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DIHEDRAL GROUPS ARE OF SCHOTTKY TYPE *

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Abstract

We show that a dihedral group H of conformal automorphisms of a closed Riemann surface S can be lifted for a suitable Schottky uniformization of S . In particular, this implies the existence of a suitable symplectic homology basis of S for which the symplectic representation of H has a simple form.

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1. Introduction

Given a closed Riemann surface S , retrosection theorem asserts that there are Schottky uniformizations $(\Omega, G, P : \Omega \rightarrow S)$ of S , that is, G is a Schottky group with region of discontinuity Ω and $P : \Omega \rightarrow S$ is a Galois covering with G as covering group.

The importance of Schottky uniformizations respect to the non-universal uniformizations of Riemann surfaces is given by the easy understanding of their geometry. Moreover, there is a conjecture that we can choose our Schottky groups to be classical ones, that is, there is a fundamental domain for the group G consisting on $2g$ round circles (g = the genus of the surface S). Positive evidence of this conjecture is given by some recent work by Maskit.

Classical Schottky groups permits easy computations of some complex analytic functions and/or forms.

Assume that H is a group of conformal automorphisms of the surface S . We may ask if there is a Schottky uniformization of S for which the group H lifts and, if this is the case, then we say that H is of Schottky type.

In [4] we have given a necessary condition, called condition (A), for H to be of Schottky type. This condition turns out to be sufficient if H is an abelian group (see [4] and [5]). If H is isomorphic to a dihedral group, then condition (A) hold trivially.

In [6] we have shown that if H acts free fixed point and it is isomorphic to a dihedral group, then it is of Schottky type. A list of Schottky uniformizations that reflect a dihedral action is also constructed in the above paper. Is natural to ask if all dihedral actions, as conformal groups, are of Schottky type. In the same paper, we constructed an example of a dihedral action of order 6, acting on a Riemann surface of genus 4, and mistakable asserted not to be of Schottky type. In trying to fill out the gap in the arguments, we found that it was in fact of Schottky type. In particular, the question if every dihedral action is of Schottky type still need an answer. The main purpose of this note is to give a definitive answer to the above question.

Theorem 1. *Let S be a closed Riemann surface and H a group of conformal automorphisms of S . If H is isomorphic to a dihedral group, then H is of Schottky type.*

In other words, theorem 1 asserts that any dihedral group of conformal automorphisms of a closed Riemann surface can be realized as a finite normal extension of a Schottky group that uniformizes the surface. We must remark that in [6] the proof is given by explicit constructions of Schottky uniformizations. In this general situation, we only show the existence of them.

In the language of 3-manifolds, the above property can be written as follows. Let M^3 be the connected sum of g copies of $D^2 \times S^1$, where D^2 is a closed 2-dimensional disc and S^1 is the unit circle. We call M^3 a handlebody of genus g . The boundary $S = \partial(M^3)$ is an orientable closed surface of genus g . Theorem 1 is equivalent to the following.

Theorem 2. *Let H be a group of orientation-preserving homeomorphisms of the boundary $S = \partial(M^3)$, where M^3 is a handlebody of genus g . If H is isomorphic to a dihedral group, then there is an orientation-preserving homeomorphism $f : S \rightarrow S$ so that fHf^{-1} can be extended as a group of orientation-preserving homeomorphisms of M^3 .*

The equivalence between both theorems relies on the Nielsen's realization theorem proved by Kerckhoff [7] and Wolpert [12]. Also this is related to the works of B. Zimmermann [13] and A. McCullough, A. Miller and B. Zimmermann [10].

2. Riemann Matrices Associated to Dihedral Groups

An object related to closed Riemann surfaces are the so called Riemann matrices. These matrices are constructed as follows. Assume S is a closed Riemann surface of genus g . A homology basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$, is called symplectic if the intersection matrix is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I denotes the identity $g \times g$ matrix.

Given a symplectic basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$, we may find a basis of holomorphic 1-forms on S , say $\omega_1, \dots, \omega_g$, such that (see [2])

$$\int_{\alpha_i} \omega_j = \delta_{ij}$$

The matrix $Z = (\int_{\beta_i} \omega_j)$ is symmetric and with positive definite imaginary part (the Riemann conditions) [2], called the Riemann matrix for S associated to the above symplectic basis.

The Siegel space of genus g , denoted by $\mathcal{H}_g$, is by definition the space of $g \times g$ matrices with complex coefficients that satisfy to be symmetric and with positive definite imaginary part.

The symplectic group of genus g , denoted by $Sp_{2g}(\mathbf{Z})$, consists of all $2g \times 2g$ matrices with integer coefficients satisfying the condition $AJA^t = J$.

The symplectic group $Sp_{2g}(\mathbf{Z})$ acts on the Siegel space $\mathcal{H}_g$ by the rule

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} (Z) = (A + ZC)^{-1}(B + ZD)$$

Torelli's theorem asserts that the Riemann surface S is determined, up to conformal equivalence, by any of its Riemann matrices.

