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Edge-to-vertex m-detour monophonic number of a graph *

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Abstract

For a connected graph $G = (V, E)$ of order at least three, the monophonic distance $d_m(u, v)$ is the length of a longest $u - v$ monophonic path in G. A $u - v$ path of length $d_m(u, v)$ is called a $u - v$ detour monophonic. For subsets A and B of V, the m-monophonic distance $D_m(A, B)$ is defined as $D_m(A, B) = max\{d_m(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $D_m(A, B)$ is called a $A - B$ m-detour monophonic path joining the sets $A, B \subseteq V$, where $u \in A$ and $v \in B$. A set $S \subseteq E$ is called an edge-to-vertex m-detour monophonic set of G if every vertex of G is incident with an edge of S or lies on a m-detour monophonic path joining a pair of edges of S. The edge-to-vertex m-detour monophonic number $Dm_{ev}(G)$ of G is the minimum order of its edge-to-vertex m-detour monophonic sets and any edge-to-vertex m-detour monophonic set of order $Dm_{ev}(G)$ is an edge-to-vertex m-detour monophonic basis of G. Some general properties satisfied by this parameter are studied. The edge-to-vertex m-detour monophonic number of certain classes of graphs are determined. It is shown that for positive integers r, d and $k \geq 4$ with $r < d$, there exists a connected graph G such that $rad_m(G) = r$, $diam_m(G) = d$ and $Dm_{ev}(G) = k$.

Key Words : *monophonic distance, m-detour monophonic path, edge-to-vertex m-detour monophonic set, edge-to-vertex m-detour monophonic basis, edge-to-vertex m-detour monophonic number.*

AMS Subject Classification : *05C12.*

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [1, 5]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent to v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

The *detour distance* $D(u, v)$ between two vertices u and v in G is the length of a longest $u - v$ path in G. An $u - v$ path of length $D(u, v)$ is called an $u - v$ *detour*. It is known that D is a metric on the vertex set V of G. The closed detour interval $I_D[x, y]$ consists of x, y, and all the vertices in some $x - y$ detour of G. For $S \subseteq V$, $I_D[S]$ is the union of the sets $I_D[x, y]$ for all $x, y \in S$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. The concept of detour number of a graph was introduced in [2, 3] and further studied in [3, 4].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ *detour monophonic path*. A set S of vertices of a graph G is a *detour monophonic set* if each vertex v of G lies on an $x - y$ detour monophonic path for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour monophonic number of a graph was introduced in [9] and further studied in [10].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, rad_mG of G is $rad_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_mG$ of G is $diam_m(G) = \max\{e_m(v) : v \in V(G)\}$. A vertex u in G is a *monophonic eccentric vertex* of a vertex v in G if $e_m(v) = d_m(u, v)$. The monophonic distance was introduced in [6] and further studied in [7].

For subsets A and B of V, the monophonic distance $d_m(A, B)$ is defined as $d_m(A, B) = min\{d_m(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $d_m(A, B)$ is called an $A-B$ detour monophonic path joining the sets $A, B \subseteq V$, where $u \in A$ and $v \in B$. A set $S \subseteq E$ is called an *edge-to-vertex detour* *monophonic set* of G if every vertex of G is incident with an edge of S or lies on a detour monophonic path joining a pair of edges of S. The *edge-to-vertex detour monophonic number* $dm_{ev}(G)$ of G is the minimum order of its edge- to-vertex detour monophonic sets and any edge-to-vertex detour monophonic set of order $dm_{ev}(G)$ is an *edge-to-vertex detour monophonic basis* of G. The edge-to-vertex detour monophonic number of a graph was introduced and studied in [8].

Throughout this paper G denotes a connected graph with at least three vertices.

