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Cosine families of operators in a class of Fréchet spaces

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Abstract

M. sova [10] proved that the infinitesimal generator of all uniformly continuous cosine family, of operators in Banach space, is a bounded operator. We show by counter-example that the result mentioned above is not true in general on Fréchet spaces, and we prove that the infinitesimal generator of an uniformly continuous cosine family of operators in a class of Fréchet spaces (quojection) is necessarily continuous.

Keywords : *Strongly continuous cosine families. Semi-groups of operators. Locally convex space. Quojection Fréchet space.*

1. Introduction

A strongly continuous cosine family, of bounded operators in Banach space X, appear as solution of the abstract Cauchy problem of second order [2][5] :

$$U'' = AU, \quad U(0) = x, \quad \text{and} \quad U'(0) = 0.$$

A link between the family $\{C(t)\}_{t\in\mathbf{R}}$ and the operator A is given by the Laplace transform : $\lambda(\lambda^2 I - A)^{-1} = \int_0^\infty e^{-\lambda s} C(s)ds$, and A is called the infinitesimal generator of the cosine family $\{C(t)\}_{t\in\mathbf{R}}$ [2][4].

We know, in classical theory, that the infinitesimal generator A of an uniformly continuous cosine family $\{C(t)\}_{t\in\mathbf{R}}$, of bounded operators in X (Banach space), is a bounded operator [4][10], moreover for all $t \in \mathbf{R}$ we have :

$$C(t) = Cosh(t\sqrt{A}) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} A^n.$$

In section 1, We give the definition of cosine family of operators in locally convex spaces, and some propositions important for our results.

An example, of uniformly continuous cosine family, of operators in a Fréchet space, whose infinitesimal generator is a closed operator, densely defined and not continuous on the space everywhere, is given in the second section; and we demonstrate that in the case where the Fréchet space is quojection [Definition 3.1], the infinitesimal generator of all uniformly continuous cosine family of operators is necessarily continuous.

2. Cosine families of operators in locally convex space :

Let X be a locally convex Hausdorff space, and Γ_X a system of continuous semi-norms determining the topology of X. The strong topology τ_s in the space $\mathcal{L}(X)$, of all continuous linear operators from X into itself, is determined by the family of seminorms :

$$q_x(S) = q(Sx), \quad S \in \mathcal{L}(X),$$

for each $x \in X$ and $q \in \Gamma_X$, $\mathcal{L}(X)$ equipped with this topology is noted $\mathcal{L}_s(X)$.

Let $B(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on the elements of $B(X)$ is defined by the family of semi-norms :

$$q_B(S) = \sup_{x \in B} q(Sx), \quad S \in \mathcal{L}(X),$$

for each $B \in B(X)$ and $q \in \Gamma_X$, $\mathcal{L}(X)$ equipped with this topology is noted $\mathcal{L}_b(X)$.

Definition 1. *Let $\{C(t)\}_{t \in \mathbf{R}} \subseteq \mathcal{L}(X)$ be a family of operators verifying the following properties:*

1. $C(0) = I$.
2. $2C(t)C(s) = C(t+s) + C(t-s)$, $\forall s, t \in \mathbf{R}$.

- *We say that $\{C(t)\}_{t \in \mathbf{R}}$ is a strongly continuous cosine family if :*

$$C(t) \longrightarrow C(t_0) \text{ in } \mathcal{L}_s(X), \text{ as } t \longrightarrow t_0, \quad \forall t_0 \in \mathbf{R}.$$

- *we say that $\{C(t)\}_{t \in \mathbf{R}}$ is an uniformly continuous cosine family if :*

$$C(t) \longrightarrow C(t_0) \text{ in } \mathcal{L}_b(X), \text{ as } t \longrightarrow t_0, \quad \forall t_0 \in \mathbf{R}.$$

Definition 2. *Let X be a sequentially complete locally convex Hausdorff space and $\{C(t)\}_{t \in \mathbf{R}}$ be a strongly continuous cosine family on X .*
Let A the operator defined on $D(A)$ by :

$$Ax = \lim_{t \longrightarrow 0} \frac{2}{t^2}(C(t)x - x), \text{ where } D(A) = \{x \in X / \lim_{t \longrightarrow 0} \frac{2}{t^2}(C(t)x - x) \text{ exists in } X\}.$$

A is called the infinitesimal generator of $\{C(t)\}_{t \in \mathbf{R}}$.

Remark 1.

