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Intuitionistic fuzzy n-normed algebra and continuous product

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Abstract

In this paper we extend the notion of intuitionistic fuzzy n-normed linear space (IFnNLS) to define an intuitionistic fuzzy n-normed algebra (IFnNA). We give a necessary and sufficient condition for an IFnNA to be with continuous product. Further, the concept of multiplicatively continuous product has been introduced and related results have been established. Illustrative examples have been provided in support of our results.

Keywords : *Intuitionistic fuzzy n-normed linear space; intuitionistic fuzzy continuous mapping; intuitionistic fuzzy continuous linear operator.*

AMS Subject Classification : *55M20, 54H25, 47H09.*

1. Introduction

The fuzzy set theory was introduced by Zadeh [30] in 1965, in order to explain the situations where data are imprecise or vague. On the other hand in 1986, a generalized form of fuzzy set namely, intuitionistic fuzzy set was introduced by Atanassaov [1] which deals with both the degree of membership (belongingness) and non-membership (non-belongingness) of an elements within a set.

Situations where the crisp norm is unable to measure the length of a vector accurately, the notion of fuzzy norm happens to be useful. The idea of a fuzzy norm on a linear space was introduced by Katsaras [17] in 1984. In 1992, Felbin [14] introduced an alternative idea of a fuzzy norm whose associated metric is of Kaleva and Seikkala [16] type. In 1994, another notion of fuzzy norm on a linear space was given by Cheng-Moderson [6] whose associated metric is that of Kramosil-Michalek [20] type. Again in 2003, following Cheng and Mordeson, Bag and Samanta [2] introduced another concept of fuzzy normed linear space. In this way, there has been a systematic development of fuzzy normed linear spaces (FNLSs) and one of the important development over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS) [25]. Vijayabalaji and Narayanan [28] extended n-normed linear space to fuzzy n-normed linear space while the concept of IFnNLS was introduced by Vijayabalaji et al [29]. Some more recent work in similar context may be found in [5, 7, 8, 10, 11, 13, 18, 19, 22, 24, 27].

The notion of intuitionistic fuzzy Banach algebra was introduced by Dinda et al. [12]. More work in this direction have been carried out recently in [4, 21].

In the current paper we introduce the notion of IFnNA and investigate the concept of continuous product.

2. Preliminaries

First we recall some basic and preliminary definitions and examples which are related to intuitionistic fuzzy n-normed linear space.

Definition 2.1. *[15] Let $n \in \mathbf{N}$ and X be a real linear space of dimension $d \geq n$ (d may be infinite). A real valued function $\|.\|$ on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ is called an n-norm on X if it satisfies the following properties:*

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent,*

(ii) $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation,*

(iii) $\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ *for any* $\alpha \in \mathbf{R}$,

(iv) $\|x_1, x_2, \ldots, x_{n-1}, y+z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$, *and the pair* $(X, \|.\|)$ *is called an n-normed linear space.*

Definition 2.2. [24] An IFnNLS is the five-tuple $(X, \mu, \nu, *, \circ)$, where X is a linear space over a field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, ν are fuzzy sets on $X^n \times (0, \infty)$, μ denotes the degree of membership and ν denotes the degree of non-membership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, 1)$ satisfying the following conditions for every $(x_1, x_2, \ldots, x_n) \in X^n$ and $s, t > 0$:

(i) $\mu(x_1, x_2, \ldots, x_n, t) + \nu(x_1, x_2, \ldots, x_n, t) \leq 1$,

(ii) $\mu(x_1, x_2, \ldots, x_n, t) > 0$,

(iii) $\mu(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(iv) $\mu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

(v) $\mu(x_1, x_2, \ldots, cx_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,

(vi) $\mu(x_1, x_2, \ldots, x_n, s) * \mu(x_1, x_2, \ldots, x_n^{'}, t) \leq \mu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,

(vii) $\mu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t,

(viii) $\lim_{t \to \infty} \mu(x_1, x_2, \ldots, x_n, t) = 1$ and $\lim_{t \to 0} \mu(x_1, x_2, \ldots, x_n, t) = 0$,

(ix) $\nu(x_1, x_2, \ldots, x_n, t) < 1$,

(x) $\nu(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(xi) $\nu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

(xii) $\nu(x_1, x_2, \ldots, cx_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,

(xiii) $\nu(x_1, x_2, \ldots, x_n, s) \circ \nu(x_1, x_2, \ldots, x_n^{'}, t) \geq \nu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,

(xiv) $\nu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,

(xv) $\lim_{t\to\infty} \nu(x_1, x_2, \ldots, x_n, t) = 0$ and $\lim_{t\to 0} \nu(x_1, x_2, \ldots, x_n, t) = 1$.

$a \wedge b = min\{a, b\}$, $a \cdot b = ab$(usual multiplication in [0,1]) and $a *_L b = max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Definition 2.3. [19] Suppose $*$, $*^{'}$ are t-norms. Then $*^{'}$ dominates $*$ if $(x_1 *^{'} x_2) * (y_1 *^{'} y_2) \leq (x_1 * y_1) *^{'} (x_2 * y_2)$ for all $x_1, x_2, y_1, y_2 \in [0, 1]$ and it is denoted by $*^{'} >> *$.

