ASYMPTOTIC BEHAVIOR OF THE WEIGHTED TRACE OF SCHRÖDINGER EQUATION WITH OPERATOR COEFFICIENT GIVEN IN n -DIMENSIONAL SPACE

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Abstract

We show that an operator formed by Schrödinger differential expression with operator coefficient on separable Hilbert space has a pure discrete spectrum. We also discuss the asymptotic behavior of the weighted trace of this operator.

Keywords : *Hilbert space, Self-adjoint operator, Spectrum, Tauberian theorem.*

1. Introduction

Let $f(x) = f(x_1, x_2, \dots, x_n)$ be a function which values belong to separable Hilbert space H . We denote the space of functions $f(x)$ defined in n -dimensional ($n \geq 1$) Euclidean space E_n by $H_1 = L_2(E_n, H)$, where the functions $f(x)$ are measurable in the sense of Bochner given in [17] and

$$\|f\|_1^2 = \int_{E_n} \|f(x)\|^2 dx < \infty.$$

Here $\|f\|$ denotes the norm in H space. If we define the inner product in H_1 space with the formula

$$(f, g)_1 = \int_{E_n} (f(x), g(x)) dx, \quad f(x), g(x) \in H_1$$

then $H_1 = L_2(E_n, H)$ space becomes also Hilbert space, where $(., .)$ denotes the inner product in H . Since H is a separable space, $L_2(E_n, H)$ is also separable space.

Let us denote by σ_1 [4] the set of compact operators which satisfying the condition

$$\sum_{i=1}^{\infty} \sqrt{\lambda_i(A^*A)} < \infty$$

and making transformation on the Hilbert space H . Here the numbers $\lambda_i(A^*A)$ are eigenvalues of the positive operator A^*A .

The norm of any operator A belonging to σ_1 is defined by the following formula (absolute norm):

$$\|A\|_{\sigma_1} = \sum_{i=1}^{\infty} \lambda_i^{1/2}(A^*A).$$

In particular, if $A^* = A \geq 0$ then

$$\|A\|_{\sigma_1} = \text{tr} A = \sum_{i=1}^{\infty} \lambda_i(A), \quad \text{tr} = \text{trace}.$$

Let us consider the differential expression

$$-\Delta + Q(x)$$

where for each $x \in E_n$, $Q(x)$ is a self-adjoint operator in H satisfying the following conditions:

(a) Let the domain D of the operator $Q(x)$ ($x \in E_n$) be independent of x , $\bar{D} = H$ ($\bar{D}$ denotes the closure of D), $Q(x) \geq I$ (I is the unit operator in H), $Q^{-1}(x) \in \sigma_1$ and for $f \in D$, let $Q(x)f$ be continuous function in H .

(b) We assume that for the number $l > 0$

$$\int_{E_n} \text{tr} Q^{-l}(x) dx = \int_{E_n} \|Q^{-l}(x)\|_{\sigma_1} dx < \infty.$$

(c) When $|x - s| \leq 1$ and $\gamma = 0$, $\gamma = \beta$ (β is any real number), let the conditions

$$\|Q^\gamma(x)e^{-ctQ(s)}\|_{\sigma_1} \leq A_1 \|Q^\gamma(x)e^{-f(c)tQ(x)}\|_{\sigma_1}$$

and

$$\|Q^\beta(x)Q^{-\beta}(s)\| \leq A_2, \quad (A_1, A_2 \text{ are constants})$$

be satisfied for all $c > 0$, $t > 0$ ($f(c) > 0$).

(d) We assume that

$$\|Q^\gamma(x)[Q(s) - Q(x)]Q^{-\gamma-b}(s)\| \leq A_3|x - s|, \quad (|x - s| \leq 1)$$

where $\gamma = 0$, $\gamma = \beta$, $0 < b < 3/2$ and $A_3 = \text{constant}$.

(e) When $M > 0$ is an arbitrary number, $\gamma = 0$, $\gamma = \beta$, let it be

$$\int_{E_n} \text{tr}[Q^\gamma(x)e^{-MtQ(x)}]dx = O(1) \int_{E_n} \text{tr}[Q^\gamma(x)e^{-tQ(x)}]dx$$

as $t \rightarrow 0^+$.