- *According to 2. of the Definition 1., we have $C(.)$ is even.*

  Indeed, for $t = 0$ we have $2C(s) = C(s) + C(-s)$ which implies that $C(s) = C(-s)$, $\forall s \in \mathbf{R}$.

- *For all $t, s \in \mathbf{R}$, we have $C(t)C(s) = C(s)C(t)$.*

  Indeed, $2C(t)C(s)=C(t+s)+C(t\text{-}s)$
  $=C(s+t)+C(s\text{-}t)=2C(s)C(t), \quad \forall t, s \in \mathbf{R}$.

We always denote by X a sequentially complete locally convex Hausdorff space, and Γ_X a system of continuous semi-norms determining the topology of X . We recall that a family $H \subset \mathcal{L}(X)$ is called equicontinuous if for all neighborhood $\mathbf{U}$ of 0, there exists a neighborhood $\mathbf{V}$ of 0 such that $T(\mathbf{V}) \subseteq \mathbf{U}$, $\forall T \in H$ [6]; and we say that a family $\{C(t)\}_{t\in\mathbf{R}} \subset \mathcal{L}(X)$ is locally equicontinuous, if for all $s \in \mathbf{R}^+$, the set $\{C(t), \quad -s \leq t \leq s\}$ is equicontinuous.

Proposition 1. *Let $\{C(t)\}_{t\in\mathbf{R}}$ be a strongly continuous cosine family on X. Then for all $x \in X$ and $t \in \mathbf{R}$ we have :*

$$\lim_{h\longrightarrow 0} \frac{2}{h^2}\int_t^{t+h}(t+h-s)C(s)xds = C(t)x.$$

Proof. Let $t \in \mathbf{R}$, $x \in X$ and $p \in \Gamma_X$. Thus for all $h \in \mathbf{R}_+^*$ we have :

$p(\frac{2}{h^2}\int_t^{t+h}(t+h-s)C(s)xds - C(t)x)$
$= p(\frac{2}{h^2}\int_t^{t+h}(t+h-s)(C(s)x - C(t)x)ds)$

$\leq \frac{2}{h^2}\int_t^{t+h}(t+h-s)p(C(s)x - C(t)x)ds$

$\leq \frac{2}{h^2}\int_t^{t+h}(t+h-s)ds \sup_{s\in[t,t+h]} p(C(s)x - C(t)x)$

$\leq \sup_{s\in[t,t+h]} p(C(s)x - C(t)x)$.

Since $C(t)x$ is continuous on $\mathbf{R}$, Then $\sup_{s\in[t,t+h]} p(C(s)x - C(t)x) \longrightarrow 0$, as $h \longrightarrow 0$.

Similarly, for $h \in \mathbf{R}_-^*$ we have $\sup_{s\in[t+h,t]} p(C(s)x - C(t)x) \longrightarrow 0$, as $h \longrightarrow 0$.
Hence the result.
□

Remark 2. *If $\{C(t)\}_{t\in\mathbf{R}}$ is an uniformly continuous cosine family, then for all $t \in \mathbf{R}$ we have:*

$$\frac{2}{h^2}\int_t^{t+h}(t+h-s)C(s)ds \longrightarrow C(t) \text{ in } \mathcal{L}_b(X), \text{ as } h \longrightarrow 0.$$

Corollary 1. *Let $\{C(t)\}_{t\in\mathbf{R}}$ be a strongly continuous cosine family on X, and A its infinitesimal generator. For all $t \in \mathbf{R}$, and $x \in X$ we have :*

$$\int_0^t (t-s)C(s)xds \in D(A), \text{ and } \quad A\int_0^t (t-s)C(s)xds = C(t)x - x.$$

Proof. Let $h \in \mathbf{R}^*$, for all $x \in X$ we have :
$\frac{2}{h^2}(C(h)-I)\int_0^t (t-s)C(s)xds$
$= \frac{2}{h^2}\int_0^t (t-s)C(h)C(s)xds - \frac{2}{h^2}\int_0^t (t-s)C(s)xds$

$= \frac{1}{h^2}\int_0^t (t-s)C(s+h)xds + \frac{1}{h^2}\int_0^t (t-s)C(s-h)xds$

$-\frac{2}{h^2}\int_0^t (t-s)C(s)xds$

$= \frac{1}{h^2}\int_h^{t+h} (t+h-s)C(s)xds + \frac{1}{h^2}\int_{-h}^{t-h} (t-h-s)C(s)xds$

$-\frac{2}{h^2}\int_0^t (t-s)C(s)xds$

$= \frac{1}{h^2}\int_t^{t+h} (t+h-s)C(s)xds + \frac{1}{h^2}\int_t^{t-h} (t-h-s)C(s)xds$

