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ON TOPOLOGICAL PROPERTIES OF SKOROKHOD'S INTEGRAL *

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Abstract

We study continuity properties of Skorokhod's integral following the definition of such an integral in [4].

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1. Notations – Preliminaries

1. We begin with some notations. The basic probability space is the set of continuous functions $\Omega = C([0, 1], \mathbf{R})$ endowed with the uniform topology. Over Ω we define the canonical projections $W_t(w) = w(t)$, and the associated σ -algebras $\mathbf{F}_t = \sigma(\mathbf{W}_s; s \leq t)$ ($t \in [0, 1]$). Let $\mathbf{F}$ denote the Borel σ -algebra of Ω , and let $\mathbf{P}$ denote the Wiener measure on $(\Omega, \mathbf{F})$ with which the process W becomes a Brownian motion.

It is well known that the space $L^2(\Omega, \mathbf{F}, \mathbf{P})$ may be decomposed into a direct sum of symmetrical tensor products $\bigoplus_{n \geq 0} H^{n\odot}$, where H is the

Hilbert space defined as the L^2 -closure of the vector space spanned by finite linear combinations $\sum_{i=1}^N a_i W_{t_i}$. This is the so-called chaos expansion of L^2 .

The chaotic representation leads to the consideration of the following special class of random variables. First we let $C_p^\infty(\mathbf{R}^m)$ denote the set of all infinitely differentiable functions $f : \mathbf{R}^m \rightarrow \mathbf{R}$ such that f together with all of its derivatives have polynomial growth order. Next we define the set of smooth random variables $\mathcal{S}$ consisting of all $F : \Omega \rightarrow \mathbf{R}$ for which there exist a finite collection $t_1, \dots, t_m \in [0, 1]$ and $f \in C_p^\infty(\mathbf{R}^m)$ such that

$$(1.1) \quad F(w) = f(W_{t_1}(w), \dots, W_{t_m}(w)).$$

By this chaotic representation we have that $\mathcal{S}$ is dense in $L^2(\Omega, \mathbf{F}, \mathbf{P})$.

2. For every $F \in \mathcal{S}$, the derivative DF is defined to be the process $DF = (D_t F)_{t \geq 0}$ given by

$$(1.2) \quad D_t F = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_m}) \cdot I_{[0, t_i]}(t)$$

where $I_{[0, t_i]}$ stands for the indicator function of the interval $[0, t_i]$, with $t \in [0, 1]$.

3. For every $u \in L^2([0, 1] \times \Omega)$, we define

$$(1.3) \quad N(u) = \inf\{c > 0 : |\mathbf{E}(\int_0^1 D_t F u_t dt)| \leq c \|F\|_{L^2(\Omega)}; F \in \mathcal{S}\}$$

where $\|\cdot\|_p$ is the norm in $L^p(\Omega, \mathbf{F}, \mathbf{P})$. We adopt the convention $\inf \phi = \infty$.

Now we introduce the domain of Skorokhod's integral

$$(1.4) \quad \text{Dom}(D^*) = \{u \in L^2([0, 1] \times \Omega) : N(u) < \infty\}.$$

The Skorokhod integral of $u \in \text{Dom}(D^*)$ is then characterized as the unique element D^*u in $L^2(\Omega)$ such that

$$(1.5) \quad \mathbf{E}(FD^*u) = \mathbf{E} \left(\int_0^1 D_t F u_t dt \right),$$

for all $F \in \mathcal{S}$.

4. We next introduce a weak topology which will be associated to the construction of Skorokhod's integral. Given a sequence $\{Z_n; n \geq 1\}$ in $L^2(\Omega)$, we say it converges weakly along $\mathcal{S}$ towards a limit Z_∞ , and write $Z_n \xrightarrow{\mathcal{W}(\mathcal{S})} Z_\infty$, if $\mathbf{E}(FZ_n)$ goes to $\mathbf{E}(FZ_\infty)$, for all $F \in \mathcal{S}$.

It is worth noting that if $Z_n \xrightarrow{\mathcal{W}(\mathcal{S})} Z_\infty$ and $\sup_n \|Z_n\|_{L^2(\Omega)} < \infty$, then $\{Z_n; n \geq 1\}$ converges weakly to Z_∞ in $L^2(\Omega)$. Thus the $\mathbf{W}(\mathcal{S})$ -topology is weaker than the usual weak topology $\mathbf{w}$ in $L^2(\Omega)$.

