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Some new triple sequence spaces over n-normed space

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Abstract

Triple sequence spaces were introduced by Sahiner et al. [27, 28]. The main objective of this paper is to define some new classes of triple sequences over n-normed space by means of Museiak-Orlicz function and difference operators. We also study some algebraic and topological properties of these new sequence spaces.

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1. Introduction

By w''' we shall denote the class of all complex triple sequence $\langle a_{ijk} \rangle$, where $i, j, k \in \mathbf{N}$, the set of positive integers. Then, w''' is a linear space under the coordinate wise addition and scalar multiplication.

A triple sequence can be represented by a matrix, in case of double sequences we write in the form of a square. In case of triple sequence it will be in the form of a box in three dimensions. The different types of notions of triple sequence was introduced and investigated initially by Sahiner et al. [27, 28] , Esi [7], Esi and Catalbas [8], Esi and Savas [9], Datta [2], Debnath [3] and many others.

The concept of 2-normed spaces was initially developed by Gähler [11], in the mid of 1960's while that of $n-$normed spaces was studied by Misiak [21]. Since then many authors have studied $n-$normed spaces and obtained various results, see Gunawan ([12, 13]) and Gunawan and Mashadi [14].

Kizmaz [19] introduced the notion of difference sequence spaces as follows
$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$
for
$Z = \ell_\infty$, c and c_0, $\Delta(x) = x_k - x_{k+1}$, $\Delta^0(x) = x_k$.
The study was further generalized by Et and Colak [10] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

The difference operator on triple sequence is defined as
$\Delta x_{mnk} = x_{mnk} - x_{(m+1)nk}$
$-x_{m(n+1)k} - x_{mn(k+1)} + x_{(m+1)(n+1)k}$

$+x_{(m+1)n(k+1)} + x_{m(n+1)(k+1)} - x_{(m+1)(n+1)(k+1)}.$

2. Definitions and preliminaries

Definition 2.1. *[27] A triple sequence $\langle a_{ijk} \rangle$ is said to be convergent to L in Pringsheim's sense if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbf{N}$ such that*

$$|a_{ijk} - L| < \epsilon \qquad \text{whenever} \quad i \geq N, j \geq N, k \geq N$$

and is written as $\lim_{i,j,k\to\infty} a_{ijk} = L$.

Note: A triple sequence is convergent in Pringsheim's sense may not be bounded [27].

Definition 2.2. *[27] A triple sequence $\langle a_{ijk} \rangle$ is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbf{N}$ such that*

$$|a_{ijk} - a_{lmn}| < \epsilon \qquad \text{whenever} \quad i \geq l \geq N, j \geq m \geq N, k \geq n \geq N$$

Definition 2.3. *[27] A triple sequence $\langle a_{ijk} \rangle$ is said to be bounded if there exists $M > 0$, such that $|a_{ijk}| < M$ for all $i, j, k \in \mathbf{N}$.*

Definition 2.4. *[2]A Triple sequence space Y is said to be Solid if $\langle \alpha_{ijk} a_{ijk} \rangle \in Y$ whenever $\langle a_{ijk} \rangle \in Y$ and for all triple sequences $\langle \alpha_{ijk} \rangle$ of scalars with $|\alpha_{ijk}| \leq 1$, for all $i, j, k \in \mathbf{N}$.*

Definition 2.5. *[2] A Triple sequence space Y is said to be monotone if it contains the canonical pre-images of all its step spaces .*

Note: A sequence space is solid implies that it is monotone.

Definition 2.6. *[18],[24]*
*(***Orlicz function and Musielak-Orlicz function***)*
An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x)+M(y)$; then this function is called modulus function.

