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A NOTE ON THE NEWTON METHOD FOR DEGREE FOUR POLYNOMIALS *

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Abstract

We study the Newton method for finding roots of real polynomial equations of degree four from a global and a dynamics point of view. We describe some representative families of four degree polynomials that contain all of the significative features of the dynamics.

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1. Introduction

Given that f is differentiable, the Newton method is a first candidate for a numerical solution of the equation $f(x) = 0$.

Let $x_0 \in \mathbf{R}$ be an initial guess for a zero of f . Compute

$$x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

which defines a discrete dynamical system called the Newton method associated to f ; N_f will be called the Newton map (also associated to f). The orbits of the dynamics of the Newton method are attracted to stable fixed points. Typically, the sequence $(x_n)_{n=0,1,2,\dots}$ will converge to a zero x^* of f and the local convergence is of second order, if x^* is a zero of f such that $f'(x^*) \neq 0$. The Newton method appears repeatedly in numerical analysis and in discrete dynamical systems literature (see references in [3]). Barna (1953) proved that, for real polynomials of degree n , $n \geq 4$, such that all roots are real and distinct, the set of exceptional points of the Newton method, where Newton map under iteration fails to converge to a fixed point which corresponds to a root of f , is homeomorphic to the union of a Cantor set and a countable set. The countable set arises from a Cantor process that is driven by the existence of two or more 2-cycles of the Newton method. Cubic polynomials with three distinct roots have a Newton map with a single 2-cycle; in this case the exceptional set is a countable set. Quadratic polynomials with two distinct roots have a Newton map with trivial dynamics.

We study the dynamical behavior of the Newton method for polynomials of degree four based on the description of generic families of four degree polynomials given in [2].

2. Basic Features on the Newton Method and Results for Two and Three Degree Polynomials

We give a brief revision of basic features on the Newton method as well as results on the dynamics of the Newton map for two and three degree polynomials.

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a C^r function, $r \geq 1$. The Newton map associated to f is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \text{when } f'(x) \neq 0.$$

Since $N'_f(x) = \frac{f(x)f''(x)}{(f'(x))^2}$, we have that critical points of $N'_f(x) = 0$ are either roots or inflection points of f . We also have that $N_f(x) = x$ if and only if $f(x) = 0$; that is, the fixed points of N_f are the roots of f . On the other hand, if x_0 is a fixed point of N_f , we have that $N'_f(x_0) = \frac{m-1}{m}$ where m the multiplicity of the root x_0 of f . Thus if x_0 is a simple root of f , then x_0 is a superattractive fixed point of N_f ; that is, $N'_f(x_0) = 0$. Therefore the convergence of the iterated $N_f^n(x) = N_f \circ N_f \circ \cdots \circ N_f(x)$ (n times) is at least quadratic in a neighborhood of x_0 .

We recall that a topological conjugacy between two maps $f, g : \mathbf{R} \rightarrow \mathbf{R}$ is a homeomorphism $\tau : \mathbf{R} \rightarrow \mathbf{R}$ such that $f \circ \tau = \tau \circ g$. It is well known that conjugacies preserve all significative features of the dynamics.

For two and three degree polynomials we have:

- 1.- Let $f(x) = ax^2 + bx + c$, $a \neq 0$, be a quadratic polynomial. If f has two distinct roots, then $\tau(x) = 2ax + b$ is a topological conjugacy between N_f and N_q , where $q(x) = x^2 - A$ and $A = b^2 - 4ac$. Also, we have that $\mathcal{W}^s(-\sqrt{A}) =]-\infty, 0[$ and that $\mathcal{W}^s(\sqrt{A}) =]0, \infty[$, where $\mathcal{W}^s(p) = \{x \in \mathbf{R} : \mathbf{N}_f^n(x) \rightarrow p \text{ as } n \rightarrow \infty\}$ is the attraction basin of the attractive fixed point p . And if f does not have real roots, then N_q has a chaotic behavior in $]-\infty, 0[\cup]0, \infty[$ (see [1]).
- 2.- Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, be a cubic polynomial. Then

$$N_f(x) = \frac{2ax^3 + bx^2 - d}{3ax^2 + 2bx + c}.$$

Let $C = 2b^3 - 9a(bc - 3ad)$, and $B = 9ac - 2b^2$. Let $g(x) = x^3 + Bx + C$, and $\tau(x) = 3ax + b$. Then $\tau \circ N_f = N_g \circ \tau$; that is, N_f and N_g are topologically conjugated by the conjugacy τ .