5. Let $\mathbf{D}$ denote the $\mathbf{D}$ closure of $\mathcal{S}$ according to the norm $\|\cdot\|_{1,2}$ where

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|D.F\|_{L^2([0,1] \times \Omega)}.$$

We say a sequence $\{Z_n; n \geq 1\}$ in $L^2(\Omega)$ is $\mathcal{W}(\mathcal{S})$ -weakly uniformly continuous at zero if

$$\lim_{\|F\|_{1,2} \downarrow 0} \sup_{n \geq 1} |\mathbf{E}(FZ_n)| = 0.$$

On the other hand, we will say the sequence $\{Z_n; n \geq 1\} \subset L^2$ is $\mathcal{W}(\mathcal{S})$ -weakly bounded if

$$\sup_{n \geq 1} |\mathbf{E}(FZ_n)| < \infty$$

for all F in $\mathcal{D}$.

2. Main Results

1. Proposition. *The function N is a norm over $\text{Dom}(D^*)$. The topology defined by N is called the natural topology in $\text{Dom}(D^*)$.*

Proof : It is clear that N is a seminorm. Hence if $N(u) = 0$, then $\mathbf{E}(\int_0^1 D_t F u_t dt) = 0$. But this means

$$(2.1) \quad \mathbf{E}(\int_0^1 (\sum_{i=1}^m \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_m}) I_{[0, t_i]}(t) u_t) dt) = 0$$

where $t_1, \dots, t_m$ vary over $[0, 1]$ and f runs over $C_p^\infty(\mathbf{R}^m)$. In particular, letting f vary over the set of polynomials, we have

$$(2.2) \quad \mathbf{E}(\int_0^1 \sum_{i=1}^m \varphi_i \otimes \psi_i(w, t) u_t(w) dt) = 0,$$

where $\varphi_i(w) = P(W_{t_1}(w), \dots, W_{t_m}(w))$ runs over polynomial functions and $\psi_i(t) = I_{[0, t_i]}(t)$. Since the φ_i 's are total in $L^2(\Omega)$ (the Wiener chaos property), and since the ψ_i 's have the same property in $L^2([0, 1])$, the tensor product of both lead to a total family in $L^2([0, 1] \times \Omega)$. Therefore (2.2) implies $u = 0$ almost surely with respect to the product measure $dt d\mathbf{P}$ on $[0, 1] \times \Omega$

We now prove some results with relation to the convergence of Skorohod's integral.

2. Theorem. *Let $\text{Dom}(D^*)$ be the domain of Skorohod's integral which we endow with the natural topology. Then the operator D^* from $\text{Dom}(D^*)$ into $L^1(\Omega, \mathbf{F}, \mathbf{P})$ is continuous. Furthermore, for every $u \in \text{Dom}(D^*)$, the following inequality holds*

$$(2.3) \quad \|D^*u\|_{L^1(\Omega)} \leq 2N(u).$$

Proof : Note that if we prove (2.3), the first statement follows immediately. For the second, let A^+ (resp. A^-) denote the set of all $w \in \Omega$ such that $D^*u(w) > 0$ (resp. $D^*u(w) < 0$). Though to define A^+ and A^-

we should choose a representative function in the class D^*u , to simplify superfluous technicalities we identify sets which are almost surely equal.

Since $\mathcal{S}$ is dense in $L^2(\Omega)$, there exists a sequence (F_n^+) (resp. (F_n^-)) of smooth random variables which approximates I_{A^+} (resp. I_{A^-}) in the L^2 -norm. Therefore

$$\begin{aligned} \mathbf{E}(|D^*u|) &= \mathbf{E}(I_A + D^*u) - \mathbf{E}(I_A - D^*u) \\ &= \lim_n \left(\mathbf{E} \left(\int_0^1 D_t F_n^+ u_t dt \right) \right) - \lim_m \left(\mathbf{E} \left(\int_0^1 D_t F_m^- u_t dt \right) \right) \\ &\leq \lim_n \left(\left| \mathbf{E} \left(\int_0^1 D_t F_n^+ u_t dt \right) \right| + \left| \mathbf{E} \left(\int_0^1 D_t F_n^- u_t dt \right) \right| \right) \\ &\leq N(u) \lim_n (\|F_n^+\|_2 + \|F_n^-\|_2) \\ &\leq 2N(u) \end{aligned}$$

□

3. Proposition. *Let $\text{Dom}(D^*)$ be endowed with the natural topology, and let $L^2(\Omega)$ be endowed with the $\mathcal{W}(\mathcal{S})$ -topology. Then the operator D^* from $\text{Dom}(D^*)$ into $L^2(\Omega)$ is continuous. Furthermore, given any sequence $\{u^n; n \geq 1\}$ in $\text{Dom}(D^*)$ such that it converges to $u \in \text{Dom}(D^*)$ in the natural topology and that $\sup_n \|D^*(u^n)\|_{L^2(\Omega)} < \infty$, we have that $D^*(u^n) \xrightarrow{w} D^*(u)$.*