Lindenstrauss and Tzafriri [20] used the idea of Orlicz function to construct Orlicz sequence space:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}$$

The space ℓ_M with the norm:

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$ $(1 \leq p < \infty)$, the space ℓ_M coincide with the classical sequence space $\ell_p (p \geq 1)$.
A sequence $f = (f_{mnk})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mnk})$ defined by

$$g_{mnk}(v) = \sup\{|v|u - (f_{mnk})(u) : u \geq 0\} \quad m, n, k = 1, 2, 3, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows:

$$t_f = \left\{x \in w''' : I_f\left(|x_{mnk}|\right)^{\frac{1}{m+n+k}} \to 0 \quad \text{as} \quad m, n, k \to \infty\right\}$$

where I_f is a convex modulus function defined by:

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(|x_{mnk}|\right)^{\frac{1}{m+n+k}}\ , \qquad x = (x_{mnk}) \in t_f.$$

Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

Where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Definition 2.7 (n-Normed Space). *Let $n \in \mathbf{N}$ and X be a linear space over the field $\mathbf{R}$ of reals of dimension d, where $2 \leq d \leq n$. A real valued function $\| \cdot, ..., \cdot \|$ on X^n satisfying the following four conditions:*

(1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation;*

(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ *for any* $\alpha \in \mathbf{R}$*;*

(4) $\|x_1 + x_1^{'}, x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1^{'}, x_2, ..., x_n\|$*;*

is called an n*-norm on* X *and* $(X, \|\cdot, ..., \cdot\|)$ *is called an* $n-$*normed space over the field* $\mathbf{R}$*.*

For example $(\mathbf{R}^n, \|\cdot, ..., \cdot\|_E)$ *where*

$\|x_1, x_2, ..., x_n\|_E =$ *the volume of the*
$n -$ *dimensional parallelopiped spanned by the vectors*
$x_1, x_2, ..., x_n$

which can also be written as

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{ij})|$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbf{R}^n$ *for each* $i = 1, 2, \cdots, n$*. Let* $(X, \|\cdot, ..., \cdot\|)$ *be an* $n-$*normed space of dimension* $2 \leq n \leq d$ *and* $\{a_1, a_2, \cdots, a_n\}$ *be linearly independent set in* X*. Then the following function* $\|\cdot, ..., \cdot\|_\infty$ *on* X^{n-1} *defined by*

$$\|x_1, x_2, ..., x_{n-1}\|_\infty = \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, 2, ..., n\}$$

defines an $(n-1)$*-norm on* X *with respect to* $\{a_1, a_2, ..., a_n\}$*.*

A sequence (x_k) *in a* n*-normed space* $(X, \|\cdot, ..., \cdot\|)$ *is said to converge to some* $L \in X$ *if*

$$\lim_{k \to \infty} \|x_k - L, z_1, ..., z_{n-1}\| = 0 \qquad \text{for every} \quad z_1, ..., z_{n-1} \in X.$$

A sequence (x_k) *in a* n*-normed space* $(X, \|\cdot, ..., \cdot\|)$ *is said to be Cauchy if*

$$\lim_{k,p \to \infty} \|x_k - x_p, z_1, ..., z_{n-1}\| = 0 \qquad \text{for every} \quad z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X *converges to some* $L \in X$*, then* X *is said to be complete with respect to the* n*-norm. Any complete* n*-normed space is said to be* n*-Banach space. The* n*-normed space has been studied in stretch ([1], [4], [5], [6], [21], [22],[23]).*

Definition 2.8. *[30] (***Paranormed Space***)*
Let X be a linear metric space. A function $p : X \to \mathbf{R}$ is called paranorm, if

(1) $p(x) \geq 0$ *, for all* $x \in X$

(2) $p(-x) = p(x)$*, for all* $x \in X$

(3) $p(x + y) \leq p(x) + p(y)$*, for all* $x, y \in X$

(4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm. For further reference on paranormed space ([15-17], [25] , [26]).

If $p = (p_{ijk})$ is a triple sequence of positive real numbers with

$$0 \leq p_{ijk} \leq \sup p_{ijk} = G, K = \max(1, 2^{G-1})$$

Then

$$|a_{ijk} + b_{ijk}|^{p_{ijk}} \leq K\{|a_{ijk}|^{p_{ijk}} + |b_{ijk}|^{p_{ijk}}\} \tag{2.1}$$

for all i, j, k and triple sequences $a_{ijk}, b_{ijk} \in \mathbf{C}$. Also $|a|^{p_{ijk}} \leq \max(1, |a|^G)$ for all $a \in \mathbf{C}$.