Let us fix a symplectic basis for the homology of S and let $Z \in \mathcal{H}_g$ be the Riemann matrix of S computed in such a basis. Assume now that we have a group H of conformal automorphisms of S . Then H induces a faithful action at the level of homology

$$\theta : H \rightarrow \theta(H) \subset Sp_{2g}(\mathbf{Z})$$

and induces a faithful action at the level of holomorphic 1-forms

$$\eta : H \rightarrow \eta(H) \subset GL(g, \mathbf{C}).$$

The group $\theta(H)$ fixes the matrix Z , that is, $\theta(h)(Z) = Z$ for all $h \in H$. This makes more easy to compute Riemann matrices for Riemann surfaces with large group of automorphisms. Now, if the representation $\theta(H)$ is more or less easy, then we are able to compute explicitly the fixed points of it in the Siegel space.

The condition for the group H to be of Schottky type is equivalent to the existence of some symplectic basis of S , say $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$, such that the action of H at the level of homology keeps invariant the submodule

generated by the loops $\alpha_1, \dots, \alpha_g$. In particular, the representation $\theta(H)$ in the symplectic group is of the form

$$\theta(h) = \begin{pmatrix} A_h & B_h \\ 0 & C_h \end{pmatrix}, \quad \text{for each } h \in H$$

A matrix of the form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

belongs to the symplectic group if and only if BA^t is symmetric and $C = (A^{-1})^t$.

In particular, theorem 1 and the above observations give us the following

Theorem 3. *Let $H = \langle \tau, \sigma : \tau^n = \sigma^2 = (\tau\sigma)^2 = 1 \rangle$ be a dihedral group of order $2n$, acting as group of conformal automorphisms of a closed Riemann surface S of genus $g \geq 1$. Then there is a symplectic basis for the homology of S so that*

$$\theta(\tau) = \begin{pmatrix} x_1 & x_2 \\ 0 & (x_1^{-1})^t \end{pmatrix} \quad \theta(\sigma) = \begin{pmatrix} y_1 & y_2 \\ 0 & (y_1^{-1})^t \end{pmatrix}.$$

such that

- (i) $x_1^n = y_1^2 = (x_1 y_1)^2 = 1$, that is, x_1 and y_1 gives a (faithful) representation of H in $SL(g, \mathbf{Z})$;
- (ii) $x_2 x_1^t$ and $y_2 y_1^t$ are both symmetric;
- (iii) $x_1 y_1 (x_1 y_2 + x_2 (y_1^{-1})^t) = -(x_1 y_2 + x_2 (y_1^{-1})^t) (x_1^{-1})^t (y_1^{-1})^t$.

The above asserts that, for dihedral groups of conformal automorphisms, the computations of Riemann matrices involves only the computations of faithful representations of dihedral groups on $SL(g, \mathbf{Z})$. In the next section we provide some explicit examples of dihedral actions. For these dihedral actions, there exist symplectic basis as in the above theorem with the extra property that $x_2 = y_2 = 0$ (see for example [14]).

3. Examples of Dihedral Actions

In this section, we consider closed Riemann surfaces with certain type of dihedral groups of conformal automorphisms which are related to certain graphs called prisms and pyramids. Riemann matrices for this class of dihedral groups have been studied in [14]. Let us start with the following result, which is consequence of the results of J.F.X. Ries in [11] and [1].

Theorem 4. *Let $h : S \rightarrow S$ be a conformal automorphism of order n of a closed Riemann surface of genus $g \geq 2$. Assume we are in one of the following two situations:*

- (1) *$n = g$ and every non-trivial power of h has exactly two fixed points with opposite rotation numbers;*
- (2) *$n = g - 1$ and no non-trivial power of h has fixed points.*

Then there is a conformal involution $j : S \rightarrow S$ such that the group generated by h and j is a dihedral group of order $2n$. Moreover, the topological action is totally determined by the group of symmetries of special graphs in $\mathbf{R}^3$ called g -pyramids and g -prisms.

We give a different proof, using Schottky uniformizations, of theorem 4. We decompose it into lemmas 1 and 2 below, which are proven in section 5. The proves are given by explicit construction of Schottky uniformizations. We also construct, at the end of this section, explicit Schottky groups uniformizing Riemann surfaces that reflects the groups of automorphisms of g -prisms.

Let us start with some notations. Let us denote by $\mathbf{R}^3$ the three dimensional euclidean space. We identify this space with $\mathbf{C} \times \mathbf{R}$, where $\mathbf{C}$ is the complex plane. In this way, we can think of the points in $\mathbf{R}^3$ as pairs (z, t) where $z \in \mathbf{C}$ and $t \in \mathbf{R}$. The one-point compactification of $\mathbf{R}^3$ is the 3-sphere $S^3 = \mathbf{R}^3 \cup \{\infty\}$ containing the Riemann sphere $\hat{\mathbf{C}}$.