Proof : Given $\{u^n; n \geq 1\} \subset \text{Dom}(D^*)$ such that $u^n \xrightarrow{N} u$ and $F \in \mathcal{S}$ arbitrary, we have

$$\begin{aligned} |\mathbf{E}(FD^*(u^n)) - \mathbf{E}(FD^*(u))| &= \left| \mathbf{E} \int_0^1 D_t F (u_t^n - u_t) dt \right| \\ &\leq \|F\|_{L^2(\Omega)} N(u^n - u) \end{aligned} \quad \square$$

and the result now follows.

Remark that the above result still holds if we consider the $L^2(\Omega \times [0, 1])$ -topology and use the Cauchy-Schwarz inequality.

4. Corollary *Let u be a process such that $uI_{[0,t]} \in \text{Dom}(D^*)$, for all $t \in [0, 1]$. Then there exists a sequence of semimartingales $(X_n)_{n \geq 1}$ such that $X_n(t)$ converges in the $\mathcal{W}(\mathcal{S})$ -topology to the integral $\int_0^t u(s) dW(s) = D^*(u1_{[0,t]})$, for all $t \in [0, 1]$.*

Proof : Since $u \in L^2(\Omega \times [0, 1])$, there exists a sequence of processes u_n that approaches u in the L^2 -norm of the form

$$u_n(t) = F_0 1_{\{0\}} + \sum_{i=1}^{N_n} F_i^n I_{[t_i^n, t_{i+1}^n]}(t)$$

where the F 's are smooth random variables and $\pi_n : 0 = t_0^n < t_1^n < \dots < t_{N_n}^n = 1$ is a sequence of partitions of $[0, 1]$ such that $|\pi_n| = \sup_i |t_{i+1}^n - t_i^n| \rightarrow 0$ as $n \rightarrow \infty$.

Each F_i^n is a function of a Brownian vector of the form $(W_{\alpha_1^{n,i}}, \dots, W_{\alpha_{m(n,i)}^{n,i}})$. Fix n and arrange all the α 's in a single finite ordered sequence, denoted $p(n)$. Set $\mathcal{H}_n = \sigma(W_\alpha; \alpha \in p(n))$. All the variables F_i^n are now $\mathcal{H}_n$ -measurable as well as their Malliavin derivatives.

Define $\mathcal{F}_t^n = \cap_{s>t} (\mathcal{F}_s \cup \mathcal{H}_n)$, for all $t \in [0, 1]$. The process u_n now becomes predictable with respect to the new filtration $\mathcal{F}_t^n$. In addition, K. Itô proved that W becomes a semimartingale with respect to the new filtration. According to the anticipative calculus, $u_n I_{[0,t]}$ may be integrated in the Skorokhod sense and we have

$$D^*(u_n 1_{[0,t]}) = (u \cdot W)(t) - \int_0^t D_s u_n(s) ds$$

where the dot $\cdot$ is used to denote stochastic integration of the simple $(\mathcal{F}_t^n)$ -predictable process u_n with respect to the semimartingale W . Remark that the second integral on the right-hand side of the above equation is adapted to $(\mathcal{F}_t^n)$ as well. So that, for each n , the process defined as $X_n(t) = D^*(u_n I_{[0,t]})$ is an $(\mathcal{F}_t^n)$ -semimartingale. Since u_n converges to u in $L^2(\Omega \times [0, 1])$, the result follows from the continuity of D^* in the $\mathcal{W}(\mathcal{S})$ -topology.

Finally we show a characterization of the $\mathcal{W}(\mathcal{S})$ -weak relative compactness.

For a sequence $\{Z_n; n \geq 1\}$ in $L^2(\Omega, \mathbf{F}, \mathbf{P})$, we can now prove the following two results.