2. Edge-to-vertex m-detour monophonic number

Definition 2.1. Let $G = (V, E)$ be a connected graph with at least three vertices. For subsets A and B of V, the *m-monophonic distance* $D_m(A, B)$ is defined as $D_m(A, B) = max\{d_m(x, y) : x \in A, y \in B\}$. A $u - v$ detour monophonic path of length $D_m(A, B)$ is called an $A - B$ *m-detour monophonic path* joining the sets A and B, where $u \in A$ and $v \in B$. For $A = \{u, v\}$ and $B = \{z, w\}$ with uv and zw edges, we write an $A - B$ m-detour monophonic path as $uv - zw$ m-detour monophonic path, and $D_m(A, B)$ as $D_m(uv, zw)$.

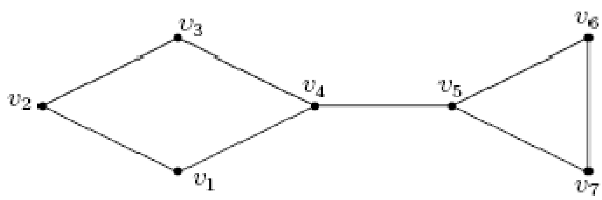

Figure 2.1: G

Example 2.2. For the graph G given in Figure 2.1, with $A = \{v_1, v_2, v_3\}$ and $B = \{v_6, v_7\}$, $P_1 : v_1, v_4, v_5, v_6$ is the only $v_1 - v_6$ detour monophonic path; $P_2 : v_1, v_4, v_5, v_7$ is the only $v_1 - v_7$ detour monophonic path; $P_3 : v_2, v_3, v_4, v_5, v_6$ and $Q_1 : v_2, v_1, v_4, v_5, v_6$ are the only $v_2 - v_6$ detour monophonic paths; $P_4 : v_2, v_3, v_4, v_5, v_7$ and $Q_2 : v_2, v_1, v_4, v_5, v_7$ are the only $v_2 - v_7$ detour monophonic paths; $P_5 : v_3, v_4, v_5, v_6$ is the only $v_3 - v_6$ detour monophonic path; $P_6 : v_3, v_4, v_5, v_7$ is the only $v_3 - v_7$ detour monophonic path. Hence, $d_m(A, B) = 3$ and $D_m(A, B) = 4$. Thus the monophonic distance and m-monophonic distance between two subsets of the vertex set are different. Also, $P_3 : v_2, v_3, v_4, v_5, v_6$, $Q_1 : v_2, v_1, v_4, v_5, v_6$, $P_4 : v_2, v_3, v_4, v_5, v_7$ and $Q_2 : v_2, v_1, v_4, v_5, v_7$ are the only four $A - B$ m-detour monophonic paths.

Definition 2.3. Let $G = (V, E)$ be a connected graph with at least three vertices. A set $S \subseteq E$ is called an *edge-to-vertex m-detour monophonic set* of G if every vertex of G is incident with an edge of S or lies on a m-detour monophonic path joining a pair of edges of S. The *edge-to-vertex m-detour monophonic number $Dm_{ev}(G)$* of G is the minimum cardinality of its edge-to-vertex m-detour monophonic sets and any edge-to-vertex m-detour monophonic set of cardinality $Dm_{ev}(G)$ is an *edge-to-vertex m-detour monophonic basis* of G.

Example 2.4. *For the graph G given in Figure 2.1, the $v_1v_2 - v_6v_7$ m-detour monophonic paths are $P_3 : v_2, v_3, v_4, v_5, v_6$, $Q_1 : v_2, v_1, v_4, v_5, v_6$, $P_4 : v_2, v_3, v_4, v_5, v_7$ and $Q_2 : v_2, v_1, v_4, v_5, v_7$, each of length 4 so that $D_m(v_1v_2, v_6v_7) = 4$. Since every vertex of G is either an internal vertex or an incident with edge of $v_1v_2 - v_6v_7$ m-detour monophonic paths, $S_1 = \{v_1v_2, v_6v_7\}$ is an edge-to-vertex m-detour monophonic basis of G so that $Dm_{ev}(G) = 2$. Also $S_2 = \{v_2v_3, v_6v_7\}$ is an edge-to-vertex m-detour monophonic bases of G. Thus there can be more than one edge-to-vertex m-detour monophonic basis for a graph.*

The following proposition is clear from the fact that an edge-to-vertex m-detour monophonic set needs at least two edges, and the set of all edges of G is an edge-to-vertex m-detour monophonic set of G.