Now let $p(x) = x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$. Then $N_p(x) = N_f(x)$. Let $f(x) = x^3 + ax + b^3$, $b \neq 0$, and let $g(x) = x^3 + cx + 1$, where $c = \frac{a}{b^2}$. Then the homeomorphism $\tau(x) = \frac{x}{b}$ is a conjugacy between N_f and N_g .

Summarizing, let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, $g(x) = x^3 + Cx + \lambda^3$, and $h(x) = x^3 + Dx + 1$, where $B = 9ac - 3b^3$, $C = 2b^3 - 9a(bc - 3ad)$. If $B \neq 0$, let $\lambda = \sqrt[3]{B}$, and let $D = \frac{C}{\lambda^2}$. Then the homeomorphism $\tau_1(x) = 3ax + b$ is a conjugacy between N_f and

N_g , and the homomorphism $\tau_2(x) = \frac{x}{\lambda}$ is a conjugacy between N_g and N_h . Therefore, if $h(x) = x^3 + \mu x + 1$, where

$$\mu = \frac{C}{\left(\sqrt[3]{B}\right)^2} = \frac{2b^3 - 9a(bc - 3ad)}{\left(\sqrt[3]{9ac - 3b^2}\right)^2}$$

is a parameter, then N_h is conjugated to N_f .

Therefore, in order to understand the dynamics of the Newton map of f , it is suffices to understand the dynamics of the Newton map of $h(x) = x^3 + \mu x + 1$.

Now let $k_\gamma(x) = x^3 + \gamma x$, and let $N_\gamma(x) = \frac{2x^3}{3x^2 + \gamma}$ be its Newton map. Then the dynamics of N_f is conjugated to the dynamics of N_h provided $b \neq 0$, or to the dynamics of N_γ provided $b = 0$.

Let us now consider the following three maps:

- (i) $p_+(x) = x^3 + x$;
- (ii) $p_-(x) = x^3 - x$;
- (iii) $p_0(x) = x^3$.

If $\gamma = 0$, $k_0(x) = p_0(x)$. If $\gamma > 0$, setting $\theta = \sqrt{\gamma}$, we have that $k_\gamma(x) = k_\theta(x) = x^3 + \theta^2 x$. Finally, if $\gamma < 0$, setting $\theta = \sqrt{|\gamma|}$, $k_\gamma(x) = k_\theta(x) = x^3 - \theta^2 x$. Thus, if $\theta \neq 0$, then $N_\theta(x) = \frac{2x^3}{3x^2 \pm \theta}$.

Let $\tau(x) = \frac{x}{\theta}$. Then $\tau^{-1} \circ N_{p_\pm} \circ \tau(x) = \frac{2x^3}{\theta(3x^2 \pm \theta^2)}$; that is, if $k_{\theta_\pm}(x) = x^3 \pm \theta^2 x$, the homeomorphism (C^∞ diffeomorphism) $\tau(x) = \frac{x}{\theta}$ is a conjugacy between $N_{\theta_\pm}$ and $N_{p_\pm}$ (see [1]).

