^\alpha)^4 = (e, e)$

Hence, Example 2.1 and Example 2.4 contain 3 elements of order 4 (another tie).

To show that Example 2.1 and Example 2.4 are non-isomorphic, let us consider the order of elements in their nuclei.

First in Example 2.1, let us consider the left nucleus.
Let $x = (x^a, y^\alpha), y = (x^b, y^\beta), u = (x^d, y^\delta)$, then considering the definition of left nucleus, by computation, it becomes

$$ux \cdot y = (x^{a+b+d+(a+d)b^2}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$u \cdot xy = (x^{a+b+d+ab^2}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \notin N_\lambda(I, \cdot)$.

Now, let us consider $N_\lambda(I, \cdot)$ in Example 2.4:
By computation we have:

$$ux \cdot y = (x^{a+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$

$$u \cdot xy = (x^{a+d}, y^{\alpha+\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \in N_\lambda(I, \cdot)$.

Since $N_\lambda(I,\cdot) - \{e\} = \emptyset$ in Example 2.1 and $N_\lambda(I,\cdot) - \{e\} \neq \emptyset$ in Example 2.4, then the Osborn loops in Example 2.1 and Example 2.4 are non-isomorphic. The proof is complete.

3. Central Properties of the Examples

Theorem 3.1. *The four examples above have trivial centers.*

Proof:

(i) Example 2.1 Let $x = (x^a, y^\alpha), y = (x^b, y^\beta), u = (x^d, y^\delta)$, then from section 2, $u \notin N_\lambda(I,\cdot)$. Examining the right nucleus, by computation, we have:

$$x \cdot yu = (x^{a+b+d}, y^{\alpha+\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2)$$
$$xy \cdot u = (x^{a+b+d+ab^2}, y^{\alpha+\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \notin N_\rho(I,\cdot)$.
Then, considering the definition of middle nucleus, by computation, it becomes

$$x \cdot uy = (x^{a+b+d+b^2d}, y^{\alpha+\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2)$$
$$xu \cdot y = (x^{a+b+(a+d)b^2}, y^{\alpha+\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \notin N_\mu(I,\cdot)$. Thus, $u \notin N(I,\cdot)$.

Remark 3.1. *Example 2.1 has trivial subloops*

Let us now consider the centrum. Let $y = (x^b, y^\beta), u = (x^d, y^\delta)$, then considering the definition of the centrum, by computation, it becomes

$$uy = (x^{a+b+b^2d}, y^{\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2)$$
$$yu = (x^{b+d}, y^{\beta+\delta}),\ \text{if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \notin C(I,\cdot)$. Then, $u \notin Z(I,\cdot)$

(ii) Example 2.2 Let us consider the left nucleus; the right nucleus and, the middle nucleus:

$$ux \cdot y = (x^{a+d+3b}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$u \cdot xy = (x^{a+d+3b}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \in N_\lambda(I, \cdot)$.
Let us now consider the right nucleus. Let $x = (x^a, y^\alpha), y = (x^b, y^\beta), u = (x^d, y^\delta)$, then considering the definition of right nucleus, by computation, it becomes

$$x \cdot yu = (x^{a+b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$xy \cdot u = (x^{a+d+3b}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \notin N_\rho(I, \cdot)$. Finally, let consider the middle nucleus.

$$xu \cdot y = (x^{a+d+3b}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$x \cdot uy = (x^{a+d+3b}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Therefore, $u \in N_\mu(I, \cdot)$. Thus, $u \notin N(I, \cdot)$ in Example 2.2.

Remark 3.2. *Example 2.2 has non-trivial $N_\lambda(I, \cdot)$ and $N_\mu(I, \cdot)$ as subloops.*

Next, we examine the centrum $C(I, \cdot)$. Consider:

$$yu = (x^{b+d}, y^{\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$uy = (x^{3b+d}, y^{\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \notin C(I, \cdot)$. Therefore, $Z(I, \cdot) = \emptyset$, i.e. $u \notin Z(I, \cdot)$.

(iii) Example 2.3 First, we check for $N_\lambda(I, \cdot)$. Consider:

$$u \cdot xy = (x^{a-b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$ux \cdot y = (x^{a-b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \in N_\lambda(I, \cdot)$. Next, we check $N_\mu(I, \cdot)$.

Consider:

$$x \cdot uy = (x^{a-b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$ux \cdot y = (x^{a-b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \in N_\mu(I, \cdot)$.

Next, we check $N_\rho(I, \cdot)$. Consider:

$$xy \cdot u = (x^{a-b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$x \cdot yu = (x^{a+b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \notin N_\rho(I, \cdot)$. Hence, $u \notin N(I, \cdot)$ in Example 2.3.

Remark 3.3. *Example 2.3 has non-trivial $N_\lambda(I, \cdot)$ and $N_\mu(I, \cdot)$ as subloops.*

Now, we check the centrum $C(I, \cdot)$. Consider:

$$uy = (x^{d-b}, y^{\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$yu = (x^{b+d}, y^{\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \notin C(I, \cdot)$. Therefore, $u \notin Z(I, \cdot)$.

Finally, we examine Example 2.4

(iv) Example 2.4 From section 2, $u \in N_\lambda(I, \cdot)$. Next, we check $N_\mu(I, \cdot)$.

Consider:

$$x \cdot uy = (x^{a+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$ux \cdot y = (x^{a+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \in N_\mu(I, \cdot)$.

Next, we check $N_\rho(I, \cdot)$. Consider:

$$xy \cdot u = (x^{a+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$x \cdot yu = (x^{a+b+d}, y^{\alpha+\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \notin N_\rho(I, \cdot)$. Hence, $u \notin N(I, \cdot)$ in Example 2.4.

Remark 3.4. *Example 2.4 has non-trivial $N_\lambda(I, \cdot)$ and $N_\mu(I, \cdot)$ as subloops.*

Now, we check the centrum $C(I, \cdot)$. Consider:

$$uy = (x^d, y^{\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2)$$
$$yu = (x^{b+d}, y^{\beta+\delta}), \text{ if } b \equiv 1(\text{mod } 2).$$

Thus, $u \notin C(I, \cdot)$. Therefore, $u \notin Z(I, \cdot)$.

From the foregoing, every element $(x^d, y^\delta) \in (I, \cdot)$ does not associate completely nor commute with every other element of $(I, \cdot)$ under the same condition. Hence, the proof follows.

Corollary 3.1. *The four examples above have no non-trivial normal subloops.*

Proof:
The proof follows from Theorem 3.1

4. Conclusion

This work presented a method of classifying descriptive examples of Osborn loops of order $4n$. These examples of Osborn loops considered, do not have their nuclei coincide. Artzy has proved that $N_\mu = N_\rho$ holds more generally in any right inverse property loop. But in these examples, $N_\mu \neq N_\rho$. Therefore, the analysis confirms that Osborn loops are not right-inverse property loops. Consequently, the above examples are not Bol loops and are not Moufang. Since they have no non-trivial normal subloops, then, it follows that they are a simple Osborn loops. It is to be noted also that constructing a loop from an indirect product of two cyclic groups gives rise to a cyclic simple loop.

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