Proposition 2.5. *For any connected graph G of size $q \geq 2$, $2 \leq Dm_{ev}(G) \leq q$.*

For the star $K_1, q(q \geq 2)$, it is clear that the set of all edges is the unique edge-to-vertex m-detour monophonic set so that $Dm_{ev}(K_{1,q}) = q$. The set of two end-edges of a path $P_n(n \geq 3)$ is its unique edge-to-vertex m-detour monophonic basis so that $Dm_{ev}(P_n) = 2$. Thus the bounds in Proposition 2.5 are sharp.

Definition 2.6. An edge e in a graph G is an *edge-to-vertex m-detour monophonic edge* in G if e belongs to every edge-to-vertex m-detour monophonic basis of G. If G has a unique edge-to-vertex m-detour monophonic basis S, then every edge in S is an edge-to-vertex m-detour monophonic edge of G.

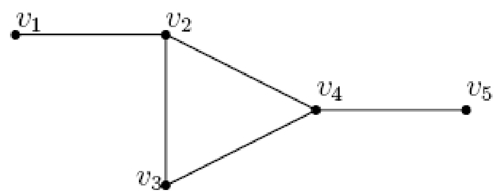

Figure 2.2: G

Example 2.7. *The two end-edges of a path $P_n (n \geq 3)$ is its unique edge-to-vertex m-detour monophonic basis of P_n so that both the end-edges in P_n are edge-to-vertex m-detour monophonic edges of P_n. For the graph G given in Figure 2.2, it is easily verified that no 2-element subset of E is an edge-to-vertex m-detour monophonic set of G. Also, it is clear that $S_1 = \{v_1v_2, v_4v_5, v_2v_3\}$ and $S_2 = \{v_1v_2, v_4v_5, v_3v_4\}$ are the only edge-to-vertex m-detour monophonic bases of G so that the edges v_1v_2, v_4v_5 are the edge-to-vertex m-detour monophonic edges of G.*

An edge of a connected graph G is called an *extreme edge* of G if one of its ends is an extreme vertex of G.

Theorem 2.8. *If v is an extreme vertex of a non-complete connected graph G, then every edge-to-vertex m-detour monophonic set of G contains at least one extreme edge that is incident with v.*

Proof. Let v be an extreme vertex of G. Let $e_1, e_2, \ldots, e_k$ be the edges incident with v. Let S be any edge-to-vertex m-detour monophonic set of G. We claim that $e_i \in S$ for some $i(1 \leq i \leq k)$. Otherwise, $e_i \notin S$ for any $i(1 \leq i \leq k)$. Since S is an edge-to-vertex m-detour monophonic set and the vertex v is not incident with any element of S, v lies on a m-detour monophonic path joining two elements, say $x, y \in S$. Let $x = v_1v_2$ and $y = v_lv_m$. Then $v \neq v_1, v_2, v_l, v_m$ and since G is non-complete, $D_m(x, y) \geq 2$. Let u and w be the neighbors of v on P. Then u and w are not adjacent and so v is not an extreme vertex, which is a contradiction. Therefore, $e_i \in S$ for some $i(1 \leq i \leq k)$. □

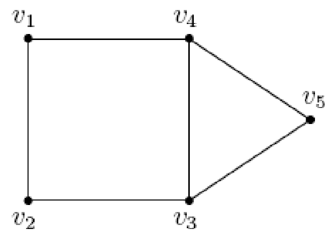

Figure 2.3: G

Remark 2.9. *For the graph G given in Figure 2.3, $S = \{v_1v_2, v_4v_5\}$ is an edge-to-vertex m-detour monophonic set of G, which does not contain the extreme edge v_3v_5. Thus all the extreme edges of a graph need not belong to an edge-to-vertex m-detour monophonic set of G.*

In the following theorem we show that there are certain edges in a connected graph G that are edge-to-vertex m-detour monophonic edges of G.