$-\frac{1}{h^2}\int_0^h (h-s)C(s)xds - \frac{1}{h^2}\int_0^{-h} (-h-s)C(s)xds.$

According to proposition 1. we obtain $\int_0^t (t-s)C(s)xds \in D(A)$, and

$$A\int_0^t (t-s)C(s)xds = C(t)x - x.$$

□

Proposition 2. *Let $\{C(t)\}_{t\in\mathbf{R}}$ be a strongly continuous cosine family on X, and A its infinitesimal generator.*

Then $D(A)$ is dense in X.

Proof. Let $(h_n)_{n\in\mathbf{N}} \subseteq \mathbf{R}^*$ be a sequence such that $h_n \longrightarrow 0$, as $n \longrightarrow \infty$.

Let $x \in X$, put :

$$x_n = \frac{2}{h_n^2}\int_0^{h_n} (h_n - s)C(s)xds.$$

According to corollary 1. we have $(x_n)_n \subseteq D(A)$, and for $t = 0$ in the Proposition 1. we have $x_n \longrightarrow x$, as $n \longrightarrow \infty$.

Then $D(A)$ is dense in X. □

Proposition 3. *If X is tonnelé , Then every strongly continuous cosine family $\{C(t)\}_{t\in\mathbf{R}}$ on X is locally equicontinuous.*

Proof. Let $s > 0$, and $\mathcal{U}$ be any convex circular closed neighborhood of 0, then $\mathcal{V} = \bigcap_{-s \leq t \leq s} (C(t))^{-1}(\mathcal{U})$ is a convex circular closed set.

Let $x \in X$, the set $\{C(t)x, -s \leq t \leq s\}$ is bounded in X, because the strongly continuity of the family $\{C(t)\}_{t \in \mathbf{R}}$ on X; since $\mathcal{U}$ is a neighborhood of 0, $\exists \lambda \geq 0$ such that : $\{C(t)x, -s \leq t \leq s\} \subseteq \lambda\mathcal{U}$. Which implies that $x \in \lambda\mathcal{V}$, this means that $\mathcal{V}$ is absorbing, and $C(t)(\mathcal{V}) \subseteq \mathcal{U}$, $\forall t \in [-s, s]$.

Since X is tonnelé, $\mathcal{V}$ is a neighborhood of 0, which means that $\{C(t)x, -s \leq t \leq s\}$ is equicontinuous. $\square$

Proposition 4. *Let $\{C(t)\}_{t \in \mathbf{R}}$ be a locally equicontinuous strongly continuous cosine family on X. Let x and y in X, then :*
$x \in D(A)$, and $Ax = y$, if and only if, $C(t)x - x = \int_0^t (t-s)C(s)y ds$. $\forall t \in \mathbf{R}$.

Proof. $\Leftarrow$ Evident (**proposition 1**).
$\Rightarrow$ Let $x, y \in X$, such that $Ax = y$.

Firstly, we have for all $t \in \mathbf{R}$, $\int_0^t (t-s)C(s)x ds \in D(A)$ and :

$$C(t)x - x = A \int_0^t (t-s)C(s)x ds = \lim_{h \longrightarrow 0} \int_0^t (t-s)C(s) \frac{2}{h^2}(C(h)x - x) ds.$$

Let $t \in \mathbf{R}_+^*$, since $\{C(t)\}_{t \in \mathbf{R}}$ is locally equicontinuous on X, $\forall p \in \Gamma_X$, $\exists q \in \Gamma_X$, $\exists M \geq 0$ such that :

$$p(C(t)x) \leq Mq(x), \quad \forall s \in [-2t, 2t].(*)$$

Then, for all $h \in \mathbf{R}^*$, with $|h| \leq t$

$p(\int_0^t (t-s)C(s)\frac{2}{h^2}(C(h)x - x)ds)$
$= p(\frac{1}{h^2}\int_t^{t-h}(t-h-s)C(s)x ds + \frac{1}{h^2}\int_t^{t+h}(t+h-s)C(s)x ds$

$-\frac{2}{h^2}\int_0^h (h-s)C(s)x ds)$

$\leq 3 \sup_{[-2t,2t]} p(C(s)x)$.

