THE DUAL BADE THEOREM IN LOCALLY CONVEX SPACES AND REFLEXIVITY OF A CLOSED UNITAL SUBALGEBRA

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Abstract

The results presented in this paper extend a dual version of the reflexivity theorem of W. Bade to locally convex spaces. Dual version of the Bade theorem in a Banach $C(K)$ -module was firstly discovered in [1]. It is our aim to extend it to a locally convex $C(K)$ -module. As a consequence, it is proven that each unital w^ operator topology closed subalgebra of the w^* operator topology closed algebra generated by a Boolean algebra of projections is reflexive.*

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1. Introduction

In this article, we investigate the role played by lattice order, locally convex $C(K)$ -module and f -module in the study of operator algebras generated by Boolean algebras of projections. The study of operator algebras generated by Bade complete (σ -complete) Boolean algebras of projections in Banach spaces was firstly given by W. G. Bade in [5]. The reflexivity result of Bade asserts that a continuous linear operator T on a Banach space X belongs to the strongly closed algebra generated by a σ -complete Boolean algebra B of projections if and only if T leaves invariant each B -invariant (closed) subspace of X . Furthermore, this idea developed by T. A. Gillespie, [10], and C. Rall, [12].

The reflexivity theorem of Bade by a systematic approach to the locally convex spaces was extended in [6], [7], [8], [13] and [14]. The reflexive operator algebras in a Banach $C(K)$ -module were studied in [11] and [1].

2. Preliminaries

Let $C(K)$ be the set of all continuous real or complex valued functions with the supremum norm on a compact Hausdorff space K . Suppose that X is a locally convex Hausdorff space. We call that X is a locally convex $C(K)$ -module if the bilinear mapping

$$(1) \quad C(K) \times X \rightarrow X$$

satisfies the following conditions:

- (i) $1.x=x$ for all $x \in X$, $1 \in C(K)$,
- (ii) $(ab).x=a.(b.x)$ for all $a, b \in C(K)$, $x \in X$,
- (iii) The bilinear mapping (1) is separately continuous.

We let X' denote the continuous dual of a locally convex Hausdorff space X , let X'' denote the second continuous dual of a locally convex Hausdorff space X and let X^* denote an algebraic dual of a locally convex Hausdorff space X . Let X be a quasicomplete locally convex Hausdorff space and let B be an equicontinuous Boolean algebra in the sequentially complete space. By $\mathcal{L}(X)$ we denote the space of all continuous linear operators from X into itself equipped with the topology of pointwise convergence in X . A Boolean algebra of projections in $\mathcal{L}(X)$ is a family B of commuting idempotents,

partially ordered with respect to range inclusion, which is a Boolean algebra with respect to the lattice operations given by $E \vee F = E + F - EF$ and $E \wedge F = EF$ for all $E, F \in B$. We assume that the unit element of B is I . We say that an equicontinuous Boolean algebra of projections B in the locally convex space B is strongly equicontinuous if E_n converges to 0 strongly for any disjoint sequence $\{E_n\}$ in B . It will always be assumed that X is quasicomplete and $\mathcal{L}(X)$ is sequentially complete. In this work all Riesz spaces will be assumed as Archimedean. For the terminology, notations and elementary properties of Riesz spaces, elementary f-algebra theory and orthomorphism not explained or proved in this paper we refer to [2],[3] and [15].

Suppose that K is the Stone space of B . Let $Q(K)$ be the set of all open and closed sets in K and let m_1 be the Boolean isomorphism from $Q(K)$ onto B . We denote by A the linear span of B and by H the linear span of $Q(K)$. As shown in [6], the closed algebra $\overline{A}$ is complete. The linear mapping m from H to $\overline{A}$ is continuous, $m(1) = I$; so we can extend it, say m , from $C(K)$ into $\overline{A}$. It is well-known that $C(K)$ is $|\sigma|(C(K)'', C(K)')$ -dense in $C(K)''$, [3] Theorem 11.16.

Then the bilinear mapping

$$(1) \quad C(K) \times X \rightarrow X$$

is separately continuous. On X' we put $\sigma(X', X)$, on $C(K)'$ we consider $\sigma(C(K)', C(K)'')$ -topology.