Let us denote by $\alpha_1(x) \leq \alpha_2(x) \leq \dots \leq \alpha_m(x) \leq \dots$ eigenvalues of the operator $Q(x)$. The above mentioned the functions $e^{-MtQ(x)}$, etc. are defined by the spectral extension of operator $Q(x)$.

Let us define the function $\rho_\gamma(\lambda)$ ($\lambda \geq 1$) by

$$\rho_\gamma(\lambda) = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2} + 1)} \sum_i \int_{\alpha_i(x) < \lambda} \alpha_i^\gamma(x) [\lambda - \alpha_i(x)]^{n/2} dx.$$

For large values of λ ($\lambda > 0$), let us assume that there exists a number $a_0 > 0$ satisfying inequality

$$\lambda \rho'_\gamma(\lambda) \leq a_0 \rho_\gamma(\lambda). \quad (1)$$

The examples belonging to the function $Q(x)$ with operator value satisfying (a)-(e) and (1) conditions have been given in Ref.s 9 and 10. (For examle,

in particular, if $\{e_k\}_{k=1}^{\infty}$ is any orthonormal basis of H and $\alpha_k(x) = k^4(1 + |x|^{1+\gamma})$, $1/2 < \gamma < 1$ then the operator function

$$Q(x) = \sum_{k=1}^{\infty} \alpha_k(x) (\cdot, e_k) e_k$$

satisfies the conditions (a)-(e) and (1).) An example related to this can be found in the Appendix.

Let us denote by D' the following sums

$$\sum_{k=1}^m \varphi_k(x) f_k$$

where f_k are any elements belonging to D and $\varphi_k(x)$ are any functions with compact support (finite functions) having a continuous secondary derivatives. D' is dense everywhere in H_1 . Let us define the operator L' by

$$L'u = -\Delta u + Q(x)u, \quad u \in D',$$

where $D' \subset H_1$. L' is the symmetric operator and $L' \geq I$ in H_1 . We denote by L the closure of the operator L' . The operator L is self-adjoint in H_1 (see [12] or [5], [6]).

Let the spectrum of L (it will be show that the spectrum is discrete) be

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots,$$

the orthonormal eigenfunctions corresponding to $\{\lambda_m\}_{m=1}^{\infty}$ be

$$\psi_1(x), \psi_2(x), \dots, \psi_m(x), \dots$$

and

$$N_{\beta}(\lambda) = \sum_{\lambda_m < \lambda} \int_{E_n} (Q^{\beta}(x) \psi_m(x), \psi_m(x)) dx, \quad (\lambda > 0).$$

As we mentioned above, we will show that the operator L has a pure discrete spectrum and when β has an arbitrary real value, we will examine asymptotic behavior of the function $N_{\beta}(\lambda)$ (weighted trace of L) as $\lambda \rightarrow \infty$.

When $Q(x)$ is a real function, that is, $H = \mathcal{R}$, the weighted trace of the operator L has been studied the first time by B. M. Levitan [11]. The asymptotic expression of the eigenvalue number and the weighted trace of the Sturm-Liouville operator equation are studied in [9] and [1], respectively. The asymptotic behavior of the eigenvalues of differential operators

with operator coefficient has been studied by many authors (see, for example, Ref. 2 and Ref.s 13, 15, 16.)

2. Examination of the Green's function of parabolic equation corresponding to the operator L

Let us consider the problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - Q(x)u, \\ u(0, x) &= \psi(x), \quad \psi(x) \in H_1.\end{aligned}\tag{2}$$

The Green's function of this problem can be found by using the method of Levi given in [3]. Since the Green's function of the problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u, \\ u(0, x) &= \psi(x)\end{aligned}\tag{3}$$

is

$$G_0(t, x - s) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x-s|^2}{4t}} I,$$

the solution of (3) will be

$$u(t, x) = \int_{E_n} G_0(t, x - s) \psi(s) ds.$$

Using the Levi's method, we shall seek the Green's function of the problem (2) such that

$$G(t, x, s) = e^{-tQ(s)} G_1(t, x - s) + \int_0^t d\tau \int_{E_n} G_1(t - \tau, x - \xi) e^{-(t-\tau)Q(s)} \varphi(\tau, \xi, s) d\xi.\tag{4}$$

Here $G_1(t, x)$ is of the form

$$G_1(t, x) = R(x) G_0(t, x)\tag{5}$$

where

$$R(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2 \\ 0 & \text{if } |x| > 1 \end{cases}\tag{6}$$

is any real-valued sufficiently smooth function and $\varphi(\tau, \xi, s)$ is a function with operator value, that will be determined later.