Definition 2.4. [22] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. We say that a sequence $x = \{x_k\}$ in X is convergent to $l \in X$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbf{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, x_k - L, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu)^n - \lim x = l$ or $x_k \stackrel{(\mu,\nu)^n}{\to} l$ as $k \to \infty$.

Definition 2.5. [22] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then the sequence $x = \{x_k\}$ in X is called a Cauchy sequence with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbf{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) < \epsilon$ for all $k, m \geq k_0$.

Definition 2.6. [3] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then $(X, \mu, \nu, *, \circ)$ is said to be complete if any Cauchy sequence in X is convergent to a point in X. A complete IFnNLS is called a intuitionistic fuzzy Banach space.

Definition 2.7. [8] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. For $t > 0$, we define an open ball $B(x, r, t)$ with centre at $x \in X$, radius $0 < r < 1$ and $y_1, y_2, \ldots, y_{n-1} \in X$ as

$$B(x, r, t) = \{y \in X : \mu(y_1, y_2, \ldots, y_{n-1}, x - y, t) > r \text{ and } \nu(y_1, y_2, \ldots, y_{n-1}, x - y, t) < 1 - r\}$$

Then, $T_{\mu,\nu} = \{T \subset X : x \in T$ if and only if, there exists $t > 0, r \in (0, 1)$ such that $B(x, r, t) \subseteq T\}$
is a topology on X.

Moreover, if the t-norm $*$ and t-conorm $\circ$ satisfies the condition $sup_{x\in(0,1)} x * x = 1$ and $inf_{x\in(0,1)} x \circ x = 0$, then $(X, T_{\mu,\nu})$ is Hausdorff.

The following result is a straightforward conclusion from [24].

Theorem 2.2. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then $(X, T_{\mu,\nu})$ is a metrizable topological vector space.*

3. Main Results

In this section we discuss our main results.

First we define an intuitionistic fuzzy n-normed algebra (IFnNA) and then explain some examples and propositions on the same.

Definition 3.1. *Suppose $(X, \mu_1, \nu_1, *_1, \circ_1)$ and $(Y, \mu_2, \nu_2, *_2, \circ_2)$ are two IFnNLSs. A mapping $T : X \longrightarrow Y$ is said to be intuitionistic fuzzy continuous (if-continuous) at $x_0 \in X$ if, for every $\epsilon > 0$, $\alpha \in (0, 1)$, $y_1, y_2, \dots, y_{n-1} \in X$ and $T(y_1), T(y_2), \dots, T(y_{n-1}) \in Y$ there exist $\delta = \delta(\epsilon, \alpha, x_0) > 0$, $\beta = \beta(\epsilon, \alpha, x_0) \in (0, 1)$, such that for all $x \in X$,*

$$\mu_1(y_1, y_2, \dots, y_{n-1}, x - x_0, \delta) > \beta \text{ and } \nu_1(y_1, y_2, \dots, y_{n-1}, x - x_0, \delta) < 1 - \beta.$$

implies,

$$\mu_2(T(y_1), T(y_2), \dots, T(y_{n-1}), T(x) - T(x_0), \epsilon) > \alpha \text{ and}$$

$$\nu_2(T(y_1), T(y_2), \dots, T(y_{n-1}), T(x) - T(x_0), \epsilon) < 1 - \alpha.$$

If T is if-continuous at each point of X, then T is called if-continuous on X.

Definition 3.2. *The seven-tuple $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ is said to be an IFnNA if*

(i) $*$ and $*_1$ are continuous t-norms and $\circ$ and $\circ_1$ are continuous t-conorms;

(ii) X is an algebra;

(iii) $(X, \mu, \nu, *, \circ)$ is an IFnNLS;

(iv) $\mu(x_1, x_2, \dots, x_{n-1}, xy, ts) \geq \mu(x_1, x_2, \dots, x_{n-1}, x, t) *_1 \mu(x_1, x_2, \dots, x_{n-1}, y, s)$ and $\nu(x_1, x_2, \dots, x_{n-1}, xy, ts) \leq \nu(x_1, x_2, \dots, x_{n-1}, x, t) \circ_1 \nu(x_1, x_2, \dots, x_{n-1}, y, s)$, for all $x_1, x_2, \dots, x_{n-1} \in X$; $x, y \in X$ and $t, s \geq 0$.

If $(X, \mu, \nu, *, \circ)$ is an intuitionistic fuzzy Banach space (IFBS), then $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ will be called intuitionistic fuzzy Banach algebra (IFBA).

Example 3.3. *Suppose $(X, \|\cdot\|)$ is an n-normed algebra, $*$ and $*_1$ are continuous t-norms, $\circ$ and $\circ_1$ are continuous t-conorms and $\mu, \nu : X \times [0, \infty) \longrightarrow [0, 1]$ defined by-*

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, x, t) = \begin{cases} 1, & t \leq \|x\| \\ 0, & t > \|x\| \end{cases}$$

Then $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ is an IFnNA.

Proof. It is a routine verification to prove that $(X, \mu, \nu, *, \circ)$ is an IFnNLS.

Next we have to show that $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ is an IFnNA.