Hence we can establish the following bilinear mappings in three steps:

$$(2) \quad X \times X' \rightarrow C(K)'; \quad (x, x') \rightarrow (x.x')(a) = x'(a.x),$$

$$(3) \quad C(K)'' \times X' \rightarrow X'; \quad (a, x') \rightarrow (a.x')(x) = a(x.x'),$$

$$(4) \quad C(K)'' \times X'^* \rightarrow X'^*; \quad (a, \gamma) \rightarrow (a.\gamma)(x') = \gamma(a.x').$$

We need to put a locally convex $C(K)''$ -module structure on X' . To do this, it is our aim to prove that the bilinear mapping (3) is separately continuous.

We fix $a \in C(K)''$ and $x' \in X'$. For if $\{x_\alpha\}$ is a net in X which converges to x , then by the separate continuity of the bilinear mapping (2) $x_\alpha.x' \rightarrow x.x'$ in $C(K)'$. $a(x_\alpha.x') \rightarrow a(x.x')$ for each $a \in C(K)''$, i.e. , $(a.x')(x_\alpha) \rightarrow (a.x')x$. Hence the bilinear mapping (3) is well-defined. To

prove the continuity of the bilinear mapping (3) in the first coordinate, we fix $x' \in X'$. Let $\{a_\alpha\}$ be a net in $C(K)''$ which converges to a with respect to $\sigma(C(K)'', C(K)')$ -topology. For $x \in X$ and $x' \in X'$, $xx' \in C(K)'$ by (2). Therefore, $a_\alpha(xx') \rightarrow a(xx')$, i.e., $a_\alpha.x' \rightarrow a.x'$. To prove the continuity of the bilinear mapping (3) in the second coordinate, we fix $a \in C(K)''$. Let $\{x'_\alpha\}$ be a net in X' which converges to x' with respect to $\sigma(X', X)$. By the bilinear mapping (2) xx'_α converges to xx' in $C(K)'$. Therefore, $a(xx'_\alpha)$ converges to $a(xx')$, i.e., $a.x'_\alpha \rightarrow a.x'$. For more information the bilinear mapping (3) we refer to Lemma 1.

The mapping (4) is only a bilinear mapping. Let X be a locally convex $C(K)$ -module and $x \in X$. Then the set $\Delta(x)$ is defined by

$$\Delta(x) = \{a.x : \|a\| \leq 1, \quad a \in C(K)\}^-,$$

where closure is taken with respect to the given topology in X . Similarly, by bilinear mapping (3) for $x' \in X'$ the set $\Delta(x')$ is defined by

$$\Delta(x') = \{a.x' : \|a\| \leq 1, \quad a \in C(K)''\}^-,$$

where closure is taken with respect to the $\sigma(X', X)$ topology on X' . A linear subspace Y of a locally convex $C(K)$ -module X is called an ideal if for each $x \in Y$ $\Delta(x) \in Y$. Similarly, an ideal for a linear subspace of a locally convex $C(K)''$ -module X' can be defined. Let Y be an ideal in a locally convex $C(K)$ -module X . We say that center of Y denoted by $Z_m(Y)$ is the set of all linear operators T from Y to Y such that $Ty \in \lambda\Delta(y)$ for each $y \in Y$, where $\lambda = \lambda(T)$ is a positive number, i. e.,

$$Z_m(Y) = \{T : Y \rightarrow Y | (\exists \lambda \geq 0) \quad (\forall y \in Y) \quad (Ty \in \lambda\Delta(y))\}.$$

Similarly, the center of Y for an ideal Y in a locally convex $C(K)''$ -module X' can be defined. Similar definitions can be found in [1].

If the bilinear mapping (1) is taken as $X = C(K)$, then the bilinear mapping (4) defined by $C(K)'' \times C(K)'' \rightarrow C(K)'', (a, b) \rightarrow (a, b)(c) = b(a.c)$ is continuous and defines the Arens multiplication on $C(K)''$ and $C(K)''$ is isomorphic to $C(S)$ with S Hyperstonian. Proof of the following Lemma was given in [9].

Lemma 1

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete and let $B \subseteq \mathcal{L}(X)$ be an equicontinuous Boolean algebra of projections. Then the following statements are true.