According to (*), $\forall p \in \Gamma_X$, $\exists q \in \Gamma_X$, $\exists M \geq 0$ we have :

$$p(\int_0^t (t-s)C(s)\frac{2}{h^2}(C(h)x - x)ds) \leq 3Mq(x).$$

Therefore,

$$\lim_{h\longrightarrow 0^+}\int_0^t (t-s)C(s)\frac{2}{h^2}(C(h)x-x)ds$$
$$=\int_0^t (t-s)C(s)\lim_{h\longrightarrow 0}\frac{2}{h^2}(C(h)x-x)ds=\int_0^t (t-s)C(s)yds.$$

Hence,

$$C(t)x-x=\int_0^t (t-s)C(s)yds, \quad t\in\mathbf{R}.$$

□

Corollary 2. *The infinitesimal generator of all locally equicontinuous strongly continuous cosine family on X is closed.*

Proof. Let $(x_n)_{n\in\mathbf{N}}\subseteq D(A)$ such that $\lim_{n\longrightarrow\infty} x_n = x$, and $\lim_{n\longrightarrow\infty} Ax_n = y$.

Let $t\in\mathbf{R}$ fix, then we have $C(t)x_n - x_n = \int_0^t (t-s)C(s)Ax_n ds$, $\forall n\in\mathbf{N}$.

Since $C(t)\in\mathcal{L}(X)$, $\lim_{n\longrightarrow\infty}(C(t)x_n - x_n) = C(t)x - x$.

As $\{C(t)\}_{t\in\mathbf{R}}$ is locally equicontinuous, $\lim_{n\longrightarrow\infty}\int_0^t (t-s)C(s)Ax_n ds = \int_0^t (t-s)C(s)yds$.

Indeed, let $p\in\Gamma_X$ and $t\in\mathbf{R}^+$, we have :

$$\begin{aligned} p(\textstyle\int_0^t (t-s)C(s)Ax_n ds - \int_0^t (t-s)C(s)yds) &= p(\textstyle\int_0^t (t-s)C(s)(Ax_n - y)ds) \\ &\leq \tfrac{t^2}{2}\sup_{0\leq s\leq t} p(C(t)(Ax_n - y)). \end{aligned}$$

Since $\{C(s)\}_{s\in\mathbf{R}}$ is locally equicontinuous, $\exists q\in\Gamma_X$, $\exists M>0$ such that
:

$$p(C(t)x)\leq Mq(x), \quad \forall x\in X \text{ et } \forall s\in[-t,t].$$

Consequently,

$$p(\int_0^t (t-s)C(s)Ax_n ds - \int_0^t (t-s)C(s)yds) \leq \frac{t^2}{2}Mq(Ax_n - y).$$

And since $\lim_{n\longrightarrow\infty} Ax_n = y$, $\frac{t^2}{2}Mq(Ax_n - y)\longrightarrow 0$, as $n\longrightarrow\infty$.

Similarly, we obtain the result for $t \in \mathbf{R}^-$.
Then,

$$C(t)x - x = \int_0^t (t-s)C(s)yds$$

which implies $x \in D(A)$ and $Ax = y$.
Hence, A is closed. $\square$

3. Strongly continuous cosine families in Quojection :

Exapmle 1. *we give an example of uniformly continuous cosine family on Fréchet space whose infinitesimal generator is not everywhere defined.*
A matrix $(a_n(i))_{i,n\in\mathbf{N}}$ of non-negative numbers is called a Köthe matrix if it satisfies the following conditions :

1. *$\forall i \in \mathbf{N}$, $\exists n \in \mathbf{N}$ such that : $a_n(i) > 0$.*
2. *$a_n(i) \leq a_{n+1}(i)$, $\forall i, n \in \mathbf{N}$.*

Let $k = (a_n(i))_{i,n\in\mathbf{N}}$ be a Köthe matrix satisfies :

$$a_n(i) \geq 1 \quad \text{and} \quad \sum_{i\in\mathbf{N}} \frac{a_n(i)}{a_{n+1}(i)} < \infty, \forall n \in \mathbf{N}.$$

.

We consider the space

$$\lambda_1(K) = \{x = (x_i)_{i\in\mathbf{N}} \in \mathbf{C}^{\mathbf{N}} : p_n(x) = \sum_{i\in\mathbf{N}} a_n(i)|x_i| < \infty, \forall n \in \mathbf{N}\}.$$

$\lambda_1(K)$ equipped with the family of semi-norms $\{p_n\}_{n=1}^{\infty}$ is a nuclear Fréchet space [7].

Let $(\mu_i)_{i\in\mathbf{N}}$ be a sequence of real numbers such that each $\mu_i > 0$ and $\lim\limits_{i\longrightarrow\infty} \mu_i = \infty$.