3. Construction of triple $n-$normed sequence spaces

Now we introduce the new class of triple sequence spaces using Orlicz functions and difference operator on $n-$ normed space, if M is an Orlicz function and $p = \langle p_{ijk} \rangle$ is a triple sequence of strictly positive real numbers and $(X, \|\cdot, ..., \cdot\|)$ is a real linear n-normed space we define the following classes of sequences:

$W'''(M,\Delta,p,\|\cdot,...,\cdot\|) = \{\langle a_{ijk}\rangle \in w''' : \lim_{l,m,n} \frac{1}{lmn}$
$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ijk}-L}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right)^{p_{ijk}} = 0,$

for each $z_1,\cdots,z_{n-1}\in X,$ for some $\rho>0$ and $L>0\},$
$W_0'''(M,\Delta,p,\|\cdot,...,\cdot\|) = \{\langle a_{ijk}\rangle \in w''' : \lim_{l,m,n} \frac{1}{lmn}$
$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right)^{p_{ijk}} = 0,$

for each $z_1,\cdots,z_{n-1}\in X,$ for some $\rho>0\},$
and
$W_\infty'''(M,\Delta,p,\|\cdot,...,\cdot\|)$
$= \{\langle a_{ijk}\rangle \in w''' : \sup_{l,m,n\ z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}$

$\left(M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right)^{p_{ijk}} < \infty,$

for some $\rho>0\},$

Some Special Cases

(i). If we take $p=(p_{ijk})=1$, we get
$W'''(M,\Delta,\|\cdot,...,\cdot\|) = \{\langle a_{ijk}\rangle \in w''' : \lim_{l,m,n} \frac{1}{lmn}$

$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ijk}-L}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right) = 0,$

for each $z_1,\cdots,z_{n-1}\in X,$ for some $\rho>0$ and $L>0\},$
$W_0'''(M,\Delta,\|\cdot,...,\cdot\|) = \{\langle a_{ijk}\rangle \in w''' : \lim_{l,m,n} \frac{1}{lmn}$

$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left(M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right) = 0,$

for each $z_1,\cdots,z_{n-1}\in X,$ for some $\rho>0\},$
and
$W_\infty'''(M,\Delta,\|\cdot,...,\cdot\|)$
$= \{\langle a_{ijk}\rangle \in w''' : \sup_{l,m,n\ z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}$

$\left(M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)\right) < \infty,$

for some $\rho>0\},$

(ii). If we take $M(x)=x$, we get

$W'''(\Delta, p, \|\cdot, ..., \cdot\|) = \{\langle a_{ijk}\rangle \in w''' :$

$\lim_{l,m,n} \frac{1}{lmn} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \left(\left\| \frac{\Delta a_{ijk} - L}{\rho}, z_1, \cdots, z_{n-1} \right\|\right)^{p_{ijk}} = 0,$

for each $z_1, \cdots, z_{n-1} \in X,$ for some $\rho > 0$ and $L > 0\},$

$W_0'''(\Delta, p, \|\cdot, ..., \cdot\|) = \{\langle a_{ijk}\rangle \in w''' : \lim_{l,m,n} \frac{1}{lmn}$
$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \left(\left\| \frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\|\right)^{p_{ijk}} = 0,$
for each $z_1, \cdots, z_{n-1} \in X,$ for some $\rho > 0\},$
and
$W_\infty'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$
$= \{\langle a_{ijk}\rangle \in w''' : \sup_{l,m,n \; z_1,\cdots,z_{n-1} \in X} \frac{1}{lmn} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n}$

$\left(\left\| \frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\|\right)^{p_{ijk}} < \infty,$

for some $\rho > 0\},$

Theorem 3.1. *Let M be an Orlicz function and $p = (p_{ijk})$ be bounded triple sequence of strictly positive real numbers. Then the classes of sequences $W'''(M, \Delta, p, \|\cdot, ..., \cdot\|), W_0'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$ and $W_\infty'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$ are linear spaces over the field of real numbers* $\mathbf{R}$.