To find $\varphi(t, x, s)$ which is an unknown function with operator value given in (4), we want that the function $G(t, x, s)$ should be satisfy the equation

$$\frac{\partial G}{\partial t} = \Delta G - Q(x)G.$$

Thus, we obtain the integral equation

$$\varphi(t, x, s) = K(t, x, s) + \int_0^t d\tau \int_{E_n} K(t - \tau, x, \xi) \varphi(\tau, \xi, s) d\xi \quad (7)$$

for the unknown $\varphi(t, x, s)$, where

$$K(t, x, s) = \bar{K}(t, x, s) - e^{-tQ(s)} \left(\frac{\partial}{\partial t} - \Delta_x \right) G_1(t, x - s) \quad (8)$$

and

$$\bar{K}(t, x, s) = [Q(s) - Q(x)] e^{-tQ(s)} G_1(t, x - s). \quad (9)$$

We will solve the integral equation (7) by the method of successive approximations. For this, it is necessary to study some properties of the function $K(t, x, s)$ with operator value. Therefore, let us give some inequalities and lemmas which are necessary for us.

The function $G_0(t, x)$ with operator value satisfies the inequalities ¹ [3]

$$\left\| \frac{\partial^k G_0(t, x)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \right\| \leq A_4 t^{-\frac{n+k}{2}} e^{-c \frac{|x|^2}{t}} \quad (10)$$

$$k = 0, 1, 2 \quad k_1 + k_2 + \dots + k_n = k, \quad A_4 = \text{constant}.$$

Lemma 1. For the constant A_5 , the inequality

$$\left\| \left(\frac{\partial}{\partial t} - \Delta_x \right) G_1(t, x - s) \right\| \leq A_5 e^{-c \frac{|x-s|^2}{t}} \quad (11)$$

is hold.

Lemma 2. Let the function $Q(x)$ with operator value be satisfy the condition (d) and the following conditions:

(i) for every $t > 0$, $x \in E_n$

$$e^{-tQ(x)} \in \sigma_1,$$

¹Let's note that the symbol c is different positive constants

(ii) when $|x - s| \leq 1$

$$\|Q^\beta(x)Q^{-\beta}(s)\| \leq A_6, \quad A_6 = \text{constant}.$$

Then the inequality

$$\|Q^\gamma(x)K(t, x, s)\|_{\sigma_1} \leq A_7 t^{-b_0-1} \|Q^\gamma(s)e^{-\frac{1}{2}tQ(s)}\|_{\sigma_1} e^{-\frac{c}{2}\frac{|x-s|^2}{t}}, \quad A_7 = \text{const}$$

is hold, where

$$b_0 = \max\left\{\frac{n-1}{2}, \frac{n-3}{2} + b\right\}.$$

The proof of these lemmas can be made by using the inequality (10) given in [3].

Lemma 3. If the function $Q(x)$ with operator value satisfies the conditions (d) and (ii) of Lemma 2, then we have

$$\|Q^\gamma(x)K(t, x, s)Q^{-\gamma}(s)\| \leq A_8 t^{-b_0-1} e^{-c\frac{|x-s|^2}{t}},$$

where $\gamma = 0$, $\gamma = \beta$, and $A_8 = \text{constant}$.

Proof. It is clear that

$$\begin{aligned} \|Q^\gamma(x)\bar{K}(t, x, s)Q^{-\gamma}(s)\| &= \|Q^\gamma(x)[Q(s) - Q(x)]e^{-tQ(s)}G_1(t, x-s)Q^{-\gamma}(s)\| \\ &= \|Q^\gamma(x)[Q(s) - Q(x)]Q^{-(\gamma+b)}(s)Q^b(s)e^{-tQ(s)}G_1(t, x-s)\| \\ &= \|Q^\gamma(x)[Q(s) - Q(x)]Q^{-(\gamma+b)}(s)t^b t^{-b}Q^b(s)e^{-tQ(s)}G_1(t, x-s)\| \\ &\leq \|Q^\gamma(x)[Q(s) - Q(x)]Q^{-(\gamma+b)}(s)\| \|t^b Q^b(s)e^{-tQ(s)}\| t^{-b} \|G_1(t, x-s)\|. \end{aligned}$$