Corollary 2.10. *All the end-edges of a connected graph G belong to every edge-to-vertex m-detour monophonic set of G. Also if the set S of all end-edges of G is an edge-to-vertex m-detour monophonic set, then S is the unique edge-to-vertex m-detour monophonic basis for G.*

Proof. This follows from Theorem 2.8. If S is the set of all end-edges of G, then by the first part of this corollary $Dm_{ev}(G) \geq |S|$. Since S is an edge-to-vertex m-detour monophonic set of G, $Dm_{ev}(G) \leq |S|$. Hence $Dm_{ev}(G) = |S|$ and S is the unique edge-to-vertex m-detour monophonic basis for G. □

Corollary 2.11. *If T is a tree with k end-edges, then $Dm_{ev}(T) = k$.*

Corollary 2.12. *For any connected graph G with k end-edges, $max\{2, k\} \leq Dm_{ev}(G) \leq q$.*

Proof. This follows from Proposition 2.5 and Corollary 2.10. □

For a cut-vertex v in a connected graph G and a component H of $G-v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v.

Theorem 2.13. *Let G be a connected graph with cut-vertices and S an edge-to-vertex m-detour monophonic set of G. Then every branch of G contains an element of S.*

Proof. Assume that there is a branch B of G at a cut-vertex v such that B contains no element of S. Then by Corollary 2.10, B does not contain any end-edge of G. Hence it follows that no vertex of B is an end-vertex of G. Let u be any vertex of B (note that $|V(B)| \geq 2$). Then u is not incident with any end-edge of G and so u lies on a $e - f$ m-detour monophonic path $P : u_1, u_2, \ldots, u, \ldots, u_t$ where u_1 is an end of e, u_t is an end of f with $e, f \in S$. Since v is a cut-vertex of G, the $u_1 - u$ and $u - u_t$ subpaths of P both contain v and so P is not a path, which is a contradiction. Hence every branch of G contains an element of S. □

Corollary 2.14. *Let G be a connected graph with cut-edges and S an edge-to-vertex m-detour monophonic set of G. Then every branch of G contains an element of S.*

Corollary 2.15. *Let G be a connected graph with cut-edges and S an edge-to- vertex m-detour monophonic set of G. Then for any cut-edge e of G, which is not an end-edge, each component of $G - e$ contains an element of S.*

Proof. Let $e = uv$. Let G_1 and G_2 be the two components of $G - e$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Since u and v are cut-vertices of G, the result follows from Theorem 2.13. □

Corollary 2.16. *If G is a connected graph with $k \geq 2$ end-blocks, then $Dm_{ev}(G) \geq k$.*

Corollary 2.17. *If G is a connected graph with a cut-vertex v and the number of components of $G - v$ is r, then $Dm_{ev}(G) \geq r$.*

Remark 2.18. *By Corollary 2.16, if S is an edge-to-vertex m-detour monophonic set of a graph G, then every end-block of G must contain at least one element of S. However, it is possible that some blocks of G that are not end-blocks must contain an element of S as well. For example, consider the graph G given in Figure 2.2, where the cycle $C_3 : v_2, v_3, v_4$ is a block of G that is not an end-block. By Corollary 2.10, every edge-to-vertex m-detour monophonic set of G must contain v_1v_2 and v_4v_5. Since the $v_1v_2 - v_4v_5$ m-detour monophonic path does not contain the vertex v_3, it follows that*

$\{v_1v_2, v_4v_5\}$ is not an edge-to-vertex m-detour monophonic set of G. Thus every edge-to-vertex m-detour monophonic set of G must contain at least one of the edges v_2v_3 or v_3v_4 from the block C_3.