Proof. Let $\langle a_{ijk}\rangle, \langle b_{ijk}\rangle \in W_\infty'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$ and $\alpha, \beta \in \mathbf{R}$. Then there exist positive real numbers ρ_1, ρ_2 such that

$$\sup_{\substack{l,m,n \\ z_1,\cdots,z_{n-1} \in X}} \frac{1}{lmn} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[M\left(\left\| \frac{\Delta a_{ijk}}{\rho_1}, z_1, \cdots, z_{n-1} \right\|\right)\right]^{p_{ijk}} < \infty,$$

for some $\rho_1 > 0$

and

$$\sup_{\substack{l,m,n \\ z_1,\cdots,z_{n-1} \in X}} \frac{1}{lmn} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[M\left(\left\| \frac{\Delta a_{ijk}}{\rho_2}, z_1, \cdots, z_{n-1} \right\|\right)\right]^{p_{ijk}} < \infty,$$

for some $\rho_2 > 0.$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$, then since $\|\cdot, ..., \cdot\|$ is a n-norm on X and M is non-decreasing, convex and so by using inequality (2.1), we have

$\sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X}$
$\frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta[\alpha a_{ijk}+\beta b_{ijk}]}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$
$\leq \sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta\alpha a_{ijk}}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right.\right.$

$\left.+\left\|\frac{\Delta\beta b_{ijk}}{\rho_3}, z_1, \cdots, z_{n-1}\right\|^{p_{ijk}}$

$\leq K \sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{2^{p_{ijk}}}\left[M\left(\left\|\frac{\Delta[a_{ijk}]}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$+K \sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{2^{p_{ijk}}}\left[M\left(\left\|\frac{\Delta[b_{ijk}]}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$\leq K \sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta[a_{ijk}]}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$+K \sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta[b_{ijk}]}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$< \infty.$

Thus we have $\alpha\langle a_{ijk}\rangle + \beta\langle b_{ijk}\rangle \in W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$.
Hence $W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$ is a linear space. Similarly it can be shown that $W'''(M, \Delta, p, \|\cdot, ..., \cdot\|), W'''_0(M, \Delta, p, \|\cdot, ..., \cdot\|)$ are linear spaces over the field of reals $\mathbf{R}$. □

Theorem 3.2. *Let M be an Orlicz function and $p = (p_{ijk})$ be bounded triple sequence of strictly positive real numbers. The sequence spaces $W'''(M, \Delta, p, \|\cdot, ..., \cdot\|), W'''_0(M, \Delta, p, \|\cdot, ..., \cdot\|)$ and $W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$ are paranormed spaces, paranormed by*

$g(\langle a_{ijk}\rangle) =$
$\sup_i |a_{i11}| + \sup_j |a_{1j1}| + \sup_k |a_{11k}|$

$+\inf\left\{\rho^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} \ {}_{z_1,\cdots,z_{n-1}\in X}\left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{\frac{p_{ijk}}{H}} \leq 1\right\}$

Where $H = \max(1, G)$, $G = \sup_{i,j,k} p_{ijk}$.

Proof. Clearly $g(0) = 0$ and $g(-\langle a_{ijk}\rangle) = g(\langle a_{ijk}\rangle)$.
Let $\langle a_{ijk}\rangle, \langle b_{ijk}\rangle \in W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$. The there exists some $\rho_1, \rho_2 > 0$

such that

$$\sup_{\substack{l,m,n \\ z_1,\cdots,z_{n-1}\in X}} \left[M\left(\left\| \frac{\Delta a_{ijk}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \leq 1.$$

and

$$\sup_{\substack{l,m,n \\ z_1,\cdots,z_{n-1}\in X}} \left[M\left(\left\| \frac{\Delta a_{ijk}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality , we have

$$\sup_{l,m,n \ z_1,\cdots,z_{n-1}\in X} \left[M\left(\left\| \frac{\Delta a_{ijk} + \Delta b_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}}$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2} \right) \sup_{l,m,n \ z_1,\cdots,z_{n-1}\in X} \left[M\left(\left\| \frac{\Delta a_{ijk}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}}$$

$$+ \left(\frac{\rho_2}{\rho_1+\rho_2} \right) \sup_{l,m,n \ z_1,\cdots,z_{n-1}\in X} \left[M\left(\left\| \frac{\Delta b_{ijk}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \leq 1.$$