- (i) The bilinear mapping (1) defines a locally convex $C(K)$ -module on X .
- (ii) The bilinear mapping (3) defines a locally convex $C(S)$ -module on X' that gives a homomorphism $m^* : C(K)'' \rightarrow \mathcal{L}(X')$ defined by $m^*(a)(x') = ax'$.
- (iii) For each $a \in C(K)$, $m^*(a) = (m(a))^*$, where $(m(a))^*$ is the adjoint of $m(a)$.
- (iv) $m(1) = I$.
- (v) $m^*(1) = I'$, where I' is the adjoint of I .
- (vi) m^* is $(w^*, w^*$ -operator) continuous.

Proof. (ii)

$(a + b).x' = a.x' + b.x'$ for $a, b \in C(K)''$, $x' \in X'$. For proof of this condition, let $x \in X$ be arbitrary, then

$$\begin{aligned} (a + b).x'(x) &= (a + b)(x.x') = a(x.x') + b(x.x') \\ &= (a.x')(x) + (b.x')(x) = (a.x' + b.x')(x). \end{aligned}$$

As x is arbitrary we conclude that $(a + b).x' = a.x' + b.x'$.

$a.(x' + y') = a.x' + a.y'$ for $a \in C(K)''$, $x', y' \in X'$. For proof of this claim, let $x \in X$ be arbitrary, then

$$\begin{aligned} (a.(x' + y'))(x) &= a(x.(x' + y')) = a(x.x' + x.y') = a(x.x') + a(x.y') = \\ &= (a.x' + a.y')(x). \end{aligned}$$

As x is arbitrary we conclude that $a.(x' + y') = a.x' + a.y'$. Therefore, (3) is a bilinear mapping. $x'.1 = x'$ for all $x' \in X'$. For this, let $x \in X$, then

$$\begin{aligned} (x'.1)(x) &= 1(x.x') = (x.x')(1) \\ &= x'(1.x) = x'(x). \end{aligned}$$

As x is arbitrary, hence condition is satisfied.

$(a.b).x' = a.(b.x')$ for $a, b \in C(K)''$, $x' \in X'$. To prove this condition let $x \in X$, then

$$(a.b).x'(x) = (a.b)(x.x') = a(b.(x.x'))$$

On the other hand, by the definition of the bilinear mapping (3) $b.x' \in X'$ and

$$a.(b.x')(x) = a((b.x').x)$$

Hence, it suffices to show that

$$b.(x.x') = (b.x').x$$

for each $x \in X, x' \in X'$ and $b \in C(K)''$. To this end, let $a \in C(K)$. Then

$$(b.(x.x'))(c) = b(c.(x.x'))$$

by the definition of bilinear mapping (3). On the other hand,

$$(x.(x'.b))(c) = (x'.b)(c.x) = b((c.x).x')$$

by the definitions of bilinear mapping (2) and (3). Therefore, it is enough to show that $c.(x.x') = (c.x).x'$ for each $c \in C(K)$, $x \in X$, $x' \in X'$. Let $d \in C(K)$ be arbitrary, then

$$(c.(x.x'))(d) = (x.x')(d.c) = x'((d.c)x).$$

Calculating the right hand side,

$$((c.x).x')(d) = x'(d(c.x)) = x'((d.c)x).$$

Therefore, $(a.b).x' = a.(b.x')$. It remains to prove the bilinear mapping (3) is separately continuous. The continuity of the bilinear mapping (3) in the second coordinate from $(X', \sigma(X', X))$ into $(X', \sigma(X', X))$ is an immediate consequence of the following paragraph.

Let us equip X' with $\sigma(X', X)$, $C(K)''$ with the norm topology and consider the space $X' \times C(K)''$ with the induced product topology. We claim the bilinear mapping (3) is continuous when the domain is equipped with the product topology and the range is equipped with $\sigma(X', X)$. Let $\{y_\alpha\} = \{(a_\alpha, x'_\alpha)\}$ be a net in $C(K)'' \times X'$ convergent to zero in the product topology. Let $x \in X$ be arbitrary, we have

$$|(y_\alpha)(x)| = |(a_\alpha.x'_\alpha)(x)| = |a_\alpha(x.x'_\alpha)| \leq \|a_\alpha\| \|x.x'_\alpha\|$$

As the bilinear mapping (2) is continuous from $(X', \sigma(X', X))$ into $(C(K)', \sigma(C(K)', C(K)))$, we have $(x.x'_\alpha)$ converges to zero in $\sigma(C(K)', C(K))$. An application of the Uniform Boundedness Principle shows that the $\{x.x'_\alpha\}$ is

bounded in the norm topology of $C(K)'$. As $x \in X$ is arbitrary, this yields that $\{y_\alpha\}$ converges to zero in X' with respect to $\sigma(X', X)$.