By using the condition (d) of $Q(x)$ and the inequality (10), one gets

$$\begin{aligned} \|Q^\gamma(x)\bar{K}(t, x, s)Q^{-\gamma}(s)\| &\leq B_1 |x-s| t^{-b-\frac{n}{2}} e^{-c\frac{|x-s|^2}{t}} \\ &\leq B_2 t^{-b_0-1} \left(\frac{|x-s|^2}{t}\right)^{\frac{1}{2}} e^{-c\frac{|x-s|^2}{2t}} e^{-c\frac{|x-s|^2}{2t}} \\ &\leq B_3 t^{-b_0-1} e^{-c\frac{|x-s|^2}{t}} \end{aligned} \tag{13}$$

where B_1, B_2 and B_3 are constants.

Since $Q(x)$ satisfies the condition (ii) of Lemma 2, we obtain

$$\begin{aligned}
 & \|Q^\gamma(x)e^{-tQ(s)}\left(\frac{\partial}{\partial t} - \Delta_x\right)G_1(t, x-s)Q^{-\gamma}(s)\| \\
 &= \|Q^\gamma(x)Q^{-\gamma}(s)e^{-tQ(s)}\left(\frac{\partial}{\partial t} - \Delta_x\right)G_1(t, x-s)\| \\
 &\leq \|Q^\gamma(x)Q^{-\gamma}(s)\| \|e^{-tQ(s)}\| \|(\frac{\partial}{\partial t} - \Delta_x)G_1(t, x-s)\| \\
 &\leq B_4\left\|\left(\frac{\partial}{\partial t} - \Delta_x\right)G_1(t, x-s)\right\|
 \end{aligned}$$

where B_4 is a constant. By using the inequality (11) and the last inequality, we can write

$$\|Q^\gamma(x)e^{-tQ(s)}\left(\frac{\partial}{\partial t} - \Delta_x\right)G_1(t, x-s)Q^{-\gamma}(s)\| \leq B_5e^{-c\frac{|x-s|^2}{t}},$$

where $B_5 = \text{constant}$. So, by using the inequality (8), (13) and the last inequality we have

$$\|Q^\gamma(x)K(t, x, s)Q^{-\gamma}(s)\| \leq B_6t^{-b_0-1}e^{-c\frac{|x-s|^2}{t}}, \quad 0 < t < 1 \quad B_6 = \text{constant}.$$

Now, using Lemma 3 let us solve the integral equation (7) by the method of successive approximations. If the first approximation is taken as

$$K_1(t, x, s) = K(t, x, s),$$

then the m -th approximation becomes

$$K_m(t, x, s) = \int_0^t d\tau \int_{E_n} K(t-\tau, x, \xi)K_{m-1}(\tau, \xi, s)d\xi. \quad (14)$$

The sum

$$\varphi(t, x, s) = \sum_{m=1}^{\infty} K_m(t, x, s) \quad (15)$$

is a solution of the integral equation (7), if the series $\sum_{m=1}^{\infty} K_m(t, x, s)$ is uniform convergent in the sense of σ_1 -norm with respect to the variables x, s , when $t \geq \delta > 0$. For this, it is necessary to estimate of the successive kernels $K_2(t, x, s), K_3(t, x, s), \dots, K_m(t, x, s)$ and the functions $Q^\gamma(x)\varphi(t, x, s)$, ($\gamma = 0, \gamma = \beta$) with operator value. By estimating the functions

$$\|Q^\gamma(x)K_m(t, x, s)\|_{\sigma_1}, \quad m = 2, 3, \dots,$$

we find that the series

$$Q^\gamma(x)\varphi(t, x, s) = \sum_{m=1}^{\infty} Q^\gamma(x)K_m(t, x, s)$$

is uniform convergent in the sense of σ_1 -norm with respect to the variables x, s , when $t \geq \delta > 0$. Hence we can write

$$\|Q^\gamma(x)\varphi(t, x, s)\|_{\sigma_1} \leq B_7 t^{-b_0-1} \left\{ \|Q^\gamma(s)e^{-ctQ(s)}\|_{\sigma_1} + \|Q^{-l}(s)\|_{\sigma_1} \right\} e^{-c\frac{|x-s|^2}{t}}, \quad (16)$$

where $\gamma = \beta$, and $B_7 = \text{constant}$. Furthermore, with the estimate on the functions

$$\|K_m(t, x, s)\|, \quad m = 2, 3, \dots, \quad (17)$$

we can write

$$\|\varphi(t, x, s)\| \leq B_8 t^{-b_0-1} e^{-c\frac{|x-s|^2}{t}}, \quad B_8 = \text{constant}.$$

Thus we have been proved the following theorem.