Now

$g(\langle a_{ijk}\rangle + \langle b_{ijk}\rangle)$
$= \sup_i |a_{i11} + b_{i11}| + \sup_j |a_{1j1} + b_{1j1}| + \sup_k |a_{11k} + b_{11k}|$

$+ \inf \Big\{ (\rho_1 + \rho_2)^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} {}_{z_1,\cdots,z_{n-1}\in X}$

$\left[M \left(\left\| \frac{\Delta a_{ijk} + \Delta b_{ijk}}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \le 1$

$\le \sup_i |a_{i11}| + \sup_i |b_{i11}| + \sup_j |a_{1j1}| + \sup_j |b_{1j1}| + \sup_k |a_{11k}| + \sup_k |b_{11k}|$

$+ \inf \Big\{ (\rho_1)^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} {}_{z_1,\cdots,z_{n-1}\in X}$

$\left[M \left(\left\| \frac{\Delta a_{ijk} + \Delta b_{ijk}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \le 1$

$+ \inf \Big\{ (\rho_2)^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} {}_{z_1,\cdots,z_{n-1}\in X}$

$\left[M \left(\left\| \frac{\Delta a_{ijk} + \Delta b_{ijk}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \le 1$

$= g(\langle a_{ijk}\rangle) + g(\langle b_{ijk}\rangle).$

Let $\lambda \in \mathbf{C}$ then the continuity of the product follows from the following inequality
$g(\lambda\langle a_{ijk}\rangle)$
$= \sup_i |\lambda a_{i11}| + \sup_j |\lambda a_{1j1}| + \sup_k |\lambda a_{11k}|$

$$+ \inf \left\{ (\rho)^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} {}_{z_1,\cdots,z_{n-1}\in X} \left[M \left(\left\| \frac{\Delta \lambda a_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \le 1 \right\}$$

$= |\lambda| (\sup_i |a_{i11}| + \sup_j |a_{1j1}| + \sup_k |a_{11k}|)$

$$+ \inf \left\{ (|\lambda| r)^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} {}_{z_1,\cdots,z_{n-1}\in X} \left[M \left(\left\| \frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \le 1 \right\}.$$

Where $\frac{1}{r} = \frac{|\lambda|}{\rho}$.
This completes the proof of the theorem. □

Theorem 3.3. *Let M be an Orlicz function and $p = (p_{ijk})$ be bounded triple sequence of strictly positive real numbers. The sequence spaces $W'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$, $W_0'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$ and $W_\infty'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$ are complete paranormed spaces, under the paranorm defined by g.*

Proof. Let $\langle a^s_{ijk} \rangle$ be a Cauchy sequence in $W_\infty'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$. Then $g(\langle a^s_{ijk} - a^t_{ijk}) \to 0$ as $s, t \to \infty$. For a given $\epsilon > 0$, choose $r > 0$ and $x_0 > 0$ be such that $\frac{\epsilon}{rx_0} > 0$ and $M\left(\frac{rx_0}{2}\right) \geq 1$. Now $g(\langle a^s_{ijk} - a^t_{ijk} \rangle) \to 0$ as $s, t \to \infty$ implies that there exists $m_0 \in \mathbf{N}$ such that

$$g(\langle a^s_{ijk} - a^t_{ijk} \rangle) < \frac{\epsilon}{rx_0} \text{ for all } s, t \geq m_0.$$