Fix an integer number $g \geq 2$, and denote by $S^1 \subset \mathbf{C}$ the unit circle. Set $v_k = (e^{2\pi(k-1)i/g}, 0)$, where $k = 1, 2, \dots, g$, and $w = (0, 0)$. Denote by e_j the arc in $S^1 \times \{0\}$ joining the vertices v_j and v_{j+1} . Set $f_k = \{tv_k; t \in [0, 1]\}$, for $j, k = 1, \dots, g$. The graph constructed in S^3 with vertices $v_1, \dots, v_g, w$, and edges $e_1, \dots, e_g, f_1, \dots, f_g$, is called a g -pyramid and denoted by P_g .

Set $T(z, t) = (2z, 2t)$ and let $C_g \subset S^3$ be the graph constructed as follows. The vertices are $u_r = (2e^{2\pi(r-1)i/(g-1)}, 0)$ and $w_r = (\frac{1}{2}e^{2\pi(r-1)i/(g-1)}, 0)$, for $r = 1, \dots, g-1$. The edges of C_g are given by the arcs $j_r \subset T(S^1)$ connecting the vertices u_r and u_{r+1} , the arcs $k_r = T^{-1}(e_r)$, and $l_r = \{tu_r + (1-t)w_r; t \in [0, 1]\}$, for $r = 1, \dots, g-1$. We call C_g a g -prism.

Let us consider the conformal automorphisms of S^3 given by $b(z, t) = (\bar{z}, -t)$, $c(z, t) = (1/\bar{z}, -t)$ and $a_k(z, t) = (e^{2\pi i/k}z, t)$, where k is a non-negative integer.

It is a well known fact (see [3]) that the group of symmetries of the graph P_g has the following description.

- (1) If $g \geq 4$, then it is isomorphic to the dihedral group of order $2g$. This group is generated by the transformations a_g and b .
- (2) If $g = 3$, then it is isomorphic to the symmetric group in four letters S_4 , generated by a_3 , b and an extra element d of order 4.

Denote in both cases the subgroup generated by a_g and b as D_g .

The group of symmetries of the graph C_g has the following description.

- (1) If $g \neq 5$, then it is the direct sum of $\mathbf{Z}/2\mathbf{Z}$ (generated by c) and a dihedral group of order $2(g-1)$ (generated by a_{g-1} and b).
- (2) If $g = 5$, then there is an extra symmetry d , of order two, such that a_{g-1} , b , c and d generate the symmetry group of P_5 . This group is isomorphic to the symmetric group S_4 in four letters.

We denote the group of symmetries of C_g by F_g . Its subgroup, generated by a_{g-1} and b , is a dihedral group of order $2(g-1)$ and we denote it by D'_{g-1} .

We can construct a tubular neighborhood V_g (respectively, W_g) of P_g (respectively, C_g), which is invariant under the action of D_g (respectively, F_g). In this case, V_g (respectively, W_g) is a compact 3-manifold homeomorphic to a handlebody of genus g . The boundary $S_g = \partial V_g$ (respectively, $R_g = \partial W_g$) is a closed orientable surface of genus g . The group D_g (respectively, F_g) acts on both V_g and S_g (respectively, W_g and R_g) as a group of orientation preserving homeomorphisms. The Nielsen realization theorem [7] asserts that

- (i) we can give to S_g (respectively, R_g) a Riemann surface structure for which D_g (respectively, F_g) acts as a group of conformal automorphisms, and

- (ii) we can give to V_g (respectively, W_g) a hyperbolic structure for which D_g (respectively, F_g) acts as group of isometries.

Let S be a closed Riemann surface of genus $g \geq 2$. Assume H is a group of conformal automorphisms of S . We say that H is of pyramidal type if:

- (1) there is an orientation preserving homeomorphism $\Phi : S \rightarrow S_g$ such that, $\Phi h \Phi^{-1} \in D_g$ for all $h \in H$ and
- (2) $\Psi : H \rightarrow D_g$, defined by $\Psi(h) = \Phi h \Phi^{-1}$, is an isomorphism of groups.

If $h : S \rightarrow S$ is a conformal automorphism of some Riemann surface S , and $p \in S$ is fixed point of it, then we can define a rotation number $\alpha(h, p) \in (-\pi, \pi]$ so that, in local coordinates z with $z(p) = 0$, we have $h(z) = e^{\alpha(h, p)i} z$.

Lemma 1. *Let S be a closed Riemann surface of genus $g \geq 2$. Assume that there is a conformal automorphism $h : S \rightarrow S$, of order g , having exactly two fixed points, say p and q . If*

- (1) $\alpha(h, p) = -\alpha(h, q)$ for $g \geq 3$, and
- (2) every non-trivial power of h has exactly two fixed point (that is, p and q),

then there is a conformal involution $j : S \rightarrow S$ so that the group generated by h and j is of pyramidal type.