For each $t \in \mathbf{R}$, define a linear operator $C(t)$ on $\lambda_1(K)$ by :

$$C(t)x = (Cos(\sqrt{\mu_i}t)x_i)_{i=1}^{\infty} \quad , \quad x \in \lambda_1(K).$$

Let $x \in \lambda_1(K)$, and $\varepsilon > 0$; for given $n \in \mathbf{N}$, $\exists i_0 \in \mathbf{N}$ such that :

$$\sum_{i>i_0} a_n(i)|x_i| < \frac{\varepsilon}{4}.$$

On the other hand, $\exists \eta > 0$ *such that* $\forall h \in \mathbf{R}^*$, *with* $|h| < \eta$, *we have :*

$$\sum_{i \leq i_0} |cos(\sqrt{\mu_i}(t+h)) - cos(\sqrt{\mu_i}t)|a_n(i)|x_i| < \frac{\varepsilon}{2}.$$

Then, for all $h \in \mathbf{R}^*$, *with* $|h| < \eta$ *we have :*

$$\begin{aligned} p_n(C(t+h)x - C(t)x) &= \textstyle\sum_{i\in\mathbf{N}} |cos(\sqrt{\mu_i}(t+h)) - cos(\sqrt{\mu_i}t)|a_n(i)|x_i| \\ &\leq \tfrac{\varepsilon}{2} + 2\textstyle\sum_{i>i_0} a_n(i)|x_i| \leq \tfrac{\varepsilon}{2} + 2\tfrac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Then $\{C(t)\}_{t\in\mathbf{R}}$ *is a strongly continuous cosine family on* $\lambda_1(K)$, *and hence, it is also uniformly continuous as* $\lambda_1(K)$ *is Montel, since it is nuclear [8].*

Let A *the infinitesimal generator of the family* $\{C(t)\}_{t\in\mathbf{R}}$, *then we have:*

$Ax = (-\mu_i x_i)_{i=1}^{\infty}$, *and* $D(A) = \{x \in \lambda_1(K);$ *tel que* $(-\mu_i x_i)_{i=1}^{\infty} \in \lambda_1(K)\}$.

Indeed, let $x \in D(A)$, *i-e* $\lim\limits_{t\longrightarrow 0} \frac{2}{t^2}(C(t)x - x)$ *exist on* X. *Thus* $\exists y \in \lambda_1(K)$ *such that for all* $n \in \mathbf{N}$ *we have:* $\lim\limits_{t\longrightarrow 0} \sum_{i=0}^{\infty} a_n(i)|(\frac{2}{t^2}(Cos(\sqrt{\mu_i}t)x_i - x_i) - y_i)| = 0$.

Hence $y_i = \lim\limits_{t\longrightarrow 0}(\frac{2}{t^2}(Cos(\sqrt{\mu_i}t)x_i - x_i) = -\mu_i x_i$, $\forall i \in \mathbf{N}$.

Therefore,

$D(A) \subset \{(x_n)_{n\in\mathbf{N}} \in \lambda_1(K); (-\mu_i x_i)_{i\in\mathbf{N}} \in \lambda_1(K)\}$, *and* $A(x_i)_{i\in\mathbf{N}} = (-\mu_i x_i)_{i\in\mathbf{N}}$.

Conversely, let $x \in \lambda_1(K)$ *such that* $(-\mu_i x_i)_{i\in\mathbf{N}} \in \lambda_1(K)$, *let* $n \in \mathbf{N}$ *then for all* $t \in \mathbf{R}^*$ *we have :*

$$p_n(\frac{2}{t^2}(C(t)x - x) + (\mu_i x_i)_{i\in\mathbf{N}}) = \sum_{i=0}^{\infty} a_n(i)|\frac{2}{\mu_i t^2}(cos(\sqrt{\mu_i}t) - 1) + 1||\mu_i x_i|.$$

Put $g(t) = \frac{2(cos(t)-1)}{t^2} + 1$, $t \in \mathbf{R}$. *The function* g *is even, then it is enough to study* g *on* $\mathbf{R}^+$. *Since* $1 - \frac{t^2}{2} \leq cos(t) \leq 1$ *for all* $t \in \mathbf{R}^+$, *which means* $0 \leq g(t) \leq 1$ $\forall t \in \mathbf{R}$.
Hence :

$$p_n(\frac{2}{t^2}(C(t)x - x) + (\mu_i x_i)_{i\in\mathbf{N}}) \leq \sum_{i=0}^{\infty} a_n(i)|\mu_i x_i| < \infty.$$