To prove the continuity of the bilinear mapping (3) in the first coordinate, we let $\{a_\alpha\}$ be a net in $C(K)''$ which converges to a in $\sigma(C(K)'', C(K)')$. Let $x \in X$, $x' \in X'$ be fixed but arbitrary elements. As $xx' \in C(K)'$ we have

$$\lim_\alpha (a_\alpha \cdot x')(x) = \lim_\alpha a_\alpha(xx') = a(xx') = (a \cdot x')(x).$$

As x is arbitrary, we conclude that the bilinear mapping (3) is continuous from $\sigma(C(K)'', C(K)')$ into $\sigma(X', X)$.

The linear mapping in Lemma 1 m^* may not be one to one. However, we can assume that m^* is one to one on $C(K)''[9]$. We say that X is cyclic with respect to B if there exists $x \in X$ such that $\overline{A(x)} = X$, where $A(x) = \{Tx : T \in A\}$. Therefore, X is a locally convex locally solid Riesz space [6] and as proven in [6], Corollary 4.9., $\overline{A(x)} = \text{Orth}(X)$ and $Z(X) = A_I$, where A_I is the order ideal generated by I in $\overline{A}$. It is easily seen that if X is cyclic with respect to the Boolean algebra B , then X is an f -module over $C(K)$ [4]. Now, we can state and easily prove the following fact.

Proposition 2

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete and let $B \subseteq \mathcal{L}(X)$ be an equicontinuous Boolean algebra of projections, and suppose that $x \in X$ is cyclic for B . Then

$$\overline{m(\overline{C(K)})} = \overline{A} = \text{Orth}(X),$$

where closure is taken with respect to the strong operator topology.

The following result follows from [15].

Proposition 3

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete and let $B \subseteq \mathcal{L}(X)$ be an equicontinuous Boolean algebra of projections, and suppose that $x \in X$ is cyclic for B . Then $T \in \text{Orth}(X)$ if and only if $T' \in \text{Orth}(X')$.

Proof.

It suffices to prove that if $T' \in \text{Orth}(X')$ then $T \in \text{Orth}(X)$. By Hahn-Banach separation theorem, X' separates the points of X , hence the order dual $X^\sim$ of X separates the points of X . Consider that X is embedded as a Riesz subspace in $X'^\sim$. Then by [15, Theorem 142.10] the order adjoint of T , $T^\sim, T^\sim \in \text{Orth}(X'^\sim)$. $T^\sim$ agrees with T' on X . For $x, y \in X$ $x \wedge y = 0$ implies that $T^\sim x \wedge y = T' x \wedge y = 0$, i. e., T' is a band preserving operator. Therefore, $T' \in \text{Orth}(X)$.

Using the continuity of m^* and $|\sigma|(C(K)'', C(K)')$ - denseness of $C(K)$ in $C(K)''$ we can observe the following fact that

$$\overline{m^*(C(K))} = \overline{m^*(C(K)'')},$$

where closure is taken with respect to the w^* operator topology in $\mathcal{L}(X')$.

Proposition 4

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete and let $B \subseteq \mathcal{L}(X)$ be an equicontinuous Boolean algebra of projections, and suppose that $x \in X$ is cyclic for B . Then

$$\overline{m^*(C(K)'')} = \text{Orth}(X'),$$

where closure is taken with respect to the w^* operator topology in $\mathcal{L}(X')$.

Proof. Continuity of m^* , the preceding observation and Lemma 1 imply that

$$\overline{m^*(C(K)'')} \subset \text{Orth}(X').$$

Suppose that T belongs to $\text{Orth}(X')$. Then by Proposition 3 or by [15, Theorem 142.10] $T^\sim \in \text{Orth}(X'^\sim)$. Since X is a Riesz subspace of $X'^\sim$ and $X^\sim$ separates points of X , the restriction of the order adjoint $T^\sim$ to X is $T' \in \text{Orth}(X')$ and hence

$$T' \in \text{Orth}(X) = \overline{m(C(K))}$$

Then there exists a net (a_α) in $C(K)$ such that

$$m(a_\alpha) \rightarrow T' \in \mathcal{L}(X),$$

i. e., $x'm(a_\alpha)x \rightarrow x'T'x$ for all $x \in X, x' \in X'$. Now,

$x'T'x = Tx'(x)$ and $m^*(a_\alpha)$ in $m^*(C(K))$ imply that $T \in \overline{m^*(C(K))}$. By the preceding fact that we have $T \in \overline{m^*(C(K)'')}$.