Let S be a closed Riemann surface of genus $g \geq 2$. Assume H is a group of conformal automorphisms of S . We say that H is of prism type if:

- (1) there is a preserving orientation homeomorphism $\Phi : S \rightarrow R_g$ so that, $\Phi h \Phi^{-1} \in D'_{g-1}$ for all $h \in H$ and
- (2) $\Psi : H \rightarrow D'_{g-1}$, defined by $\Psi(h) = \Phi h \Phi^{-1}$, is an isomorphism of groups.

Lemma 2. *Let S be a closed Riemann surface of genus $g \geq 2$. Assume that there is a conformal automorphism $h : S \rightarrow S$ of order $g - 1$. If no non-trivial power of it has fixed points, then there is a conformal involution $j : S \rightarrow S$ so that the group generated by h and j is of prism type.*

The examples

We proceed to construct Schottky uniformizations of Riemann surfaces of genus g admitting a conformal group topologically equivalent to F_g .

1.- Case $g \neq 5$

We have that $F = F_g$ has generators a, b and c , with relations

$$a^{g-1} = b^2 = c^2 = (ab)^2 = caca^{-1} = (cb)^2 = I.$$

Let $P : R \rightarrow R/F$ the holomorphic branched covering induced by the action of F on R . We have that R/F has five branch values of order two. They are as follows

- (1) two of them, say p and q , are projections of two fixed points of b ;
- (2) one, say r , is projection of a fixed point of c ;
- (3) one, say s , is projection of a fixed point of ab ; and
- (4) one, say t , is projection of a fixed point of cab .

We can uniformize the surface R/F by a Kleinian group K constructed from the first Maskit-Klein combination theorem [8] with the finite groups $K_1 = \langle J, T; J^2 = T^2 = (JT)^2 = I \rangle$ and $K_2 = \langle N, N^2 = I \rangle$.

A fundamental domain D of K is drawn in figure 1. We also assume that the fixed points of N are projected onto the points p and q , the fixed points of J and T contained in the boundary of D are projected onto the points r and s , respectively, and the fixed point of JT is projected onto t .

FIGURE 1

The existence of such a group is due to (i) quasiconformal deformation theory and (ii) the fact that we can construct a particular group K_0 as desired (the quotient $\Omega(K_0)/K_0$ does not need to be R/F).

We have the following properties:

- (5) K is free product of K_1 and K_2 ;
- (6) K is geometrically finite;
- (7) the region of discontinuity Ω of K is connected;
- (8) every non-loxodromic element of K is conjugate to either J , T , JT or N .

We have a natural surjective homomorphism $\Phi : K \rightarrow F$ defined by $\Phi(J) = c$, $\Phi(T) = ab$ and $\Phi(N) = b$. This homomorphism is constructed by the lifting of a suitable base for the fundamental group of $R/F - B$, where $B = \{p, q, r, s, t\}$.

The kernel of Φ is the smallest normal subgroup of K containing the transformations A^{g-1} , $(JN)^2$ and $JAJA^{-1}$, where $A = TN$. Let us denote this subgroup by G .

We have that G is torsion-free since neither J , T , JT nor N belong to it.

Since G has finite index in K , we have that its region of discontinuity is also Ω , and that G is geometrically finite.

As a consequence of [9], we have that G is in fact a Schottky group.

The Riemann-Hurwitz formula and the fact that K/G has order $4(g-1)$, imply that the genus of G is g .

The construction asserts that there is a conformal equivalence $f : \Omega/G \rightarrow R$ such that $f(K/G)f^{-1} = F$.

2.- Case $g = 5$

The group $F = F_5$ has generators a and b with relations

$$a^4 = b^2 = (ba)^3 = I.$$

Let $P : R \rightarrow R/F$ the holomorphic branched covering induced by the action of F on R . We have that R/F has two branch values of order two and two of order three. They are

- (1) two of them, say p and q , are projections of two fixed points of b ; and
- (2) the other two, say r and s , is projection of a fixed point of ba .

We can uniformize the surface R/F by a Kleinian group K constructed from the first Maskit-Klein combination theorem [8] with the finite groups $K_1 = \langle T; T^3 = I \rangle$ and $K_2 = \langle N, N^2 = I \rangle$. A fundamental domain D of K is drawn in figure 2. We also assume that the fixed points of N are projected onto the points p and q , and the fixed points T are projected onto the points r and s .

FIGURE 2

We have the following properties:

- (3) K is free product of K_1 and K_2 ;
- (4) K is geometrically finite;
- (5) the region of discontinuity of K , say Ω , is connected;
- (6) every non-loxodromic element of K is conjugate to either N , T or T^2 .

We have a natural surjective homomorphism $\Phi : K \rightarrow F$ defined by $\Phi(N) = b$, $\Phi(T) = ba$. This homomorphism is constructed by the lifting of a suitable base for the fundamental group of $R/F - B$, where $B = \{p, q, r, s\}$. The kernel of Φ is the smallest normal subgroup of K containing the transformation A^4 , where $A = NT$. Let us denote this subgroup by G and denote by B the transformation N .

We have that G is torsion-free since neither T , T^2 nor N belong to it. Since G has finite index in K , we have that its region of discontinuity is also Ω , and that G is geometrically finite.