Theorem 2.19. *Let G be a connected graph with cut-edges. Then no cut-edge which is not an end-edge in G belongs to any edge-to-vertex m-detour monophonic basis of G.*

Proof. Suppose that S is an edge-to-vertex m-detour monophonic basis that contains a cut-edge $e = uv$ which is not an end-edge of G. Let G_1, G_2 be the two components of $G - e$ such that $u \in G_1$ and $v \in G_2$. Then by Corollary 2.15, each of G_1 and G_2 contains an element of S. Let $S' = S - \{uv\}$. We show that S' is an edge-to-vertex m-detour monophonic set of G. Since S is an edge-to-vertex m-detour monophonic set of G and $uv \in S$, let s be any vertex of G that lies on a m-detour monophonic path P joining the edges, say xy and uv of S. We may assume that $xy \in E(G_1)$ and so $V(P) \subseteq V(G_1)$. Let P_1 be the $xy - uv$ m-detour monophonic path that contains the vertex s and let P_2 be any $uv - wz$ m-detour monophonic path in G, where $wz \in E(G_2) \cap S$. Then, since uv is a cut-edge of G, the m-detour monophonic path P_1 followed by the edge uv and the m-detour monophonic path P_2 is an $xy - wz$ m-detour monophonic path which contains the vertex s. Thus it is shown that a vertex that lies on a m-detour monophonic path joining a pair of edges xy and uv of S also lies on a m-detour monophonic path joining a pair of edges xy and wz of S'. Hence it follows that S' is an edge-to-vertex m-detour monophonic set of G. Since $|S'| = |S| - 1$, this contradicts the fact that S is an edge-to-vertex m-detour monophonic basis of G. Hence the proof is complete. □

3. Edge-to-Vertex m-Detour Monophonic Numbers of Some Standard Graphs

Theorem 3.1. *For p even, a set S of edges of $G = K_p (p \geq 4)$ is an edge-to-vertex m-detour monophonic basis of K_p if and only if S consists of $p/2$ independent edges.*

Proof. Let S be any set of $p/2$ independent edges of K_p. Since each vertex of K_p is incident with an edge of S, it follows that $Dm_{ev}(G) \leq p/2$. If $Dm_{ev}(G) < p/2$, then there exists an edge-to-vertex m-detour monophonic set S' of K_p such that $|S'| < p/2$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of S'. For independent edges e and f, $D_m(e,f) = 1$. Hence it follows that v is neither incident with any edge of S' nor lies on a m-detour monophonic path joining a pair of edges of S' and so S' is not an edge-to-vertex m-detour monophonic set of G, which is a contradiction. Thus S is an edge-to-vertex m-detour monophonic basis of K_p.

Conversely, let S be an edge-to-vertex m-detour monophonic basis of K_p. Let S' be any set of $p/2$ independent edges of K_p. Then, as in the first part of this theorem, S' is an edge-to-vertex m-detour monophonic basis of K_p. Therefore, $|S| = p/2$. If S is not independent, then there exists a vertex v of K_p such that v is not incident with any edge of S and it follows that S is not an edge-to-vertex m-detour monophonic set of G, which is a contradiction. Therefore, S consists of $p/2$ independent edges. □

Corollary 3.2. *For the complete graph $K_p(p \geq 4)$ with p even, $Dm_{ev}(K_p) = p/2$.*

For any real x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Theorem 3.3. *For the complete graph $G = K_p(p \geq 3)$ with p odd, $Dm_{ev}(G) = \dfrac{p+1}{2}$.*

Proof. Let S be any set of $\dfrac{p-1}{2}$ independent edges of G. Then there exists a unique vertex v which is not incident with an edge of S. Let S_1 be the union of S and an edge incident with v. Then S_1 is an edge-to-vertex m-detour monophonic set of G so that $Dm_{ev}(G) < \dfrac{p-1}{2} + 1$. Now, if $Dm_{ev}(G) \leq \dfrac{p-1}{2}$, then let S_2 be an edge-to-vertex m-detour monophonic set of G such that $|S_2| \leq \dfrac{p-1}{2}$. Then there exists a vertex u such that u is not incident with any edge of S_2. Obviously, u does not lie on a m-detour monophonic path joining a pair of edges of S_2, which is a contradiction to S_2 an edge-to-vertex m-detour monophonic set of G. Hence $Dm_{ev}(G) = \dfrac{p-1}{2} + 1 = \dfrac{p+1}{2}$. □