Thus, we have

$$\sup_i |a^s_{i11} - a^t_{i11}| + \sup_j |a^s_{1j1} - a^t_{1j1}| + \sup_k |a^s_{11k} - a^t_{11k}|$$

$$+ \inf \left\{ (\rho)^{\frac{p_{ijk}}{H}} > 0 : \sup_{l,m,n} \, {}_{z_1,\cdots,z_{n-1} \in X} \right.$$

$$\left[M \left(\left\| \frac{\Delta a^s_{ijk} - \Delta a^t_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{\frac{p_{ijk}}{H}} \leq 1$$

$$< \frac{\epsilon}{rx_0}. \tag{3.1}$$

This shows that $\langle a^s_{i11} \rangle, \langle a^s_{1j1} \rangle$ and $\langle a^s_{11k} \rangle$ are Cauchy sequences of real numbers. As the set of real numbers is complete so there exists real numbers $a_{i11}, a_{1j1}, a_{11k}$ such that

$$\lim_{s \to \infty} a^s_{i11} = a_{i11} \quad , \quad \lim_{s \to \infty} a^s_{1j1} = a_{1j1} \quad , \quad \lim_{s \to \infty} a^s_{11k} = a_{11k}.$$

Then from (3.1) we have

$$\left(\left\| \frac{\Delta a^s_{ijk} - \Delta a^t_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1$$

$$\Rightarrow \sup_{i,j,k} \left[M \left(\left\| \frac{\Delta a^s_{ijk} - \Delta a^t_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] \leq 1 \leq M\left(\frac{rx_0}{2}\right)$$

$$\Rightarrow \frac{\|(\Delta a^s_{ijk} - \Delta a^t_{ijk}), z_1, \cdots, z_{n-1}\|}{g(\langle a^s_{ijk} - a^t_{ijk} \rangle)} \leq \frac{rx_0}{2}$$

$$\Rightarrow \|(\Delta a^s_{ijk} - \Delta a^t_{ijk}), z_1, \cdots, z_{n-1}\| < \frac{rx_0}{2} \cdot \frac{\epsilon}{rx_0} = \frac{\epsilon}{2}.$$

This implies that $\langle \Delta a^s_{ijk} \rangle$ is a Cauchy sequence of real numbers. Let $\lim_{s\to\infty} \Delta a^s_{ijk} = y_{ijk}$ for all $i, j, k \in \mathbf{N}$. Now

$$\Delta a^s_{111} = a^s_{111} - a^s_{112} - a^s_{121} - a^s_{211} + a^s_{122} + a^s_{212} + a^s_{221} - a^s_{222}.$$

So
$$\begin{aligned} &\lim_{s\to\infty} a^s_{222} \\ &= \lim_{s\to\infty} [a^s_{111} - a^s_{112} - a^s_{121} - a^s_{211} + a^s_{122} + a^s_{212} + a^s_{221} - \Delta a^s_{111}] \\ &= a_{111} - a_{112} - a_{121} - a_{211} + a_{122} + a_{212} + a_{221} - y_{111}. \end{aligned}$$

Thus $\lim_{s\to\infty} a^s_{222}$ exists. Proceeding in this way we conclude that $\lim_{s\to\infty} a^s_{ijk}$ exists. Using the continuity of M, we have

$$\lim_{t\to\infty} \sup_{\substack{i,j,k \\ z_1,\cdots,z_{n-1}}} \left[M \left(\left\| \frac{\Delta a^s_{ijk} - \Delta a^t_{ijk}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] \leq 1.$$

Let $s \geq m_0$, then taking infimum of such ρ's we have $g(\langle a^s_{ijk} - a^t_{ijk} \rangle) < \epsilon$. Thus $\langle a^s_{ijk} - a_{ijk} \rangle \in W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$. By linearity of the space $W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$ we have $\langle a_{ijk} \rangle \in W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$. Hence $W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$ is complete. □

Theorem 3.4. *Let M be an Orlicz function and $p = (p_{ijk})$ be bounded triple sequence of strictly positive real numbers. Then*

(i) $W'''(M, \Delta, p, \|\cdot, ..., \cdot\|) \subset W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$

(ii) $W'''_0(M, \Delta, p, \|\cdot, ..., \cdot\|) \subset W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$

Proof. The proof is easy so we omit it. □

Theorem 3.5. *Let M be an Orlicz function and $p = (p_{ijk})$ be bounded triple sequence of strictly positive real numbers. Then the following relation holds*