As a consequence of [9], we have that G is in fact a Schottky group.

The Riemann-Hurwitz formula together to the fact that K/G has order 24 assert that the genus of G is 5.

The construction asserts that there is a conformal equivalence $f : \Omega/G \rightarrow R$ such that $f(K/G)f^{-1} = F$.

4. Proof of Theorem 1

Assume we are able to construct a collection of pairwise disjoint simple loops on the surface S such that they cut off S into planar surfaces. Then we can find a sub-collection of these loops such that they do not disconnect S but they cut off S into a planar surface (a $2g$ -connected surface). The sub-collection define a Schottky uniformization of S (take the normal subgroup generated by these loops in the fundamental group of S).

If such a collection of loops has the extra property that it is invariant under the action of H , then basic lifting criterion asserts that H lifts to the Schottky uniformization defined by them.

Construction of a Collection of Simple Loops

Let us consider a presentation for the group H of the form $H = \langle \tau, \sigma : \tau^n = \sigma^2 = (\tau\sigma)^2 = 1 \rangle$.

We denote by $P : S \rightarrow R = S/\langle \tau \rangle$ the canonical (branched) Galois covering induced by the cyclic group $\langle \tau \rangle$ of order n generated by τ .

Since σ has order two and conjugate $\langle \tau \rangle$ in itself, we have the existence of a conformal automorphism $\hat{\sigma} : R \rightarrow R$ satisfying $P\sigma = \hat{\sigma}P$. All the liftings of $\hat{\sigma}$ are the n involutions $\tau^k\sigma$, $k = 0, \dots, n-1$.

The topological action of the involution $\hat{\sigma} : R \rightarrow R$ is well understand and it is shown in figure 3. We have two possibilities:

- (1) $\text{Fix}(\hat{\sigma})$ is non-empty; or
- (2) $\text{Fix}(\hat{\sigma})$ is empty.

 FIGURE 3. The topological action of $\hat{\sigma}$

We proceed to show the existence of a collection of loops as desired in each situation. We must remark that a branch value of $P : S \rightarrow R$ cannot be a fixed point of $\hat{\sigma}$. In fact, if this is the case, then there is a point $y \in S$ which is fixed by some non-trivial power of τ and fixed by some element of the form $\tau^a \sigma$, where $a \in \{0, 1, 2\}$. In any case, it will assert that the stabilizer of y in H is not cyclic, a contradiction.

Case 1: Let us assume $\text{Fix}(\hat{\sigma}) \neq \emptyset$. In this case, we proceed to draw a collection of simple loops as shown in figure 4. Observe that if no non-trivial power of τ has fixed points, then the loops η_j and the regions A_j are not there.

FIGURE 4. A collection of simple loops on R

Each of these loops lifts to exactly n simple loops, permuted by τ . The reason is that $l, \hat{\sigma}(l), \theta_1, \dots, \theta_k$ are product of commutators. The loop η_1 bounds a disc containing only the branch values p_1 and $\hat{\sigma}(p_1)$. The rotation numbers at these points are the same but with opposite sign (this because σ conjugates each τ^a into τ^{-a}). It follows that η_1 lifts to exactly n loops. In similar fashion we see that the other loops $\eta_2, \dots, \eta_m$, each one lifts to exactly n loops.

Let us consider a connected component Y_1 of $P^{-1}(X_1)$, where X_1 is the subsurface of R bounded by the loop l as shown in figure 3. Since the loop l lifts to exactly n loops, we have that such a component is a closed Riemann surface with r pairwise disjoint discs removed, on which a cyclic group of order r acts free fixed points and permutes cyclically the boundary components. As a consequence of the results in [4], we are able to find a collection of pairwise disjoint simple loops on Y_1 that is invariant under the cyclic group and cut off Y_1 into genus zero surfaces. We consider on each translate of Y_1 under the group H the translates of the loops on Y_1 . The action on the connected components of $P^{-1}(\hat{\sigma}(X_1))$ is obtained by conjugating the above by σ .

A connected component of $P^{-1}(X_3)$, where X_3 is the subsurface bounded by the simple loops θ_k , l η_m and $\hat{\sigma}(l)$ is homeomorphic to X_3 . This because each of the boundary loops lifts to exactly n simple loops.

Let B_1 be a connected component of $P^{-1}(A_1)$. This surface is precisely invariant under a cyclic subgroup of $\langle \tau \rangle$ of order r_1 (exactly the order of branching of the two points enclosed by η_1 , and a involution $\tau^a \sigma$. Since the boundary loop η_1 lifts to exactly n loops, we have that by gluing discs to the boundaries of both A_1 and B_1 we do not add branching. We may apply Riemann-Hurwitz's formula to obtain that B_1 is a genus zero surface with exactly r_1 boundary loops (permuted cyclically by the above cyclic subgroup).

For the subsurface A_j bounded by the loops η_{j-1} and η_j the situation is similar to the above case.