(i) $*$ and $*_1$ are continuous t-norms and $\circ$ and $\circ_1$ are continuous t-conorms;

(ii) As $(X, \|\cdot\|)$ is n-normed algebra, therefore X is an algebra as well;

(iii) $(X, \mu, \nu, *, \circ)$ is an IFnNLS.

Finally, we have to show that

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, ts) \geq \mu(x_1, x_2, \ldots, x_{n-1}, x, t) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, s),$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, xy, ts) \leq \nu(x_1, x_2, \ldots, x_{n-1}, x, t) \circ_1 \nu(x_1, x_2, \ldots, x_{n-1}, y, s).$$

Let $x_1, x_2, \ldots, x_{n-1} \in X$; $x, y \in X$ and $t, s \in [0, \infty)$. If $\|xy\| \geq ts$, then $t \leq \|x\|$ or $s \leq \|y\|$ (if $t > \|x\|$ and $s > \|y\|$, then $ts > \|x\|.\|y\| \geq \|xy\|$, contradiction.)

If $t \leq \|x\|$, then $\mu(x_1, x_2, \ldots, x_{n-1}, x, t) = 0$ and $\nu(x_1, x_2, \ldots, x_{n-1}, x, t) = 1$.

If $s \leq \|y\|$, then $\mu(x_1, x_2, \ldots, x_{n-1}, y, s) = 0$ and $\nu(x_1, x_2, \ldots, x_{n-1}, y, s) = 1$.

Thus,

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, t) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, s) = 0,$$

and
$$\nu(x_1, x_2, \ldots, x_{n-1}, x, t) \circ_1 \nu(x_1, x_2, \ldots, x_{n-1}, y, s) = 1.$$

Therefore,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, ts) \geq \mu(x_1, x_2, \ldots, x_{n-1}, x, t) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, s),$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, xy, ts) \leq \nu(x_1, x_2, \ldots, x_{n-1}, x, t) \circ_1 \nu(x_1, x_2, \ldots, x_{n-1}, y, s).$$

Next, we consider $\|xy\| < ts$, then we have $t > \|x\|$ or $s > \|y\|$.

Then, if $t > \|x\|$, we have $\mu(x_1, x_2, \ldots, x_{n-1}, x, t) = 1$ and $\nu(x_1, x_2, \ldots, x_{n-1}, x, t) = 0$,
and if $s > \|y\|$, we have $\mu(x_1, x_2, \ldots, x_{n-1}, y, s) = 1$ and $\nu(x_1, x_2, \ldots, x_{n-1}, y, s) = 0$.

Thus,

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, t) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, s) = 1$$

and
$$\nu(x_1, x_2, \ldots, x_{n-1}, x, t) \circ_1 \nu(x_1, x_2, \ldots, x_{n-1}, y, s) = 0.$$

Therefore,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, ts) = 1 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, ts) = 0.$$

Thus,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, ts) \geq \mu(x_1, x_2, \ldots, x_{n-1}, x, t) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, s),$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, xy, ts) \leq \nu(x_1, x_2, \ldots, x_{n-1}, x, t) \circ_1 \nu(x_1, x_2, \ldots, x_{n-1}, y, s).$$

Therefore the inequality holds.

Hence $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ is an IFnNA.

$\square$

Example 3.4. *Suppose $(X, \|\cdot\|)$ is a n-normed algebra and $\mu, \nu : X \times [0, \infty) \longrightarrow [0, 1]$ be defined by-*

$$\mu(x_1, x_2, \ldots, x_{n-1}, x_n, t) = \begin{cases} \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}, & t > 0 \\ 0, & t = 0, \end{cases}$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, x, t) = \begin{cases} \frac{\|x_1,x_2,\ldots,x_n\|}{t+\|x_1,x_2,\ldots,x_n\|}, & t > 0 \\ 1, & t = 0. \end{cases}$$

Then $(X, \mu, \nu, \wedge, \vee, \cdot, \cdot)$ *is an IFnNA.*

The next theorem gives a necessary and sufficient condition for an IFnNA to be with continuous product.

Theorem 3.5. *Let* $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ *be an IFnNA. Then* X *is with continuous product if and only if, for all* $\alpha \in (0,1)$ *there exist* $\beta = \beta(\alpha) \in (0,1)$ *and* $M = M(\alpha) > 0$ *such that for all* $x, y \in X$*;* $s, t > 0$ *and* $x_1, x_2, \ldots, x_n \in X$*:*

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \beta \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \beta,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \beta \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1 - \beta.$$

imply

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) < 1 - \alpha.$$

Proof. Suppose $\alpha \in (0,1)$ and $V = \{u \in X : \mu(x_1, x_2, \ldots, x_{n-1}, u, 1) > \alpha$ and $\nu(x_1, x_2, \ldots, x_{n-1}, u, 1) < 1 - \alpha\}$ be an open neighborhood of zero.

Define a function $X \times X \longrightarrow X$ such that $(x, y) \longrightarrow x.y$, where $(x, y) \in X \times X$ and $x.y \in X$.