Theorem 1. Let the function $Q(x)$ with operator value be satisfy the conditions (a), (c) and (d). In addition, let $Q^{-l}(x)$ belong to σ_1 for each $x \in E_n$. Then the solution $\varphi(t, x, s)$ of the integral equation (7) satisfies the inequalities (16) and (17).

The following theorem may be proved analogously to the proof of Theorem 1.

Theorem 2. Let the function $Q(x)$ with operator value be satisfy the conditions (a), (c) and (d). If, in addition, let $Q^{-l}(x)$ belong to σ_1 for each $x \in E_n$, then the function $Q^\gamma(x)G(t, x, s)$ with operator value ($\gamma = 0, \gamma = \beta$) has the asymptotic expression

$$\begin{aligned} Q^\gamma(x)G(t, x, s) &= Q^\gamma(x)e^{-tQ(s)}G_1(t, x-s) + O(1)t^{-b_0}e^{-c\frac{|x-s|^2}{t}} \times \\ &\quad \left\{ \|Q^\gamma(s)e^{-ctQ(s)}\|_{\sigma_1} + \|Q^{-l}(s)\|_{\sigma_1} \right\} \end{aligned}$$

for small values of $t > 0$. Here, $O(1)$ denotes the operator which has a finite σ_1 -norm for each values $x, s \in E_n$ and $t > 0$.

3. Discreteness of spectrum of the operator L and asymptotic behavior of the weighted trace of the operator L

Since $G(t, x, s)$ is the Green's function of the problem (2), we have

$$U(t, x) = \int_{E_n} G(t, x, s) \psi(s) ds$$

and, moreover, since

$$U(t, x) = e^{-tL} \psi(x),$$

we can write

$$e^{-tL} \psi(x) = \int_{E_n} G(t, x, s) \psi(s) ds.$$

Therefore, e^{-tL} becomes an integral operator with kernels $G(t, x, s)$ ($0 < t < 1$). According to the Theorem 2, we obtain

$$\begin{aligned} \text{tr}[Q^\gamma(x)G(t, x, x)] &= \text{tr}[Q^\gamma(x)e^{-tQ(x)}G_0(t, 0)] + O(1) \left\{ \text{tr}[Q^\gamma(x)e^{-ctQ(x)}] \right. \\ &\quad \left. + \text{tr}[Q^{-l}(x)] \right\} t^{-b_0}, \end{aligned} \quad (18)$$

as $t \longrightarrow 0^+$. Here, since

$$G_0(t, 0) = \frac{1}{(2\sqrt{\pi t})^n} I,$$

the integration of the equation (18) gets

$$\begin{aligned} \int_{E_n} \text{tr}[Q^\gamma(x)G(t, x, x)] dx &= (2\sqrt{\pi t})^{-n} \int_{E_n} \text{tr}[Q^\gamma(x)e^{-tQ(x)}] dx \\ &\quad + O(1)t^{-b_0} \int_{E_n} \text{tr}[Q^\gamma(x)e^{-ctQ(x)}] dx \\ &\quad + O(1)t^{-b_0} \int_{E_n} \text{tr}[Q^{-l}(x)] dx. \end{aligned}$$

Since the function $Q(x)$ with operator value satisfies the conditions (b), (c) and since $b_0 < n/2$, in the limit $t \longrightarrow 0^+$, we, from the last expression, obtain

$$\int_{E_n} \text{tr}[Q^\gamma(x)G(t, x, s)] dx \sim (2\sqrt{\pi t})^{-n} \int_{E_n} \text{tr}[Q^\gamma(x)e^{-tQ(x)}] dx, \quad (19)$$

where $\gamma = 0$, $\gamma = \beta$. From (19), for each $t > 0$, we have

$$\int_{E_n} \text{tr}[G(t, x, x)] dx < \infty.$$

According to the theorem which is given in [7] (see, Chapter XIII.6, Theorem 2), we can write

$$\mathrm{tr} e^{-tL} = \int_{E_n} \mathrm{tr}[G(t, x, x)] dx.$$

Thus, for each $t > 0$, e^{-tL} becomes a positive self-adjoint kernel operator in the H_1 space. Therefore, the spectrum of L is pure discrete:

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots, \quad \lim_{m \rightarrow \infty} \lambda_m = \infty.$$

The following theorem can be proved.