Let Z_1 be a connected component of $P^{-1}(T_1)$. Again, the fact that θ_1 lifts to exactly n loops, the stabilizer of Z_1 is a dihedral group generated by some power of τ and some element of order two of the form $\tau^a \sigma$. By gluing discs to the boundaries of Z_1 and T_1 we only add one branched value of order two on the surface obtained from T_1 . Each gluing disc to the component Z_1 produces exactly one fixed point of a conjugate of the element $\tau^a \sigma$. Applying Riemann-Hurwitz's formula, we obtain that Z_1 is a genus one Riemann surface admitting a dihedral action. This dihedral action in a closed torus is generated by a rotation (acting free fixed points) and an involution that fixes exactly 4 points. There is a collection of pairwise disjoint simple loops on Z_1 , invariant under the dihedral action, and cutting off this torus with holes into genus zero surfaces. We translate these loops by the action of the cyclic group generated by τ to the other components of $P^{-1}(T_1)$.

For the subsurfaces T_j we proceed in similar fashion.

Putting all together, we obtain a collection of pairwise disjoint simple loops on S that cut off it into genus zero surfaces and it is invariant under the action of H .

Case 2: Let us assume $Fix(\hat{\sigma}) = \emptyset$. In this case, we proceed to draw a simple loop γ that is invariant under $\hat{\sigma}$, disjoint from the branched values of the (branched) covering $P : S \rightarrow R$ and homologically non-trivial as shown in figure 5. If we show that the loop γ lifts to exactly n loops, then we can proceed as in the above case (think of such a loop as the fixed point of $\hat{\sigma}$ enclosed by the loops η_1 in that situation).

FIGURE 5. A simple loop γ on R

To show the lifting property of γ , we consider a connected component η of $P^{-1}(\gamma)$. If our claim is not true, then the stabilizer in H of η is a dihedral group acting free fixed points. But this will asserts that a dihedral group is a surjective homomorphism image of $\mathbf{Z}$, a contradiction.

Let us consider the collection of loops as shown in figure 6. Now we can proceed as in case 1 to construct the desired collection of loops. The proof of theorem 1 is now finished.

FIGURE 6. A collection of simple loops on R

5. Proof of Theorem 4

In this section, we first prove lemmas 1 and 2. Theorem 4 is just combination of both lemmas. The prove of both lemmas are constructive and give explicit Schottky uniformizations.

5.1. Proof of Lemma 1

Let us consider the branched covering $P : S \rightarrow R = S/H$, where H is the cyclic group generated by h . We may assume that h satisfies the following minimal condition

$$0 < \alpha(h, p) \leq \alpha(h^r, p), \quad r = 1, 2, \dots, g-1.$$

Set $u = P(p)$, $v = P(q)$, $\tilde{S} = S - \{p, q\}$ and $\tilde{R} = R - \{u, v\}$. We still denoting by P the induced covering obtained by restriction of the above one. The Riemann-Hurwitz formula asserts that R is a genus one surface.

Consider oriented simple loops α , β , δ and η , through a point $x \in \tilde{R}$ as shown in figure 7.

FIGURE 7

The fundamental group of $\tilde{R}$, based at x , is generated by α , β , δ and η , with relation $[\alpha, \beta]\eta\delta = 1$, where $[\alpha, \beta]$ denotes the commutator $\beta^{-1}\alpha^{-1}\beta\alpha$.

Let $y \in \tilde{S}$ be such that $P(y) = x$. We have a natural surjective homomorphism

$$\Psi_g : \Pi_1(\tilde{R}, x) \rightarrow H,$$

defined by lifting loops, based at x , with starting point y .

Since the group H is Abelian and $\alpha(h, p) = -\alpha(h, q)$ (if order of h is bigger than two), we have that

$$\Psi_g(\eta) = h^r, \quad \Psi_g(\delta) = h^{-r}, \quad \Psi_g([\alpha, \beta]) = 1,$$

where h^r generates H . As a consequence of proposition 6 of [5], we may assume that $\Psi_g(\beta) = 1$ and $\Psi_g(\alpha) = h$.

By quasiconformal deformation theory, we can find a (branched) holomorphic covering $Q : \Omega \rightarrow R$ such that:

- (1) its covering group is a Kleinian group $\tilde{G}$ with Ω as its region of discontinuity;
- (2) $\tilde{G}$ is a cyclic normal extension of a Schottky group of genus one, freely generated by a loxodromic element L and an elliptic element E of order g ;
- (3) the fixed points of E are projected by Q onto the points u and v , respectively;
- (4) a fundamental domain D (see figure 8) for $\tilde{G}$ is bounded by three disjoint simple loops γ^+ , γ^- and γ , satisfying

$$(4.1) \quad L(\gamma^-) = \gamma^+, \quad E(\gamma) = \gamma,$$

$$(4.2) \quad \text{both fixed points of } E \text{ belong to } \gamma,$$

$$(4.3) \quad \text{the loops } \gamma^\pm \text{ are projected onto the loop } \beta,$$

$$(4.4) \quad \text{the loop } \gamma \text{ is projected onto a simple path connecting } u \text{ and } v \text{ and disjoint from } \alpha.$$

FIGURE 8

Consider the subgroup G generated by the transformations $L_k = E^k L E^{-k}$, $k = 0, 1, \dots, g - 1$. We have that G is a normal subgroup of $\tilde{G}$ of index g and $\tilde{G}/G$ is isomorphic to the cyclic group H .