As this function is continuous at $(0,0)$, there exist $\epsilon_1 = \epsilon_1(\alpha) > 0$, $\epsilon_2 = \epsilon_2(\alpha) > 0$, $\gamma_1 = \gamma_1(\alpha) \in (0,1)$, $\gamma_2 = \gamma_2(\alpha) \in (0,1)$ such that for all $u_1, u_2 \in X$:

$$\mu(x_1, x_2, \ldots, x_{n-1}, u_1, \epsilon_1) > \gamma_1 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, u_1, \epsilon_1) < 1 - \gamma_1,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, u_2, \epsilon_2) > \gamma_2 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, u_2, \epsilon_2) < 1 - \gamma_2.$$

imply

$\mu(x_1, x_2, \ldots, x_{n-1}, u_1.u_2, 1) > \alpha$ and $\nu(x_1, x_2, \ldots, x_{n-1}, u_1.u_2, 1) < 1 - \alpha$.

Let $\beta = max\{\gamma_1, \gamma_2\} \in (0, 1)$, $M = \frac{1}{\epsilon_1.\epsilon_2} > 0$.
Let $x, y \in X$, $s, t > 0$ and $x_1, x_2, \ldots, x_{n-1} \in X$ such that

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \beta \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \beta,$$

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \beta \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1 - \beta.$$

Then

$$\mu(x_1, x_2, \ldots, x_{n-1}, \frac{x}{s}, 1) > \beta \geq \gamma_1 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, \frac{x}{s}, 1) < 1 - \beta \leq 1 - \gamma_1,$$

$$\mu(x_1, x_2, \ldots, x_{n-1}, \frac{y}{t}, 1) > \beta \geq \gamma_2 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, \frac{y}{t}, 1) < 1 - \beta \leq 1 - \gamma_2.$$

Let $u_1 = \frac{\epsilon_1.x}{s}$, $u_2 = \frac{\epsilon_2.y}{s}$. Then we have,

$$\mu(x_1, x_2, \ldots, x_{n-1}, u_1, \epsilon_1) = \mu(x_1, x_2, \ldots, x_{n-1}, \frac{u_1}{\epsilon_1}, 1)$$
$$= \mu(x_1, x_2, \ldots, x_{n-1}, \frac{x}{s}, 1) > \gamma_1,$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, u_1, \epsilon_1)$$
$$= \nu(x_1, x_2, \ldots, x_{n-1}, \frac{u_1}{\epsilon_1}, 1) = \nu(x_1, x_2, \ldots, x_{n-1}, \frac{x}{s}, 1) < 1 - \gamma_1.$$

And,

$$\mu(x_1, x_2, \ldots, x_{n-1}, u_2, \epsilon_2)$$
$$= \mu(x_1, x_2, \ldots, x_{n-1}, \frac{u_2}{\epsilon_2}, 1) = \mu(x_1, x_2, \ldots, x_{n-1}, \frac{y}{t}, 1) > \gamma_2,$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, u_2, \epsilon_2) = \nu(x_1, x_2, \ldots, x_{n-1}, \frac{u_2}{\epsilon_2}, 1) = \nu(x_1, x_2, \ldots, x_{n-1}, \frac{y}{t}, 1) < 1 - \gamma_2.$$

Hence, $\mu(x_1, x_2, \ldots, x_{n-1}, u_1.u_2, 1) > \alpha \Rightarrow \mu(x_1, x_2, \ldots, x_{n-1}, \frac{\epsilon_1 x}{s}.\frac{\epsilon_2 y}{t}, 1) > \alpha$
$\Rightarrow \mu(x_1, x_2, \ldots, x_{n-1}, xy, \frac{st}{\epsilon_1.\epsilon_2} > \alpha$
$\Rightarrow \mu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) > \alpha.$

And

$\nu(x_1, x_2, \ldots, x_{n-1}, u_1.u_2, 1) < 1-\alpha \Rightarrow \nu(x_1, x_2, \ldots, x_{n-1}, \frac{\epsilon_1 x}{s}.\frac{\epsilon_2 y}{t}, 1) < 1-\alpha$
$\Rightarrow \nu(x_1, x_2, \ldots, x_{n-1}, xy, \frac{st}{\epsilon_1.\epsilon_2}) < 1-\alpha$
$\Rightarrow \nu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) < 1-\alpha.$ i.e.

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) < 1-\alpha.$$

Converse Part:

First we will prove that for each $y_0 \in X$ the mapping $X \longrightarrow X$ such that $x \longrightarrow xy_0$, where $x \in X$ and $xy_0 \in X$ is continuous.

Let $\epsilon > 0$, $\alpha \in (0,1)$. Thus there exist $\beta = \beta(\alpha) \in (0,1)$, $M = M(\alpha) > 0$ such that

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \beta \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1-\beta,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \beta \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1-\beta.$$

This implies,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) < 1-\alpha.$$

As, $lim_{t\longrightarrow\infty}\mu(x_1, x_2, \ldots, x_{n-1}, y_0, t) = 1$, there exists $t_0 > 0$ such that

$$\mu(x_1, x_2, \ldots, x_{n-1}, y_0, t_0) > \beta,$$

and as $lim_{t\longrightarrow\infty}\nu(x_1, x_2, \ldots, x_{n-1}, y_0, t) = 0$, there exists $t_0 > 0$ such that

$$\nu(x_1, x_2, \ldots, x_{n-1}, y_0, t_0) < 1-\beta.$$

Let $\delta = \delta(\alpha, \epsilon) = \frac{\epsilon}{t_0.M}$ and $\beta(\alpha, \epsilon) = \beta$.