Therefore,

$$\lim_{t\longrightarrow 0} p_n(\frac{2}{t^2}(C(t)x-x)+(\mu_i x_i)_{i\in\mathbf{N}})$$
$$=\sum_{i=0}^{\infty}\lim_{t\longrightarrow 0}|\frac{2}{\mu_i t^2}(cos(\sqrt{\mu_i}t)-1)+1|a_n(i)|\mu_i x_i|=0.$$

Then

$$\{(x_n)_{n\in\mathbf{N}}\in\lambda_1(K);\ (-\mu_i x_i)_{i\in\mathbf{N}}\in\lambda_1(K)\}\subset D(A)\ ,\ \text{and}$$
$$A(x_i)_{i\in\mathbf{N}}=(-\mu_i x_i)_{i\in\mathbf{N}}.$$

Finally, for each $n\in\mathbf{N}$ we put $\mu_i=\sum_{n=1}^{i}a_n(i)$, $\forall i\in\mathbf{N}$.

we have $\mu_i\longrightarrow\infty$ and the sequence $\frac{1}{\mu}=(\frac{1}{\mu_i})_{i\in\mathbf{N}}\in\lambda_1(K)$, because for all $m\in\mathbf{N}$,

$$\begin{aligned} p_m(\tfrac{1}{\mu})=\textstyle\sum_{i\in\mathbf{N}}a_m(i)\frac{1}{\mu_i} &= \textstyle\sum_{i=1}^{m}a_m(i)\frac{1}{\mu_i}+\sum_{i=m+1}^{\infty}\frac{a_m(i)}{\sum_{n=1}^{i}a_n(i)}\\ &\le \textstyle\sum_{i=1}^{m}a_m(i)\frac{1}{\mu_i}+\sum_{i=m+1}^{\infty}\frac{a_m(i)}{a_{m+1}(i)}<\infty.\end{aligned}$$

but, $(-\mu_i.\frac{1}{\mu_i})_{i\in\mathbf{N}}=(-1)_{i\in\mathbf{N}}\notin\lambda_1(K)$, then $\frac{1}{\mu}\notin D(A)$ and $D(A)\neq\lambda_1(K)$.

Proposition 5. *Suppose that X is a Fréchet space which contains a complemented copy of some nuclear Köthe sequence space $\lambda_1(K)$. Then there exists an equicontinuous, uniformly continuous cosine family in X whose infinitesimal generator is not everywhere defined.*

Proof. Let $P:X\longrightarrow X$ be any projection satisfying $Im(P)=\lambda_1(K)$, and define $Ker(P)=Y$. According to **Example 1.**, for each $t\in\mathbf{R}$, define a linear operator $C_1(t)$ on $\lambda_1(K)$ by :

$$C_1(t)x=(Cos(\sqrt{\mu_i}t)x_i)_{i=1}^{\infty}\quad,\quad x\in\lambda_1(K).$$

The family $\{C_1(t)\}_{t\in\mathbf{R}}$ is an equicontinuous, uniformly continuous cosine family in $\lambda_1(K)$.

Let $A\in\mathcal{L}(Y)$, such that $\{A^n\}_{n=1}^{\infty}$ is equicontinuous in $\mathcal{L}(Y)$. i.e. $\forall p\in\Gamma_X$, $\exists q\in\Gamma_X$, $\exists M\ge 0$, such that :

$$p(A^n x)\le Mq(x),\quad\forall n\in\mathbf{N},\quad\forall x\in X.\qquad(*)$$

for each $t \in \mathbf{R}$, define a linear operator $C_2(t)$ on Y by :

$$C_2(t)x = \frac{e^{-t}e^{tA} + e^{t}e^{-tA}}{2} = \frac{e^{-t}}{2}\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n + \frac{e^{t}}{2}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}A^n \quad , \quad x \in Y.$$

According to $(*)$, the family $\{C_2(t)\}_{t\in\mathbf{R}}$ is an equicontinuous, uniformly continuous cosine family in Y.