5. Proposition. *The sequence $\{Z_n; n \geq 1\}$ is $\mathcal{W}(\mathcal{S})$ -weakly uniformly continuous at zero if the sequence $\mathbf{E}(\mathbf{F} Z_n)$ converges as n tends to infinity, for all $F \in \mathbf{D}$.*

Proof : Fix $\alpha > 0$ and define

$$L_j = \{F \in \mathcal{D} : \forall n, m \geq j, \quad |\mathbf{E}(F(Z_n - Z_m))| \leq \alpha\}.$$

The sets L_j are closed in $\mathcal{D}$ and the hypothesis of the proposition implies that the union $\cup_{j \geq 1} L_j$ is equal to $\mathcal{D}$. By Baire's theorem there exists a closed set L_j with interior nonempty: that is, that there exist an index $j \geq 1$, a real number $r > 0$, and a random variable $F_0 \in L_j$ such that

$$\|G - F_0\|_{1,2} < r \text{ implies } : G \in L_j ;$$

that is, $|\mathbf{E}(G(Z_n - Z_m))| \leq \alpha, \quad n, m \geq j$.

Consequently if $n \geq j$ and F is such that $\|F\|_{1,2} < r$, then

$$\begin{aligned} |\mathbf{E}(FZ_n)| &= |\mathbf{E}((F + F_0)Z_n) - \mathbf{E}(F_0Z_n)| \\ &\leq |\mathbf{E}((F + F_0)Z_n) - \mathbf{E}((F + F_0)Z_j)| \\ &\quad + |\mathbf{E}(FZ_j)| + |\mathbf{E}(F_0Z_n) - \mathbf{E}(F_0Z_j)| \\ &\leq 2\alpha + |\mathbf{E}(FZ_j)|. \end{aligned}$$

Therefore

$$\sup_{n \geq 1, \|F\|_{1,2} \leq r} |\mathbf{E}(FZ_n)| \leq 2\alpha + \sup_{n \leq k \leq j, \|F\|_{1,2} \leq r} |\mathbf{E}(FZ_k)|$$

converges to zero as r tends to zero. The proof is now complete.

Using this proposition we can provide necessary and sufficient conditions for a sequence of square integrable random variables to be weakly relatively compact in the sense of the $\mathcal{W}(\mathcal{S})$ -topology.

6. Theorem. *A sequence of square integrable random variables*

$\{Z_n; n \geq 1\}$ is $\mathcal{W}(\mathcal{S})$ -weakly relatively compact if and only if the following conditions are satisfied:

- (i) *The sequence $\{Z_n; n \geq 1\}$ is $\mathcal{W}(\mathcal{S})$ -weakly uniformly bounded.*
- (ii) *The sequence $\{Z_n; n \geq 1\}$ is $\mathcal{W}(\mathcal{S})$ -weakly uniformly continuous at zero.*
- (iii) *For any sequence $\{Z_{n(k)}; n \geq 1\}$ such that the limit $\lim_{k \rightarrow \infty} \mathbf{E}(FZ_{n(k)})$ exists, for all $F \in \mathcal{D}$, we have $\left| \lim_{k \rightarrow \infty} \mathbf{E}(FZ_{n(k)}) \right| \leq c \|F\|_2$.*

Proof : We first show that the above three conditions are necessary for the weak $\mathcal{W}(\mathcal{S})$ -relative compactness of the sequence $\{Z_n; n \geq 1\}$. Assume that condition (i) does not hold. In this case we can find a sequence

$\{Z_{n(k)}; k \geq 1\}$ such that, for some random variable $F \in \mathcal{D}$, we have that $|\mathbf{E}(FZ_{n(k)})|$ increases to infinity. Then the sequence $\{Z_{n(k)}; k \geq 1\}$ does not have any $\mathcal{W}(\mathcal{S})$ -weakly convergent subsequence.

If condition (ii) is not true, any subsequence $\{Z_{n(k)}; k \geq 1\}$ is not $\mathcal{W}(\mathcal{S})$ -weakly uniformly continuous at zero, and by Proposition 5 it cannot be $\mathcal{W}(\mathcal{S})$ -weakly convergent.

Clearly condition (iii) is satisfied, since Definition 1.5 implies that all limit points of the $\mathcal{W}(\mathcal{S})$ -weakly convergent subsequence must be square integrable random variables.

Conversely, we now assume that conditions (i) truth (iii) are satisfied. Fix a countable and dense subset $\{F_j; j \geq 1\}$ on $\mathcal{D}$. Using condition (i) and the classical diagonal argument we can find a subsequence $\{Z_{n(k)}; k \geq 1\}$ such that $\mathbf{E}(F_j Z_{n(k)})$ converges as k tends to infinity, for all $j \geq 1$. The limit, denoted $\phi(F_k)$, is continuous in the $\|\cdot\|_{1,2}$ - norm by condition (ii). Hence the sequence $\mathbf{E}(F Z_{n(k)})$ converges, for all $F \in \mathcal{D}$, and the limit is continuous in the L_2 - norm by condition (iii). $\square$

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