Noting that $\text{Alglat}A$ denotes the set of all linear operators T such that T leaves invariant each A -invariant $\sigma(X', X)$ closed subspace of X' , we can state one of the principal results of this paper.

Proposition 5

Let B be an equicontinuous Boolean algebra of projections in the quasi-complete locally convex Hausdorff space X such that $\mathcal{L}(X)$ is sequentially complete, and suppose that $x \in X$ is a cyclic vector for B . Then $m^*(C(K))$ is a reflexive algebra, i. e., $\overline{m^*(C(K))} = \text{Alglat}m^*(C(K))$, where closure is taken with respect to the w^* operator topology in $\mathcal{L}(X')$.

Proof. Let $m^*(C(K)) = W$. It is sufficient to show that $\text{Alglat}W \subseteq \overline{W}$. Let $T \in \text{Alglat}W$ and suppose that Y is a band in X' . For each $S \in W$, $SY \subseteq Y$. Therefore, $TY \subset Y$, i. e., T is a band preserving operator. Dedekind completeness of X' implies that $T \in \text{Orth}(X')$. Hence, $T \in \overline{W}$.

The following Lemmas can be easily proven by using definitions.

Lemma 6

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete and let $B \subseteq \mathcal{L}(X)$ be an equicontinuous Boolean algebra of projections, and let $x, y \in X$. Then the following statements are true.

- (i) A functional $x' \in X'$ belongs to $\overline{m(C(K))(x)}^o$ if and only if $x.x' = 0$, where $\overline{m(C(K))(x)}^o$ is the polar of $\overline{m(C(K))(x)}$ in X' .
- (ii) $y \in \Delta(x)$ if and only if $|y.x'| \leq |x.x'|$ for each $x' \in X'$.

Similar statements can be given in the following Lemma.

Lemma 7

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete and let $B \subseteq \mathcal{L}(X)$ be an equicontinuous Boolean algebra of projections, and let $x', y' \in X'$. Then the following statements are true.

- (i) An element $x \in X$ belongs to $\overline{m^*(C(K)'')(x')}^o$ if and only if $x.x' = 0$, where $\overline{m^*(C(K)'')(x')}^o$ is the polar of $\overline{m^*(C(K)'')(x')}$ in X .

- (ii) $y' \in \Delta(x')$ if and only if $|x.y'| \leq |x.x'|$ for all $x \in X$.

Lemma 8

Let B be an equicontinuous Boolean algebra of projections in the quasi-complete locally convex Hausdorff space X such that $\mathcal{L}(X)$ is sequentially complete and $T \in \mathcal{L}(X')$. Then the following statements are equivalent.

- (i) T leaves invariant all B -invariant $\sigma(X', X)$ closed subspaces of X' .
(ii) If $x.x' = 0$, then $x.Tx' = 0$, where $x' \in X', x \in X$.

Proof. It is easily seen that X is a locally convex $C(K)$ -module; therefore, we can prove equivalencess.

Noting that with the assumption below $m^*(C(K)'')$ is an order ideal in $Orth(X')$, the following result follows from [4].

Proposition 9

Let B be an equicontinuous Boolean algebra of projections in the quasicomplete locally convex Hausdorff space X such that $\mathcal{L}(X)$ is sequentially complete, and suppose that $x \in X$ is a cyclic vector for B . Then $m^*(U) = Z(X')$, where U is the order ideal generated by unit element in $C(K)''$ and

$$Z(X') = \{T \in Orth(X') : |T| \leq nI \text{ for some } n\}.$$

The one of the main results of this paper, called Dual Bade Theorem, is given by the following Theorem.