2.1. Main Results

Let $P(x) = ax^4 + bx^3x + cx^2x^2 + dx + e$, $a \neq 0$, be a quartic polynomial. Let $\tau: \mathbf{R} \rightarrow \mathbf{R}$ be given by $\tau(x) = x - \frac{b}{4a}$, $c_0 = -\frac{b}{4a}$. Then

$$P_1(x) = P \circ \tau(x) = P\left(x - \frac{b}{4a}\right) = \tilde{b}_0 + \frac{D_1}{8a^2}x - \frac{D_2}{8a}x^2 + ax^4,$$

where $\tilde{b}_0 = P(c_0)$, $D_1 = b^3 - 4acb + 8a^2d$ and $D_2 = 3b^2 - 8ca$.

We have the following

Proposition 1. Let τ be as above. Then τ is conjugacy between N_P and N_{P_1} .

Proof.

$$\begin{aligned}\tau^{-1} \circ N_P \circ \tau(x) &= \tau^{-1} \left(\tau(x) - \frac{P(\tau(x))}{P'(\tau(x))} \right) \\ &= \tau(x) - \frac{P(\tau(x))}{P'(\tau(x))\tau'(x)} + \frac{b}{4a} \\ &= x - \frac{P \circ \tau(x)}{(P \circ \tau)'(x)} \\ &= N_{P_1}(x) .\end{aligned}$$

Proposition 2. Let $\sigma : \mathbf{R} \rightarrow \mathbf{R}$, $\sigma(x) = c_1x$, $c_1 \neq 0$, and let $P_2(x) = P_1 \circ \sigma(x) = P_1(c_1x) = P(c_1x + c_0)$. Then σ is a conjugacy between N_{P_1} and N_{P_2} (i.e., $N_{P_1}(x) = \sigma \circ N_{P_2} \circ \sigma^{-1}(x)$).

Proof. It is a straightforward computation.

Remark. If $f(x) = b\varphi(x)$, where $b \neq 0$ is a constant then $N_f = N_\varphi$, i.e., the identity map is a conjugacy between N_f and N_φ .

Theorem 1. Let $P_3(x) = b_1P_2(x) = b_1P_1(c_1x) = b_1P(c_1x + c_0)$. Then

$$P_3(x) = b_1P(c_1x + c_0) = x^4 - \frac{D_2}{8a^2 + c_1^4}x^2 + \frac{D_1}{8a^3c_1^3}x + b_1P(c_0) ,$$

and $N_{P_2} = N_{P_3}$;

i.e., the following diagram commutes

Proof. It is a straightforward computation.

Now for the Newton map associated to each polynomial in the above commutative diagram, we have the following commutative diagram

Thus we have proven the following

Theorem 2. *Let P and P_3 be as above. Then N_P is conjugated to N_{P_3} .*

We have that any four degree polynomial P may be reduced to

$$P_3(x) = b_1 P(c_1 x + c_0) = x^4 - \frac{D_2}{8a^2 c_1^2} x^2 + \frac{D_1}{8a^3 c_1^3} x + b_0$$

Now the dynamics of N_P and N_{P_3} are conjugated, thus by Theorem 2 it suffices to describe the dynamics of N_{P_3} . Note that P_3 depends on two parameters D_1 and D_2 , and hence we have the following families of polynomials of degree four whose Newton maps contain all of the significative features of the Newton maps of four degree polynomials.

- (1) If $D_1 = D_2 = 0$, then $P_3(x) = x^4 + b_0$;
- (2) If $D_1 = 0$ and $D_2 > 0$, choosing $c_1 = \sqrt{\frac{D_2}{8a^2}}$, we obtain $P_3(x) = x^4 - x^2 + b_0$;
- (3) If $D_1 = 0$ and $D_2 < 0$, choosing $c_1 = \sqrt{-\frac{D_2}{8a^2}}$, we obtain $P_3(x) = x^4 + x^2 + b_0$;
- (4) If $D_1 \neq 0$, choosing $c_1 = \frac{\sqrt[3]{D_1}}{2a}$, we obtain $P_3(x) = x^4 + rx^2 + x + b_0$, where

$$r = -\frac{D_2}{2\sqrt[3]{D_1^2}}$$

is a parameter.

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