Then, the family $\{C(t)\}_{t\in\mathbf{R}}$ of continuous linear operators in X defined via:

$$C(t)x = C_1(t)Px + C_2(t)(I-P)x, \quad t \in \mathbf{R}, \quad x \in X.$$

is an equicontinuous, uniformly continuous cosine family in X whose infinitesimal generator is not everywhere defined. $\square$

Definition 3. *A Fréchet space X is a quojection if it is the projective limit of a projective system of Banach spaces $((X_n, ||.||_n)_{n=1}^{\infty}, (\Pi_n^m)_{n\le m})$, with $\Pi_n^m : X_m \longrightarrow X_n$ is surjective,$\forall m \ge n$.(i.e $X = Proj_n(X_n, \Pi_n^m)$).*

Theorem 1. *Let X be a quojection.*

The infinitesimal generator of every uniformly continuous cosine family is continuous, (i.e. $A \in \mathcal{L}(X)$). Moreover, we have :

$$C(t)x = \sum_{k=1}^{\infty}\frac{t^{2k}}{(2k)!}A^k x, \quad \forall x \in X.$$

Proof. Since X is a quojection, then it is the projective limit of projective system $((X_n, ||.||_n)_{n=1}^{\infty}, (\Pi_n^m)_{n\le m})$ of Banach spaces, with $\Pi_n^m : X_m \longrightarrow X_n$ is surjective, $\forall n \le m \in \mathbf{N}$, and we have $\Pi_m : X \longrightarrow X_m$ is sujective, $\forall m \in \mathbf{N}$.

X is tonnelé, according to **proposition 3.** the family $\{C(t)\}_{t\in\mathbf{R}}$ is locally equicontinuous.

i.e. : for all $t_0 > 0$ fix, $\forall n \in \mathbf{N}$, $\exists m \in \mathbf{N}$, $m \ge n$, $\exists M > 0$ such that :

(3.1) $\quad ||\Pi_n(C(t)x)||_n \le M||\Pi_m(x)||_m, \quad \forall t \in [-t_0, t_0], \quad \forall x \in X.$

On the other hand, for all $0 < |t| \le t_0$ we define the operator :

$$\varphi_t(y) = \frac{2}{t^2}\int_0^t (t-s)C(s)y ds, \quad y \in X.$$

According to **Corollary 1.** we have $\varphi_t(y) \in D(A)$, $\forall y \in X$, and $A\varphi_t(y) = \frac{2}{t^2}(C(t)y - y)$.

Since the family $\{C(t)\}_{t\in \mathbf{R}}$ is uniformly continuous, According to **Remark 2.** we have : $\varphi_t \longrightarrow I$, as $t \longrightarrow 0$, uniformly on bounded subsets of X.

For all $t \in [-t_0, t_0]$, we define the operator $\tilde{\varphi}_t$ on X_m by :

$$\begin{aligned} \tilde{\varphi}_t : \quad & X_m \longrightarrow X_n \quad (m \geq n) \\ & \Pi_m(x) \longrightarrow \Pi_n(\varphi_t(x)). \end{aligned}$$

$\tilde{\varphi}_t$ is continuous, Indeed, let $x \in X$ we have :

$$||\tilde{\varphi}_t(\Pi_m(x))||_n = ||\Pi_n(\varphi_t x)||_n = ||\frac{2}{t^2}\int_0^t (t-s)\Pi_n(C(s)x)ds||_n.$$

We obtain

$$||\tilde{\varphi}_t(\Pi_m(x))||_n \leq M||\Pi_m(x)||_n, \quad \forall t \in [-t_0, t_0].$$

Since X is a quojection, $\exists B \in B(X)$ such that $B_m \subseteq \Pi_m(B)$, with B_m is the unit ball of X_m [3].

Thus, we have :

$$\begin{aligned} \sup_{z\in B_m} ||\tilde{\varphi}_t(z) - \Pi_n^m(z)||_n &\leq \sup_{z\in \Pi_m(B)} ||\tilde{\varphi}_t(z) - \Pi_n^m(z)||_n \\ &\leq \sup_{y\in\ B} ||\tilde{\varphi}_t(\Pi_m(y)) - \Pi_n^m(\Pi_m(y))||_n \\ &\leq \sup_{y\in\ B} ||\Pi_n(\varphi_t(y) - y)||_n \end{aligned}$$

Since $\varphi_t \longrightarrow I$, as $t \longrightarrow 0$, uniformly on bounded subsets of X, $\tilde{\varphi}_t$ is uniformly converges on B_m to Π_n^m.

Since the set of surjective operators is open in $\mathcal{L}(X_m, X_n)$ [9], $\exists t_1 \in \mathbf{R}^*$ such that $\tilde{\varphi_{t_1}}$ is surjective (for small $|t_1|$).

Let $n, m \in \mathbf{N}$, with $m \geq n$, such that (3.1) is verified, and we take $m_0 \geq m$ again with (3.1) satisfies.