Corollary 3.4. *For the complete graph $K_p(p \geq 3)$, $Dm_{ev}(K_p) = \left\lceil \dfrac{p}{2} \right\rceil$.*

Theorem 3.5. *For the cycle $C_p(p \geq 3)$, $Dm_{ev}(C_p) = 2$.*

Proof. It is easily seen that, any two adjacent edges of C_p is an edge-to-vertex m-detour monophonic set of C_p so that $Dm_{ev}(C_p) = 2$. □

4. Monophonic Diameter and Edge-to-Vertex m-Detour Monophonic Number

Theorem 4.1. *For each pair of integers k and q with $2 \leq k \leq q$, there exists a connected graph G of order $q + 1$ and size q with $Dm_{ev}(G) = k$.*

Proof. For $2 \leq k \leq q$, let P be a path of order $q - k + 3$. Then the graph G obtained from P by adding $k - 2$ new vertices to P and joining them to any cut-vertex of P is a tree of order $q + 1$ and size q with k end-edges. Hence by Corollary 2.11, $Dm_{ev}(G) = k$. □

Remark 4.2. *If G is a connected graph of size $q \geq 2$, then by Proposition 2.5, $2 \leq Dm_{ev}(G) \leq q$. Indeed, by Theorem 4.1, for each pair k, q of integers with $2 \leq k \leq q$, there is a tree of size q with edge-to-vertex m-detour monophonic number k. An improved upper bound for the edge-to-vertex m-detour monophonic number of a graph can be given in terms of its size q and detour monophonic diameter. For convenience, we denote the detour monophonic diameter $diam_m(G)$ by d_m itself.*

Theorem 4.3. *If G is a connected graph of size q and monophonic diameter d_m, then $Dm_{ev}(G) \leq q - d_m + 2$.*

Proof. Let u and v be vertices of G such that $d_m(u, v) = d_m$ and let $P : u = v_0, v_1, v_2, \ldots, v_{d_m - 1}, v_{d_m} = v$ be a $u - v$ detour monophonic path of length d_m. Let $S = (E(G) - E(P)) \cup \{uv_1, v_{d_m - 1}v\}$. Then it is clear that S is an edge-to-vertex m-detour monophonic set of G so that $Dm_{ev}(G) \leq |S| = q - d_m + 2$. □

We give below a characterization theorem for trees.

Theorem 4.4. *For any tree T of size $q \geq 2$ and monophonic diameter d_m, $Dm_{ev}(T) = q - d_m + 2$ if and only if T is a caterpillar.*

Proof. Let T be any tree of size $q \geq 2$ and $P : v_0, v_1, \ldots, v_{d_m - 1}, v_{d_m}$ be a monophonic diametral path of T. Let $e_1, e_2, \ldots, e_{d_m - 1}, e_{d_m}$ be the edges of P, where $e_i = v_{i-1}v_i (1 \leq i \leq d_m)$, k the number of end-edges of T and l the number of internal edges of T other than $e_2, \ldots, e_{d_m - 1}$. Then $k + l + d_m - 2 = q$. By Corollary 2.11, $Dm_{ev}(T) = k = q - d_m - l + 2$. Hence $Dm_{ev}(T) = k = q - d_m + 2$ if and only if $l = 0$, if and only if all the internal edges of T lie on the monophonic diametral path P, if and only if T is a caterpillar. □

Corollary 4.5. *For a wounded spider T of size $q \geq 2$, $Dm_{ev}(T) = q - d_m + 2$ if and only if T is obtained from $K_{1,t}(t \geq 2)$ by subdividing at most two of its edges.*

Proof. Since a wounded spider T is a caterpillar if and only if T is obtained from $K_{1,t}(t \geq 2)$ by subdividing at most two of its edges, the result follows from Theorem 4.4. □

For any connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [6] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the edge-to-vertex m-detour monophonic number can be prescribed when $rad_m(G) < diam_m(G)$.