Let $x \in X$ such that $\mu(x_1, x_2, \ldots, x_{n-1}, x, \delta) > \beta$ and $\nu(x_1, x_2, \ldots, x_{n-1}, x, \delta) < 1-\beta$.

As $\mu(x_1, x_2, \ldots, x_{n-1}, y_0, t_0) > \beta$ and $\nu(x_1, x_2, \ldots, x_{n-1}, y_0, t_0) < 1 - \beta$ we obtain that

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy_0, Mt_0\delta) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy_0, Mt_0\delta) < 1-\alpha.$$

This implies that

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy_0, \epsilon) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy_0, \epsilon) < 1 - \alpha.$$

Similarly, we can establish that, for each $x_0 \in X$, the mapping $y \longrightarrow x_0y$, where $y \in X$ and $x_0y \in X$ is continuous.

Next, we will prove that $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ is with continuous product.

Let $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$. Thus $x_ny_0 \longrightarrow x_0y_0$ and $x_0y_n \longrightarrow x_0y_0$.

Hence,

$$lim_{n\longrightarrow\infty}\mu(x_1, x_2, \ldots, x_{n-1}, x_ny_0 - x_0y_0, s) = 1 \text{ and}$$

$$lim_{n\longrightarrow\infty}\nu(x_1, x_2, \ldots, x_{n-1}, x_ny_0 - x_0y_0, s) = 0,$$

and

$$lim_{n\longrightarrow\infty}\mu(x_1, x_2, \ldots, x_{n-1}, x_0y_n - x_0y_0, t) = 1 \text{ and}$$

$$lim_{n\longrightarrow\infty}\nu(x_1, x_2, \ldots, x_{n-1}, x_0y_n - x_0y_0, t) = 0,$$

for all $s, t > 0$.

Therefore,

$\mu(x_1, x_2, \ldots, x_{n-1}, x_ny_n - x_0y_0, t) = \mu(x_1, x_2, \ldots, x_{n-1}, (x_n - x_0)(y_n - y_0) + (x_n - x_0)y_0 + x_0(y_n - y_0), t)$
$\geq \mu(x_1, x_2, \ldots, x_{n-1}, (x_n-x_0)(y_n-y_0), \frac{t}{3}) * \mu(x_1, x_2, \ldots, x_{n-1}, (x_n-x_0)y_0, \frac{t}{3})$
$* \mu(x_1, x_2, \ldots, x_{n-1}, x_0(y_n - y_0), \frac{t}{3})$
$\geq (\mu(x_1, x_2, \ldots, x_{n-1}, x_n - x_0, \frac{\sqrt{t}}{\sqrt{3}}) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y_n - y_0, \frac{\sqrt{t}}{\sqrt{3}}))$
$* \mu(x_1, x_2, \ldots, x_{n-1}, (x_n - x_0)y_0, \frac{t}{3}) * \mu(x_1, x_2, \ldots, x_{n-1}, x_0(y_n - y_0), \frac{t}{3})$
$\longrightarrow 1.$

And

$\nu(x_1, x_2, \ldots, x_{n-1}, x_ny_n - x_0y_0, t)$
$= \nu(x_1, x_2, \ldots, x_{n-1}, (x_n - x_0)(y_n - y_0) + (x_n - x_0)y_0 + x_0(y_n - y_0), t)$
$\leq \nu(x_1, x_2, \ldots, x_{n-1}, (x_n-x_0)(y_n-y_0), \frac{t}{3}) \circ \nu(x_1, x_2, \ldots, x_{n-1}, (x_n-x_0)y_0, \frac{t}{3}) \circ$
$\nu(x_1, x_2, \ldots, x_{n-1}, x_0(y_n - y_0), \frac{t}{3})$
$\leq (\nu(x_1, x_2, \ldots, x_{n-1}, x_n - x_0, \frac{\sqrt{t}}{\sqrt{3}}) \circ_1 \nu(x_1, x_2, \ldots, x_{n-1}, y_n - y_0, \frac{\sqrt{t}}{\sqrt{3}})) \circ$
$\nu(x_1, x_2, \ldots, x_{n-1}, (x_n - x_0)y_0, \frac{t}{3}) \circ \nu(x_1, x_2, \ldots, x_{n-1}, x_0(y_n - y_0), \frac{t}{3})$
$\longrightarrow 0.$

Hence, $x_ny_n \longrightarrow x_0y_0$.

Hence X is with continuous product. $\square$

Lemma 3.6. *Any continuous t-norm $*$ and t-conorm $\circ$ satisfy:*

for all $\gamma \in (0,1)$ there exist $\alpha, \beta \in (0,1)$ such that $\alpha * \beta = \gamma$ and $(1-\alpha) \circ \beta = 1-\gamma$.

Theorem 3.7. *Let $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ be any IFnNA. Then X is with continuous product.*

Proof. Let $\alpha \in (0,1)$. Then there exists $\epsilon > 0$ such that $\alpha + \epsilon \in (0,1)$ and $(1-\alpha) - \epsilon \in (0,1)$.