Let $x \in X$, then $\exists y \in X$ such that $\Pi_m(x) = \tilde{\varphi_{t_1}}(\Pi_{m_0}(y)) = \Pi_m(\varphi_{t_1}(y))$.

Which implies, $\Pi_n(C(h)x) = \Pi_n(C(h)\varphi_{t_1}(y))$, $h \in [-t_0, t_0]$. In particular, for $h = 0$ we have $\Pi_n(x) = \Pi_n(\varphi_{t_1}(y))$.

Thus, for all $0 < |h| \leq t_0$ we have :

$$\Pi_n(\frac{2}{h^2}(C(h)x - x)) = \Pi_n(\frac{2}{h^2}(C(h)\varphi_{t_1}(y) - \varphi_{t_1}(y))).$$

Since,

$$A(\varphi_t(y)) = \frac{2}{t^2}(C(t)y - y)), \quad \forall y \in X.$$

Then, as $h \longrightarrow 0$ we obtain :

$$\Pi_n(Ax) = \Pi_n(\frac{2}{t_1^2}(C(t_1)y - y)).$$

Since n is arbitrary, A is defined for every $x \in X$. Moreover, A is closed because $\{C(t)\}_{t\in\mathbf{R}}$ is locally equicontinuous.

Hence, A belongs to $\mathcal{L}(X)$.

For each $n \in \mathbf{N}^*$, we define the family $\{C_n(t)\}_{t\in\mathbf{R}}$, of operators, in X_n by :

$$C_n(t)\Pi_n x = \Pi_n C(t)x, \quad t \in \mathbf{R}, \quad x \in X.$$

Each $\{C_n(t)\}_{t\in\mathbf{R}}$, $n \in \mathbf{N}$, form a strongly continuous cosine family in X_n. Actually, it is also uniformly continuous in X_n, indeed, let B_n the unit ball of X_n, then $\exists B \in B(X)$ such that $B_n \subseteq \Pi_n(B)$, and we have :

$$\sup_{x_n\in B_n} ||C_n(t)x_n - x_n||_n \leq \sup_{\Pi_n x\in B_n} ||C_n(t)\Pi_n x - \Pi_n x||_n \leq \sup_{x\in B} ||\Pi_n(C(t)x - x)||_n.$$

Since the family $\{C(t)\}_{t\in\mathbf{R}}$ is uniformly continuous in X, $\{C_n(t)\}_{t\in\mathbf{R}}$ is uniformly continuous in X_n. Hence A_n, the infinitesimal generator of $\{C_n(t)\}_{t\in\mathbf{R}}$, is a bounded operator in X_n, moreover, for each $n \in \mathbf{N}^*$, we have :

$$A_n\Pi_n x = \Pi_n Ax, \quad \forall x \in X.$$

Since X_n is a Banach space, the family $\{C_n(t)\}_{t\in\mathbf{R}}$ is written in the form :

$$C_n(t)\Pi_n x = \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} A_n^k \Pi_n x, \quad \forall x \in X.$$

Therefore,

$$\Pi_n C(t)x = \Pi_n \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} A^k x, \quad \forall x \in X.$$

Since n is arbitrary, and $X = Proj_n(X_n, \Pi_n^m)$,

$$C(t)x = \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} A^k x, \quad \forall x \in X.$$

□

Corollary 3. *Let X be a prequojection, then the infinitesimal generator of every uniformly continuous cosine family is continuous. (i.e. $A \in \mathcal{L}(X)$).*

Proof. Since X is prequojection, X_β^{tt} is a quojection.

Let $\{C(t)\}_{t\in\mathbf{R}}$ be a uniformly continuous cosine family on X, then, according to **Lemma 2.1.** [1], the bi-dual operators $\{C(t)^{tt}\}_{t\in\mathbf{R}}$ form an uniformly continuous cosine family in X_β^{tt}.

Hence, According to **Theorem 1**, the infinitesimal generator of $\{C(t)^{tt}\}_{t\in\mathbf{R}}$, is belong to $\mathcal{L}(X_\beta^{tt})$.

Actually, the infinitesimal generator of $\{C(t)^{tt}\}_{t\in\mathbf{R}}$, noted by A^{tt} ($D(A^{tt}) = X_\beta^{tt}$), is the bi-dual of infinitesimal generator A of $\{C(t)\}_{t\in\mathbf{R}}$.

Since $A^{tt}/_{D(A)} = A$, and $D(A)$ dense in X, it follows that A is also everywhere defined, and $A \in \mathcal{L}(X)$. □

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