Theorem 4.6. *For positive integers r, d and $k \geq 4$ with $r < d$, there exists a connected graph G such that $rad_m(G) = r$, $diam_m(G) = d$ and $Dm_{ev}(G) = k$.*

Proof. We prove this theorem by considering two cases.
Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ be a cycle of order d+2. Let G be the graph obtained by adding $k - 2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ to C_{d+2} and joining each of the vertices $u_1, u_2, \ldots, u_{k-2}, v_3, v_4, \ldots, v_{d+1}$ to the vertex v_1. The graph G is shown in Figure 4.1. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1, e_m(v_2) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$. Let $S = \{v_1u_1, v_1u_2, \ldots, v_1u_{k-2}\}$ be the set of all pendant edges of G. By Corollary 2.10, S is contained in every edge-to-vertex m-detour monophonic set of G. It is clear that S is not an edge-to-vertex m-detour monophonic set of G. It is also seen that $S \cup \{e\}$, where $e \in E(G) - S$ is not an edge-to-vertex m-detour monophonic set of G. However, the set $S' = S \cup \{v_1v_2, v_1v_{d+2}\}$ is an edge-to-vertex m-detour monophonic set of G so that $Dm_{ev}(G) = k$.

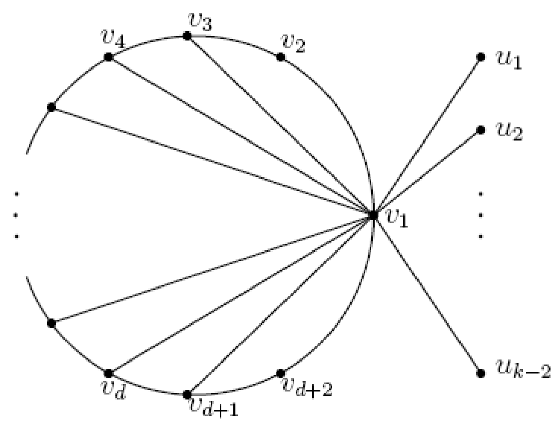

Figure 4.1: G

Case 2. $r \geq 2$. Let $C : v_1, v_2, \ldots, v_{r+2}, v_1$ be a cycle of order $r + 2$ and $W = K_1 + C_{d+2}$ be the wheel with $V(C_{d+2}) = \{u_1, u_2, \ldots, u_{d+2}\}$. Let H be the graph obtained from C and W by identifying v_1 of C and the central vertex K_1 of W. Now, add $k-3$ new vertices $w_1, w_2, \ldots, w_{k-3}$ to the graph H and join each $w_i (1 \leq i \leq k - 3)$ to the vertex v_1 and obtain the graph G of Figure 4.2. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$ and $e_m(u_1) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. Let $S = \{v_1w_1, v_1w_2, \ldots, v_1w_{k-3}\}$ be the set of all pendant edges of G. By Corollary 2.10, every edge-to-vertex m-detour monophonic set of G contains S. It is clear that S is not an edge-to-vertex m-detour monophonic set of G. Also, for any $x, y \in E(H)$, $S \cup \{x\}$ and $S \cup \{x, y\}$ are not edge-to-vertex m-detour monophonic sets of G. Let $T = S \cup \{u_1u_2, u_2u_3, v_2v_3\}$. It is easily verified that T is a minimum edge-to-vertex m-detour monophonic set of G and so $Dm_{ev}(G) = k$. $\square$

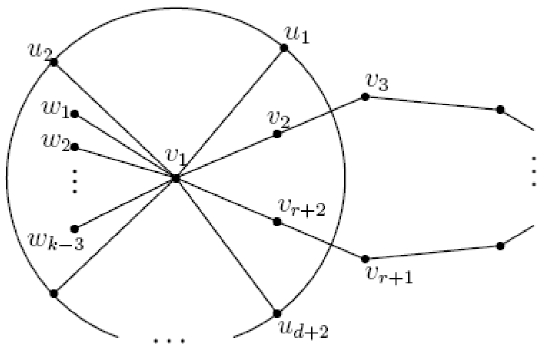

Figure 4.2: G

Problem 4.7. *For any three positive integers r, d and $k \geq 4$ with $r = d$, does there exist a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $Dm_{ev}(G) = k$?*

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