As $*_1$ is a continuous t-norm and $\circ_1$ is a continuous t-conorm by lemma 3.6, we obtain that there exist $\beta_\alpha, \gamma_\alpha \in (0,1)$ such that

$$\alpha + \epsilon = \beta_\alpha *_1 \gamma_\alpha \text{ and } (1-\alpha) - \epsilon = (1-\beta_\alpha) \circ_1 \gamma_\alpha$$

We suppose that $\beta_\alpha \geq \gamma_\alpha$ and $(1-\beta_\alpha) \leq \gamma_\alpha$ (the case $\beta_\alpha \leq \gamma_\gamma$ is similar).

We choose $M = M(\alpha) = 1$.

Let $x, y \in X$, $s, t > 0$ and $x_1, x_2, \ldots, x_{n-1} \in X$ such that

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \beta_\alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \beta_\alpha,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \beta_\alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1 - \beta_\alpha.$$

Then

$\mu(x_1, x_2, \ldots, x_{n-1}, xy, Mst)$
$\geq \mu(x_1, x_2, \ldots, x_{n-1}, x, s) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, t)$
$\geq \beta_\alpha *_1 \beta_\alpha$
$\geq \beta_\alpha *_1 \gamma_\alpha$
$= \alpha + \epsilon$
$> \alpha$.

And

$\nu(x_1, x_2, \ldots, x_{n-1}, xy, Mst)$
$\leq \mu(x_1, x_2, \ldots, x_{n-1}, x, s) \circ_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, t)$
$< (1-\beta_\alpha) \circ_1 (1-\beta_\alpha)$
$< (1-\beta_\alpha) \circ_1 \gamma_\alpha$
$= (1-\alpha) - \epsilon$
$< 1 - \alpha$.

Hence X is with continuous product.

□

Definition 3.8. *Let $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ be the IFnNA. Then X is said to be with multiplicatively continuous product if for all $\alpha \in (0,1)$; $x, y \in X$; $s, t > 0$ and $x_1, x_2, \ldots, x_{n-1} \in X$:*

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \alpha,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1 - \alpha.$$

imply,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, st) \geq \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, st) \leq 1 - \alpha.$$

Example 3.9. *(IFnNA with multiplicatively continuous product) Let $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ be the IFnNA from the example 3.3 is with multiplicatively continuous product.*

Proof. Let $\alpha \in (0,1)$, for all $x, y \in X$, for all $x_1, x_2, \ldots, x_{n-1} \in X$, and $s, t > 0$ such that

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \alpha,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1 - \alpha.$$

Then,

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) = 1 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) = 0,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) = 1 \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) = 0.$$

Thus we have,
$\|x\| < s$, $\|y\| < t$.
Therefore, $\|xy\| \leq \|x\|.\|y\| < st$.
Hence,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, st) = 1 > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, st) = 0 < 1 - \alpha.$$

Therefore X is with multiplicatively continuous product. $\square$

Example 3.10. *(IFnNA which is not with multiplicatively continuous product) We consider the IFnNA from example 3.4 where $X = \mathbf{R}$ and the norm on X is the absolute value $\|\cdot\|$. Then $(\mathbf{R}, \mu, \nu, *, \circ, *_1, \circ_1)$ is not with multiplicatively continuous product.*

Proof. Suppose, $\alpha = \frac{1}{7}$, $1 - \alpha = \frac{6}{7}$, $x = \frac{7}{2}s$, $y = \frac{7}{2}t$ and $s, t > 0$.
Then we have,

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) = \frac{s}{s + \|x\|} = \frac{s}{s + \frac{7}{2}s} = \frac{2}{9} > \frac{1}{7}.$$

i.e.

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \alpha,$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, x, s) = \frac{\|x\|}{s + \|x\|} = \frac{\frac{7}{2}s}{s + \frac{7}{2}s} = \frac{7}{9} < \frac{6}{7}.$$

i.e.

$$\nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \alpha$$

.

But,

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, st) = \frac{st}{st + \|xy\|} = \frac{st}{st + \frac{7}{2} \cdot \frac{7}{2}st} = \frac{4}{53} < \frac{1}{7} = \alpha,$$

and

$$\nu(x_1, x_2, \ldots, x_{n-1}, xy, st) = \frac{\|xy\|}{st + \|xy\|} = \frac{\frac{49}{4}st}{st + \frac{49}{4}st} = \frac{49}{53} > \frac{1}{7} = 1 - \alpha.$$

i.e.

$$\mu(x_1, x_2, \ldots, x_{n-1}, xy, st) < \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, xy, st) > 1 - \alpha.$$

Thus $(\mathbf{R}, \mu, \nu, *, \circ, *_1, \circ_1)$ is not with multiplicatively continuous product. □

Theorem 3.11. *Let $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ be the IFnNA such that $\alpha *_1 \alpha \geq \alpha$ and $(1 - \alpha) \circ_1 (1 - \alpha) \leq 1 - \alpha$ for all $\alpha \in (0, 1)$. Thus $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ is with multiplicatively continuous product.*