Theorem 3. If the function $Q(x)$ with operator value satisfies the conditions (a)-(e), then the operator $Q^\beta(x)e^{-tL}$ becomes a kernel operator in H_1 for each $t > 0$ and for the values of β ($-\infty < \beta < \infty$). Now, we give the fundamental theorem:

Theorem 4. Let the function $Q(x)$ with operator value satisfy the conditions (a)-(e) and (1). Then the weighted trace $N_\beta(\lambda)$ of the operator L has the asymptotic behavior

$$N_\beta(\lambda) \sim \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2} + 1)} \sum_j \int_{\alpha_j(x) < \lambda} \alpha_j^\beta(x) [\lambda - \alpha_j(x)]^{n/2} dx,$$

as $\lambda \rightarrow \infty$.

Proof. Since the orthonormal eigenfunctions $\psi_1(x), \psi_2(x), \dots, \psi_m(x), \dots$ corresponding to the eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots$ of the operator L form a base in H_1 space and since the $Q^\beta(x)e^{-tL}$ ($t > 0$) is the kernel operator (according to the Theorem 3), we have

$$\begin{aligned} \mathrm{tr}[Q^\beta(x)e^{-tL}] &= \sum_{m=1}^{\infty} \left(Q^\beta(x)e^{-tL}\psi_m(x), \psi_m(x) \right)_1 \\ &= \sum_{m=1}^{\infty} e^{-t\lambda_m} \left(Q^\beta(x)\psi_m(x), \psi_m(x) \right)_1. \end{aligned}$$

Thus, one obtains

$$\sum_{m=1}^{\infty} e^{-t\lambda_m} \left(Q^\beta(x)\psi_m(x), \psi_m(x) \right)_1$$

$$\begin{aligned}
&= \int_0^\infty e^{-\lambda t} d \left\{ \sum_{\lambda_m < \lambda} \int_{E_n} (Q^\beta(x) \psi_m(x), \psi_m(x)) dx \right\} \\
&= \int_0^\infty e^{-\lambda t} dN_\beta(\lambda).
\end{aligned}$$

So we have

$$\operatorname{tr}[Q^\beta(x)e^{-tL}] = \int_0^\infty e^{-\lambda t} dN_\beta(\lambda). \quad (21)$$

Since $Q^\beta(x)e^{-tL}$ is an integral operator with kernel $Q^\beta(x)G(t, x, s)$, according to the theorem which is given in [7] (Chapter XIII.6, Theorem 2), one has

$$\operatorname{tr}[Q^\beta(x)e^{-tL}] = \int_{E_n} \operatorname{tr}[Q^\beta(x)G(t, x, x)] dx. \quad (22)$$

Using (19), (21) and (22), we obtain

$$\int_0^\infty e^{-\lambda t} dN_\beta(\lambda) \sim (2\sqrt{\pi t})^{-n} \int_{E_n} \operatorname{tr}[Q^\beta(x)e^{-tQ(x)}] dx,$$

as $t \rightarrow 0^+$. On the other hand, we can write [10]

$$(2\sqrt{\pi t})^{-n} \int_{E_n} \alpha_j^\beta(x) e^{-t\alpha_j(x)} dx = \int_0^\infty e^{-\lambda t} dg_j(\lambda)$$

where

$$g_j(\lambda) = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2} + 1)} \int_{\alpha_j(x) < \lambda} \alpha_j^\beta(x) [\lambda - \alpha_j(x)]^{n/2} dx.$$