Theorem 10

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete, and suppose that B is an equicontinuous Boolean algebra of projections in X . Then

$$Z_m(X') = Alglatm^*(C(K)).$$

Proof. It is sufficient to show that $Alglatm^*(C(K)) \subseteq Z_m(X')$. Let $T \in Alglatm^*(C(K))$ and fix an $x \in X$. The space $\overline{m(C(K))(x)}'$ is identified with $X'/\overline{m(C(K))(x)}^o$, where $\overline{m(C(K))(x)}^o$ is the polar of $\overline{m(C(K))(x)}$ in

X' . Since $\overline{m(C(K))(x)}^o$ is $\sigma(X', X)$ closed and B -invariant subspace, T leaves invariant $\overline{m(C(K))(x)}^o$. We define a quotient map T_1 from $\overline{m(C(K))(x)}^o$ to $\overline{m(C(K))(x)}'$ by means of $T_1(x' + \overline{m(C(K))(x)}^o) = Tx' + \overline{m(C(K))(x)}^o$. Taking an element $x' + \overline{m(C(K))(x)}^o \in \overline{m(C(K))(x)}'$ and an element $y \in \overline{m(C(K))(x)}$ such that $y.(x' + \overline{m(C(K))(x)}^o) = 0$, we find $y.x' = 0$. Hence, by Lemma 8 $y.Tx' = 0$. It follows that $y.T_1(x' + \overline{m(C(K))(x)}^o) = 0$. One more using Lemma 8, we see that $T_1 \in Z_m(\overline{m(C(K))(x)}')$. In particular, for each $x' \in X'$, we have $Tx' \in \Delta(x')$, i. e., $T \in Z_m(X')$.

In order to prove the next result we need the following Lemma from [8].

Lemma 11

Let X be a locally convex Hausdorff space, and let D be a closed unital subalgebra of $\mathcal{L}(X)$, and suppose that $T \in \mathcal{L}(X)$ is given.

- (i) T leaves invariant every D -invariant subspace of X if and only if $\langle Tx, x' \rangle = 0$, whenever $x \in X$ and $x' \in X'$ are such that $\langle Sx, x' \rangle = 0$ for all $S \in D$.

- (ii) $T \in D$ if and only if

$$\sum_{i=1}^n \langle Tx_i, x'_i \rangle = 0$$

whenever $x_1, x_2, \dots, x_n \in X$ and $x'_1, x'_2, \dots, x'_n \in X'$ are such that

$$\sum_{i=1}^n \langle Sx_i, x'_i \rangle = 0$$

for all $S \in D$.

Let B_1 be a Boolean algebra in $C(K)''$. Then $m^*(B_1)$ is strongly equicontinuous in $\mathcal{L}(X')$ [9]. As shown in [6], Proposition 3.6., the completion of linear span of $m^*(B_1)$ called $\hat{M}$ is a Dedekind complete locally solid locally convex Riesz space and has Lebesgue topology. Considering these results we will conclude by presenting an application of the Dual Bade Theorem.

Proposition 12

Let X be a quasicomplete locally convex Hausdorff space such that $\mathcal{L}(X)$ is sequentially complete, and let B be an equicontinuous Boolean algebra

of projections in X . Suppose that D is a closed unital subalgebra of $\hat{M}$. Then the following statements are equivalent.

- (i) $T \in D$.
- (ii) T leaves invariant each D -invariant $\sigma(X', X)$ closed subspace of X' and $T \in \mathcal{L}(X')$.

Proof. It is sufficient to prove that $(ii) \Rightarrow (i)$. Suppose that T leaves invariant D -invariant $\sigma(X', X)$ closed subspace of X' and $T \in \mathcal{L}(X')$. It is obvious that $T \in Z_m(X')$ by Theorem 10. By Lemma 11, we have the following: $T \in D \iff$

$$\sum_{i=1}^n \langle x_i, Tx'_i \rangle = 0,$$

whenever

$$\sum_{i=1}^n \langle x_i, Sx'_i \rangle = 0$$

for $x_1, x_2, \dots, x_n \in X$, $x'_1, x'_2, \dots, x'_n \in X'$ and all $S \in D$. As shown in [8], Proposition 3.2., there exist $x' \in X'$, $x \in X$ such that $\langle x, Sx' \rangle = 0$ for all $S \in D$. Let Y be a $\sigma(X', X)$ closed subspace of X' generated by $\{Sx' : S \in D\}$. Clearly $SY \subseteq Y$, and therefore $TY \subseteq Y$. Obviously the element $x' \in Y$. It implies that $x \perp Y$, i. e., $\langle x, y \rangle = 0$ for all $y \in Y$. We conclude that $\langle x, Tx' \rangle = 0$. By Lemma 11, we have $T \in D$.

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