Proof. Let $\alpha \in (0,1)$, for all $x, y \in X$, for all $x_1, x_2, \ldots, x_{n-1} \in X$, and $s, t > 0$ such that

$$\mu(x_1, x_2, \ldots, x_{n-1}, x, s) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, x, s) < 1 - \alpha,$$

and

$$\mu(x_1, x_2, \ldots, x_{n-1}, y, t) > \alpha \text{ and } \nu(x_1, x_2, \ldots, x_{n-1}, y, t) < 1 - \alpha.$$

Then,

$\mu(x_1, x_2, \ldots, x_{n-1}, xy, st) \geq \mu(x_1, x_2, \ldots, x_{n-1}, x, s) *_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, t)$
$> \alpha *_1 \alpha$
$\geq \alpha.$
and
$\nu(x_1, x_2, \ldots, x_{n-1}, xy, Mst) \leq \mu(x_1, x_2, \ldots, x_{n-1}, x, s) \circ_1 \mu(x_1, x_2, \ldots, x_{n-1}, y, t)$
$< (1 - \alpha) \circ_1 (1 - \alpha)$
$\leq 1 - \alpha.$

Thus X is with multiplicatively continuous product. □

Remark 3.12. *Theorem 3.11 is sufficient but not necessary.*

Theorem 3.13. *Let $(X, \mu_1, \nu_1, *, \circ, *_1, \circ_1)$ and $(Y, \mu_2, \nu_2, *, \circ, *_1, \circ_1)$ be two IFnNAs. If t-norm $*'$ denotes both $*$ and $*_1$ and t-conorm $\circ'$ denotes both $\circ$ and $\circ_1$, then $((X \times Y), \mu, \nu, *', \circ', *_1, \circ_1)$ is a IFnNA where,*

$$\mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x, y), t)$$

$$= \mu_1(x_1, x_2, \ldots, x_{n-1}, x, t) *' \mu_2(y_1, y_2, \ldots, y_{n-1}, y, t),$$

and

$$\nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x, y), t)$$

$$= \nu_1(x_1, x_2, \ldots, x_{n-1}, x, t) \circ' \nu_2(y_1, y_2, \ldots, y_{n-1}, y, t),$$

for all $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}) \in X \times Y$, $x_1, x_2, \ldots, x_{n-1} \in X$, $y_1, y_2, \ldots, y_{n-1} \in Y$, $(x, y) \in X \times Y$ and $s, t > 0$.

Proof. We have to prove that

$\mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(xx^{'},yy^{'}),st)$
$\geq \mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x,y),s)$
$*_1\, \mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x^{'},y^{'}),t)$,
and
$\nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(xx^{'},yy^{'}),st)$
$\leq \nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x,y),s)$
$\circ_1\, \nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x^{'},y^{'}),t)$,
for all $(x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}) \in X\times Y$, $x,x^{'}\in X$, $y,y^{'},\in Y$, and $s,t>0$.

Now,
$\mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(xx^{'},yy^{'}),st)$
$= \mu_1(x_1,x_2,\dots,x_{n-1},xx^{'},st) *^{'} \mu_2(y_1,y_2,\dots,y_{n-1},yy^{'},st)$
$\geq (\mu_1(x_1,x_2,\dots,x_{n-1},x,s) *_1 \mu_1(x_1,x_2,\dots,x_{n-1},x^{'},t)) *^{'}$
$(\mu_2(y_1,y_2,\dots,y_{n-1},y,s) *_1 \mu_2(y_1,y_2,\dots,y_{n-1},y^{'},t))$
$\geq [\mu_1(x_1,x_2,\dots,x_{n-1},x,s) *^{'} \mu_2(y_1,y_2,\dots,y_{n-1},y,s)] *_1$
$[\mu_1(x_1,x_2,\dots,x_{n-1},x^{'},t) *^{'} \mu_2(y_1,y_2,\dots,y_{n-1},y^{'},t)]$
$= \mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x,y),s) *_1$
$\mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x^{'},y^{'}),t)$.

Therefore,
$\mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(xx^{'},yy^{'}),st)$
$\geq \mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x,y),s)$
$*_1\, \mu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x^{'},y^{'}),t)$.

And
$\nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(xx^{'},yy^{'}),st)$
$= \nu_1(x_1,x_2,\dots,x_{n-1},xx^{'},st) \circ^{'} \nu_2(y_1,y_2,\dots,y_{n-1},yy^{'},st)$
$\leq (\nu_1(x_1,x_2,\dots,x_{n-1},x,s) \circ_1 \nu_1(x_1,x_2,\dots,x_{n-1},x^{'},t)) \circ^{'}$
$(\nu_2(y_1,y_2,\dots,y_{n-1},y,s) \circ_1 \nu_2(y_1,y_2,\dots,y_{n-1},y^{'},t))$
$\leq [\nu_1(x_1,x_2,\dots,x_{n-1},x,s) \circ^{'} \nu_2(y_1,y_2,\dots,y_{n-1},y,s)] \circ_1$
$[\nu_1(x_1,x_2,\dots,x_{n-1},x^{'},t) \circ^{'} \nu_2(y_1,y_2,\dots,y_{n-1},y^{'},t)]$
$= \nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x,y),s) \circ_1$
$\nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x^{'},y^{'}),t)$.