The normality of G asserts that Ω is also the region of discontinuity of G .

By the choice of the loops α , β , δ and η , we have that the coverings $P : S \rightarrow R$ and the natural covering $\Omega/G \rightarrow R$, induced by Q , are equivalent. The automorphism $h : S \rightarrow S$ corresponds to the automorphism induced by E .

Up to conjugation, we may assume that

$$E = \begin{pmatrix} e^{\pi i/g} & 0 \\ 0 & e^{-\pi i/g} \end{pmatrix},$$

and one fixed of L is 1. Denote by r the other fixed point of L . Then $r \in \mathbb{C} - \{0, 1\}$. Let us consider the transformation

$$J = \begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix}.$$

Direct computations show that

- (1) $J^2 = I$
- (2) $JEJ = E^{-1}$
- (3) $JLJ = L^{-1}$

Let K be the group generated by E , L and J . We have that K is a finite normal extension of

- (1) $\tilde{G}$ of index two, and
- (2) G of index $2g$.

Moreover, K/G is isomorphic to a dihedral group of order $2g$.

The transformation J induces an automorphism $j : S \rightarrow S$ of order two. We have then that j and h generates a group $\mathcal{D}$ isomorphic to a dihedral group of order $2g$.

By construction, it is not hard to see that the topological action of $\mathcal{D}$ on S is topologically the same as the one given by D_g on S_g .

We must note that the automorphism j corresponds to the automorphism $ab \in D_g$ and h (with the minimal rotation condition) corresponds to $a \in D_g$. We have proved lemma 1.

Remark. Another approach to this theorem is the following. Let us denote by

- (1) H the cyclic group generated by $h : S \rightarrow S$;
- (2) R the quotient Riemann surface obtained by the action of H on S ;
- (3) $P : S \rightarrow R$ the natural holomorphic branched covering induced by H .

By the formula of Riemann-Hurwitz and the assumptions on the fixed points of the elements of H , we have that R is a torus with exactly two branch values of order g . Set $x = P(p)$ and $y = P(q)$ these two branch values.

Since $x \neq y$, there is a unique conformal involution $J : R \rightarrow R$ such that $J(x) = y$ and having exactly four fixed points. As a consequence of our hypothesis on the rotation numbers, we can lift J to a conformal automorphism of S .

If $j : S \rightarrow S$ denotes one of such liftings, that is, $Pj = JP$, then $j(p) = q$. In fact, since $J(x) = y$, we must have that $j(p)$ and q are H equivalent. But q is a fixed point of every element in H , that is, $j(p) = q$.

The fact that J has fixed points implies that there is a point $s \in S - \{p, q\}$ with $j(s) = h^r(s)$. In particular, $h^{-r}j$ fixes the point s . Since $h^{-r}j$ is again a lifting of J , we may assume that is the case for j .

The property $J^2 = I$ implies that j^2 belongs to H and, in particular, by our hypothesis on H and the above fact that $j^2 \in H$ fixes $s \notin \{p, q\}$, we have $j^2 = I$.

We also have that $j h j = h^t$, where $t \in \{1, \dots, g-1\}$ and $(g, t) = 1$. By looking at the rotation numbers, we have the following:

$$-t\alpha(h, q) = t\alpha(h, p) = \alpha(h^t, p) = \alpha(j h j, p) = \alpha(h, j(p)) = \alpha(h, q).$$

It follows that $t = -1$ and, in particular, j and h generate a dihedral group of order $2g$.

5.2. Proof of Lemma 2

Let us consider the unbranched covering $P : S \rightarrow R = S/H$, where H is the cyclic group generated by h . The Riemann-Hurwitz formula asserts that R is a genus two surface.

Consider oriented simple loops $\alpha_1, \alpha_2, \beta_1$ and β_2 , through a point $x \in R$ as shown in figure 9.

FIGURE 9

The fundamental group of R , based at x , has generators $\alpha_1, \alpha_2, \beta_1$ and β_2 , and relation $[\alpha_1, \beta_1][\alpha_2, \beta_2] = 1$, where $[\alpha_i, \beta_i] = \beta_i^{-1}\alpha_i^{-1}\beta_i\alpha_i$.

If $y \in S$ is such that $P(y) = x$, then we have a natural surjective homomorphism

$$\Psi_g : \Pi_1(R, x) \rightarrow H,$$

defined by lifting loops at y .

Since the group H is Abelian, we have that

$$\Psi_g([\alpha_i, \beta_i]) = 1.$$

By proposition 6 of [5], we may assume that $\Psi_g(\beta_1) = \Psi_g(\beta_2) = \Psi_g(\alpha_2) = 1$ and $\Psi_g(\alpha_1) = h$.