(i) If $0 < \inf p_{ijk} \leq p_{ijk} < 1$, *then* $W'''(M, \Delta, p, \|\cdot, ..., \cdot\|) \subseteq W'''(M, \Delta, \|\cdot, ..., \cdot\|)$

(ii) If $0 < p_{ijk} \leq \sup p_{ijk} < \infty$, *then* $W'''(M, \Delta, \|\cdot, ..., \cdot\|) \subseteq W'''(M, \Delta, p, \|\cdot, ..., \cdot\|)$

Proof. (i) Let $\langle a_{ijk}\rangle \in W'''(M,\Delta,p,\|\cdot,...,\cdot\|)$; since $0 < \inf\{p_{ijk}\} \leq p_{ijk} < 1$, we have
$\sup_{l,m,n_{z_1,\cdots,z_{n-1}\in X}}$
$\frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]$

$\leq \sup_{l,m,n_{z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}$

$\left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_{ijk}}$
and hence $\langle a_{ijk}\rangle \in W'''(M,\Delta,\|\cdot,...,\cdot\|)$.
(ii) Let $p_{ijk} > 1$ for each (ijk) and $\sup_{i,j,k} p_{ijk} < \infty$. Let $\langle a_{ijk}\rangle \in W'''(M,\Delta,\|\cdot,...,\cdot\|)$. Then, for each $0,\epsilon < 1$, there exists a positive integer $\mathbf{N}$ such that

$$\sup_{\substack{l,m,n\\ z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right] \leq \epsilon < 1,$$

for all $m,n \geq \mathbf{N}$. Since
$\sup_{l,m,n_{z_1,\cdots,z_{n-1}\in X}}$
$\frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$\leq \sup_{l,m,n_{z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]$
Hence $\langle a_{ijk}\rangle \in W'''(M,\Delta,p,\|\cdot,...,\cdot\|)$ which completes the proof. □

Theorem 3.6. *Let M_1 and M_2 be Orlicz functions, then we have*
$W'''_\infty(M_1,\Delta,p,\|\cdot,...,\cdot\|)$
$\cap W'''_\infty(M_2,\Delta,p,\|\cdot,...,\cdot\|)$

$\subseteq W'''_\infty(M_1+M_2,\Delta,p,\|\cdot,...,\cdot\|)$

Proof. Let $\langle a_{ijk}\rangle \in W'''_\infty(M_1,\Delta,p,\|\cdot,...,\cdot\|) \cap W'''_\infty(M_2,\Delta,p,\|\cdot,...,\cdot\|)$. Then

$$\sup_{\substack{l,m,n\\ z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[M_1\left(\left\|\frac{\Delta a_{ijk}}{\rho_1},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_{ijk}} < \infty, \quad \text{for some } \rho_1 > 0$$

and

$$\sup_{\substack{l,m,n \\ z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M_2\left(\left\|\frac{\Delta a_{ijk}}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}} < \infty, \quad \text{for some } \rho_1 > 0$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality
$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}$
$\left[(M_1+M_2)\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$= \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M_1\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) + M_2\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$\leq K\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M_1\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$+K\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M_1\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

□

Theorem 3.7. *The sequence space $W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$ is solid.*

Proof. Let $\langle a_{ijk}\rangle \in W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$, that is

$$\sup_{\substack{l,m,n \\ z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}} < \infty$$

Let (α_{ijk}) be a triple sequence of scalars such that $|\alpha_{ijk}| \leq 1$ for all $i, j, k \in \mathbf{N}$.

Then we get
$\sup_{l,m,n_{z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn}$
$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M\left(\left\|\frac{\Delta \alpha_{ijk} a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$

$\leq \sup_{l,m,n_{z_1,\cdots,z_{n-1}\in X}} \frac{1}{lmn} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \left[M\left(\left\|\frac{\Delta a_{ijk}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_{ijk}}$
and this completes the proof. □

Theorem 3.8. *The sequence space $W'''_\infty(M, \Delta, p, \|\cdot, ..., \cdot\|)$ is monotone.*

Proof. The result is obvious. □

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