Therefore,
$\nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(xx^{'},yy^{'}),st)$
$\leq \nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x,y),s)$
$\circ_1\, \nu((x_1,y_1),(x_2,y_2),\dots,(x_{n-1},y_{n-1}),(x^{'},y^{'}),t)$.

Therefore $X\times Y$ is an IFnNA.

□

Theorem 3.14. *Let* $*_1$ *be a t-norm and* $\circ_1$ *be a t-conorm satisfying* $\alpha *_1 \alpha \geq$

α and $(1-\alpha)\circ_1(1-\alpha) \leq (1-\alpha)$ for all $\alpha \in (0,1)$ and let $(X, \mu_1, \nu_1, *, \circ, *_1, \circ_1)$ and $(Y, \mu_2, \nu_2, *, \circ, *_1, \circ_1)$ be two IFnNAs with multiplicatively continuous product. If a t-norm $*^{'}$ denotes both $*$ and $*_1$ and t-conorm $\circ^{'}$ denotes both $\circ$ and $\circ_1$, then $((X \times Y), \mu, \nu, *^{'}, \circ^{'}, *_1, \circ_1)$ is a IFnNA with multiplicatively continuous product.

Proof. Let $\alpha \in (0,1)$, $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}) \in X \times Y$,$(x, y) \in X \times Y$,$(x^{'}, y^{'}) \in X \times Y$ and $s, t > 0$ such that

$$\mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x, y), s) > \alpha;$$
$$\nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x, y), s) < 1 - \alpha.$$

and

$$\mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x^{'}, y^{'}), t) > \alpha;$$
$$\nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x^{'}, y^{'}), t) < 1 - \alpha.$$

Then we have successively:

$\mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (xx^{'}, yy^{'}), st)$
$= \mu_1(x_1, x_2, \ldots, x_{n-1}, xx^{'}, st) *^{'} \mu_2(y_1, y_2, \ldots, y_{n-1}, yy^{'}, st)$
$\geq (\mu_1(x_1, x_2, \ldots, x_{n-1}, x, s) *_1 \mu_1(x_1, x_2, \ldots, x_{n-1}, x^{'}, t)) *^{'}$
$(\mu_2(y_1, y_2, \ldots, y_{n-1}, y, s) *_1 \mu_2(y_1, y_2, \ldots, y_{n-1}, y^{'}, t))$
$\geq [\mu_1(x_1, x_2, \ldots, x_{n-1}, x, s) *^{'} \mu_2(y_1, y_2, \ldots, y_{n-1}, y, s)] *_1$
$[\mu_1(x_1, x_2, \ldots, x_{n-1}, x^{'}, t) *^{'} \mu_2(y_1, y_2, \ldots, y_{n-1}, y^{'}, t)]$
$= \mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x, y), s) *_1$
$\mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x^{'}, y^{'}), t)$
$> \alpha *_1 \alpha$
$\geq \alpha.$

i.e.

$$\mu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (xx^{'}, yy^{'}), st) \geq \alpha.$$

And

$\nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (xx^{'}, yy^{'}), st)$
$= \nu_1(x_1, x_2, \ldots, x_{n-1}, xx^{'}, st) \circ^{'} \nu_2(y_1, y_2, \ldots, y_{n-1}, yy^{'}, st)$
$\leq (\nu_1(x_1, x_2, \ldots, x_{n-1}, x, s) \circ_1 \nu_1(x_1, x_2, \ldots, x_{n-1}, x^{'}, t)) \circ^{'}$
$(\nu_2(y_1, y_2, \ldots, y_{n-1}, y, s) \circ_1 \nu_2(y_1, y_2, \ldots, y_{n-1}, y^{'}, t))$
$\leq [\nu_1(x_1, x_2, \ldots, x_{n-1}, x, s) \circ^{'} \nu_2(y_1, y_2, \ldots, y_{n-1}, y, s)] \circ_1$
$[\nu_1(x_1, x_2, \ldots, x_{n-1}, x^{'}, t) \circ^{'} \nu_2(y_1, y_2, \ldots, y_{n-1}, y^{'}, t)]$
$= \nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x, y), s) \circ_1$

$\nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x^{'}, y^{'}), t)$
$< (1-\alpha) \circ_1 (1-\alpha)$
$\leq (1-\alpha).$
i.e.

$$\nu((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (xx^{'}, yy^{'}), st) \leq (1-\alpha).$$

Therefore $X \times Y$ is an IFnNA with multiplicatively continuous product.
□

Example 3.15. *Let $(X, \mu, \nu, *, \circ, *_1, \circ_1)$ be an IFnNA with multiplicatively continuous product and let $S \subset X$ be a linear closed algebra of X. Then $(S, \mu, \nu, *, \circ, *_1, \circ_1)$ is an IFnNA with multiplicatively continuous product.*

4. Conclusion

In this paper we have introduced the notion of IFnNA and given a characterization for an IFnNA to be with continuous product. Further, the notion of multiplicatively continuous product has also been introduced and studied. The spectral properties of an IFnNLS will be an interesting topic for future research in this regard which may be applied in the study of dynamical systems and particle physics.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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