By quasiconformal deformation theory, we can find a Kleinian group $\tilde{G}$ (a cyclic normal extension of a Schottky group of genus two), with region of discontinuity Ω , generated by loxodromic elements A, L and an elliptic element E of order $g - 1$, such that

- (1) $EA = AE$,
- (2) there is a holomorphic covering $Q : \Omega \rightarrow R$ with $\tilde{G}$ as covering group, and
- (3) $\tilde{G}$ has a fundamental domain D (as shown in figure 10) bounded by four arcs $\eta^\pm, \delta^\pm$ and two curves $\gamma^\pm$, such that
 - (3.1) $L(\gamma^+) = \gamma^-, A(\eta^+) = \eta^-, E(\delta^+) = \delta^-$.
 - (3.2) The loop $\gamma^\pm$ is projected onto a simple loop free homotopic to α_2 .
 - (3.3) The arc $\eta^\pm$ is projected to a simple loop free homotopic to α_1 .
 - (3.4) The arc $\delta^\pm$ is projected to a simple loop free homotopic to β_1 .
 - (3.5) The loop β_2 can be lifted, in D , to a simple path connecting the loops γ^+ and γ^- .

FIGURE 10

Consider the subgroup G generated by the transformations A and $L_k = E^k L E^{-k}$, $k = 0, 1, \dots, g-2$. We have that G is a normal subgroup of $\tilde{G}$ of index $g-1$ and $\tilde{G}/G$ is isomorphic to the cyclic group H .

The normality of G asserts that Ω is also the region of discontinuity of G .

By the choice of the loops $\alpha_1, \alpha_2, \beta_1, \beta_2$, we have that the coverings $P : S \rightarrow R$ and the natural covering $\Omega/G \rightarrow R$, induced by Q , are equivalent.

The automorphism $h : S \rightarrow S$ corresponds to the automorphism induced by E .

Up to conjugation, we may assume that

$$E = \begin{pmatrix} e^{\pi i/(g-1)} & 0 \\ 0 & e^{-\pi i/(g-1)} \end{pmatrix},$$

and one fixed of L is 1. Denote by r the other fixed point of L . Then $r \in \mathbb{C} - \{0, 1\}$. Let us consider the transformation

$$J = \begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix}.$$

In particular, the transformation A has the form $A(z) = \lambda z$, for some $\lambda \in \mathbf{C}$ with absolute value different from 0 and 1. Direct computations show that

- (1) $J^2 = I$
- (2) $JEJ = E^{-1}$
- (3) $JLJ = L^{-1}$
- (4) $JAJ^{-1} = A^{-1}$

Let K be the group generated by E , A , L and J . The group K is a finite normal extension of

- (1) $\tilde{G}$ of index two,
- (2) G of index $2(g-1)$.

Moreover, K/G is isomorphic to the dihedral group of order $2(g-1)$. The automorphism $j : S \rightarrow S$ induced by J is an involution that together h generates a dihedral group.

By the construction, it is not hard to see that the topological action of this dihedral group on S is the same topological action of D'_{g-1} on R_g . We have proved lemma 2.

Remark. Another proof of this is the following. Let us denote by

- (1) H the cyclic group generated by $h : S \rightarrow S$;
- (2) R the quotient Riemann surface obtained by the action of H on S ;
- (3) $P : S \rightarrow R$ the natural holomorphic branched covering induced by H .

By the formula of Riemann-Hurwitz and the assumptions on the no existence of fixed points for the elements of H , we have that R is a genus two surface.

Let us consider the hyperelliptic involution $J : R \rightarrow R$. By Farkas result in [1], we can lift it to a conformal automorphism of S .

Let $j : S \rightarrow S$ be a lifting of J , that is, $Pj = JP$. Since J has fixed points, we have the existence of a point $s \in S$ with $j(s) = h^r(s)$. In

particular, $h^{-r}j$ fixes the point s . Since $h^{-r}j$ is also a lifting of J , we may assume that is the case for j .

The property $J^2 = I$ asserts that j^2 belongs to H . But j^2 fixes s . By our hypothesis on the fixed points of the elements of H , the above is only possible if $j^2 = I$.

Let us now consider the lifting of J given by hj . We claim that it has a fixed point. In fact, the group K generated by j and H has the property that $S/K = R/J$ is a sphere with exactly 6 branch values of order two.

The elements of order two in K are the liftings of J and maybe one element of H . Since no element of H has fixed points, we have that the only elements of order two in K that may have fixed points are the liftings of J .

The covering induced by K is of degree $2(g-1)$. The Riemann-Hurwitz formula asserts that on S there are exactly $6(g-1)$ fixed points of order two.

Since each lifting of J must have at most 6 fixed points on S , and there are exactly $g-1$ liftings, we have that every lifting has exactly 6 fixed points.

Now we can use the same argument to show that hj has order two. In consequence, the group generated by h and j is a dihedral group of order $2(g-1)$.

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