Therefore, one gets

$$\int_0^\infty e^{-\lambda t} dN_\beta(\lambda) \sim \int_0^\infty e^{-\lambda t} d\rho_\beta(\lambda), \quad \text{as } t \rightarrow 0^+,$$

where

$$\rho_\beta(\lambda) = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2} + 1)} \sum_j \int_{\alpha_j(x) < \lambda} \alpha_j^\beta(x) [\lambda - \alpha_j(x)]^{n/2} dx.$$

If we consider that

$$\lambda \rho'_\beta(\lambda) \leq a_0 \rho_\beta(\lambda), \quad a_0 > 0$$

for large values of $\lambda > 0$, then according to the Tauberian type theorem of Keldysh and Korenblum [8], we have

$$N_\beta(\lambda) \sim \rho_\beta(\lambda). \quad (23)$$

If we consider the above expression of $\rho_\beta(\lambda)$ with $\lambda \rightarrow \infty$, from (23), we can write

$$N_\beta(\lambda) \sim \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2} + 1)} \sum_j \int_{\alpha_j(x) < \lambda} \alpha_j^\beta(x) [\lambda - \alpha_j(x)]^{n/2} dx. \quad (24)$$

From this formula especially when $\beta = 0$, the asymptotic expression of the eigenvalue number not exceeding λ of the operator L is obtained.

Appendix

Example. Let $\Omega \subset \mathcal{R}^m$ ($m \geq 1$) be any finite region with uniformly bounded.

Let us consider the boundary value problem of

$$-\Delta_x u + q(x)(-\Delta_y)^s u = \lambda u \quad (a)$$

$$u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = \dots = \frac{\partial^{s-1} u}{\partial \nu^{s-1}}|_{\partial\Omega} = 0 \quad (b)$$

in $L_2(\mathcal{R}^m \times \Omega)$ space. Here

$$-\Delta_x = -\frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2}, \quad -\Delta_y = -\frac{\partial^2}{\partial y_1^2} - \dots - \frac{\partial^2}{\partial y_m^2},$$

s is any positive integer, $\partial\Omega$ is the boundary of Ω region, ν is the normal of $\partial\Omega$ and $q(x)$ is a continuous function satisfying the conditions

$$c_1(1 + |x|^\alpha) \leq q(x) \leq c_2(1 + |x|^\alpha) \quad (c)$$

where $1 < \alpha < 3/2$, c_1, c_2 are positive constants and

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

Let us define self-adjoint A operator (corresponding to the Dirichlet problem) in $H = L_2(\Omega)$ space by $(-\Delta_y)^s$ with (b) condition. The spectrum of this operator consists only eigenvalues

$$\mu_1 \leq \mu_2 \leq \dots \leq \mu_n \dots, \quad \lim_{n \rightarrow \infty} \mu_n = \infty$$

and it has the asymptotic expression

$$\mu_k \sim (c_m)^s k^{\frac{2s}{m}} \quad (d)$$

as $n \rightarrow \infty$ [14]. Here c_m is the given constant depending on m [14].

Now, let us write the boundary value problem (a), (b) in the space

$$H_1 = L_2(\mathcal{R}^n \times \Omega) = L_2(\mathcal{R}^m, H)$$

as an equation with operator coefficient as follows:

$$-\Delta_x u + Q(x)u = \lambda u, \quad Q(x) = q(x)A. \quad (e)$$

In each value of x taken from E_n , the spectrum of the self-adjoint operator $Q(x) = q(x)A$ making transformation in $H = L_2(\Omega)$ consists only eigenvalues

$$\alpha_1(x) = \mu_1 q(x), \alpha_2(x) = \mu_2 q(x), \dots, \alpha_n(x) = \mu_n q(x), \dots$$

If we consider the eigenvalues of $Q^{-1}(x)$ being

$$\alpha_k^{-1}(x) = q^{-1}(x)\mu_k^{-1}, \quad k = 1, 2, \dots$$

it is seen, from the asymptotic expression (d), that for $l > m/2s$, the series $\sum_{k=1}^{\infty} \alpha_k^{-l}(x)$ is convergent, that is, $Q^{-l}(x)$ is the kernel operator.

One can show that the operator $Q(x)$ described in this way, satisfies the other conditions which given in the beginning of the work.

Thus, in the space $H_1 = L_2(E_n, H)$, it is shown that the operator described by the differential expression $-\Delta_x u + q(x)Au$ has been a pure discrete spectrum and the asymptotic expression of the number of eigenvalues can be written with respect to the formula (24).

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