CYCLIC GROUPS OF AUTOMORPHISMS OF SCHOTTKY TYPE *

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Abstract

Given an abstract group of order n , we call its Schottky genus to the minimum genus $g \geq 2$ on which it acts as group of conformal automorphisms of Schottky type. In this note, we compute the Schottky genus for both cyclic and dihedral groups. In particular, we obtain that the Schottky genus of the dihedral group of order $2n$ is the same as for the cyclic group of order n . Since every dihedral group is of Schottky type, we have that the Schottky genus of a dihedral group of order $2n$ is also its minimum genus.

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1. Introduction

The retrosection theorem [1], [8], [9] asserts that for every closed Riemann surface S there is a Schottky group G , with region of discontinuity Ω , and a holomorphic covering $P : \Omega \rightarrow S$ with G as covering group. We say that the three-tuple $(\Omega, G, P : \Omega \rightarrow S)$ is a Schottky uniformization of S .

The importance of Schottky uniformizations respect to the non-universal uniformizations of Riemann surfaces is given by the easy understanding of their geometry. Moreover, there is a conjecture that we can choose our Schottky groups to be classical ones, that is, there is a fundamental domain for the group G consisting on $2g$ round circles (g = the genus of the surface S). Positive evidence of this conjecture is given by some recent work by Maskit.

Let us consider a pair (S, H) , where H is a group of conformal automorphisms of the closed Riemann surface S . In general it is not true the existence of a Schottky uniformization $(\Omega, G, P : \Omega \rightarrow S)$ for which H can be lifted. If this happens, then we say that H is of Schottky type.

A Necessary condition for a (finite) group of conformal automorphisms of a closed Riemann surface to be of Schottky type is given in [4]. This necessary conditions is called the condition (A) which is given as follows.

The condition (A). Let H be a finite group of conformal automorphism of some closed Riemann surface S and $\mathcal{F}$ the set of fixed points of the non-trivial elements of H . For each $p \in \mathcal{F}$ we have the stabilizer of p in H defined by $H(p) = \{h \in H : h(p) = p\}$. For $h \in H(p)$, different from the identity, we have a well defined number $\alpha(h, p) \in [-\pi, \pi)$, called the rotation number of h at p .

We say that H satisfies the condition (A) if there is a collection $\mathcal{C} = \{\{p_\alpha, q_\alpha\} : \alpha \in \mathcal{A}\}$ of pairwise disjoint subsets of $\mathcal{F}$ such that:

$$(1) \bigcup_{\alpha \in \mathcal{A}} \{p_\alpha, q_\alpha\} = \mathcal{F};$$

$$(2) \text{ If } \{p_\alpha, q_\alpha\} \in \mathcal{C}, \text{ then}$$

$$(2.1) \ p_\alpha \neq q_\alpha,$$

$$(2.2) \ H(p_\alpha) = H_\alpha = H(q_\alpha),$$

$$(2.3) \ \alpha(h, p) = -\alpha(h, q) \text{ for each } h \in H_\alpha \text{ of order greater than two,}$$

$$(2.4) \text{ if there is some } h \in H \text{ satisfying } h(p_\alpha) = q_\alpha, \text{ then } h^2 = I.$$

Remarks.

- (1) If the group H is cyclic, then conditions (2.2) and (2.3) assert that there is no $h \in H - \{I\}$ such that $h(p_\alpha) = q_\alpha$.
- (2) If H is abelian, then condition (2.4) can be replaced by: if there is some $h \in H$ satisfying $h(p_\alpha) = q_\alpha$, then $h^2 = I$ and H_α has order two. In this case the group generated by h and H_α is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.
- (3) The condition (A) is sufficient for H to be of Schottky type for abelian groups [5], for dihedral groups [6], the alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ and the symmetric group $\mathcal{S}_4$ [7]
- (4) If the group $H \cong \mathbf{Z}_n$ is a cyclic group of automorphisms on a closed Riemann surface of genus $g \geq 2$ satisfying the condition (A), then Riemann-Hurwitz's formula implies

$$n \leq \begin{cases} 2(g+1), & \text{if } g \text{ is even} \\ 2g, & \text{if } g \text{ is odd} \end{cases}$$

A general problem is to obtain the minimum genus on which a given finite group acts as a group of conformal automorphisms. This has been solved, for instance, in the cyclic and abelian cases by Harvey in [3] and MacLachlan in [10], respectively. For the class of closed Klein surfaces, this problem has been solved for cyclic groups by E. Bujalance in [2].

Given a finite abstract group H , we define the Schottky genus of H , denoted by $\mathcal{S}(H)$, as the minimum genus (greater or equal to two) on which H acts as a Schottky type group. Clearly, the Schottky genus of H is greater or equal to the minimum genus of H . For dihedral groups they coincide.

In this note, we compute the Schottky genus for cyclic and dihedral groups. In particular, we obtain the following.

Theorem 1. *Let H be cyclic group of order $n \geq 2$. If $n > 2$, let $2 \leq t$, $u \leq n$ be divisors of n so that $\frac{n}{t}$ and $\frac{n}{u}$ are relatively prime, maximizing the value $\frac{1}{t} + \frac{1}{u} < 1$. Then the Schottky genus of H is given by*

$$\mathcal{S}(H) = \begin{cases} 2, & \text{if } n = 2 \\ 1 + n(1 - \frac{1}{t} - \frac{1}{u}), & \text{if } n > 2 \end{cases}$$

Theorem 2. *The Schottky genus of the Dihedral group H of order $2n$ is the Schottky genus of the cyclic group of order n . Moreover, this is the same as the minimum genus of H .*

2. Some Types of Schottky Groups

For each $n > 2$ we construct finite normal extensions K of a Schottky group G of genus $S(n) \geq 2$ so that K/G is isomorphic to a dihedral group of order $2n$, where

$$S(n) = \begin{cases} 2, & \text{if } n = 2 \\ 1 + n(1 - \frac{1}{t} - \frac{1}{u}), & \text{if } n > 2 \end{cases}$$

where $2 \leq t, u \leq n$ are divisors of n so that $\frac{n}{t}$ and $\frac{n}{u}$ are relatively prime and $\frac{1}{t} + \frac{1}{u} < 1$. For $n > 2$ prime, we have that $S(n) = n - 1 = S(\mathbf{Z}_n)$, and $S(D_n), S(\mathbf{Z}_n) \leq S(n)$. From now on, we fix some integer $n \geq 2$ and set $D_n = \langle x, y : x^n = y^2 = (yx)^2 = 1 \rangle$.

2.1. Main Construction

Let $2 \leq t, u \leq n$ be divisors of n (it may happen that $t = u$), so that $\frac{n}{t}$ and $\frac{n}{u}$ are relatively prime and $\frac{1}{t} + \frac{1}{u} < 1$.

We construct, using Klein-Maskit's combination theorems I and II [11], a Kleinian group K generated by Möbius transformations $A(z) = e^{2\pi i/t} z$, $B(z) = \frac{1}{z}$ and C of order u with fixed points $e^{\pm \pi i/2t}$, so that $D_u = \langle B, C : B^2 = C^u = (CB)^2 = 1 \rangle$ (see figures 1 ($u = 2$) and 2 ($u \geq 3$)).

Figure 1

Figure 2

The group K is a free product of the dihedral groups $D_t = \langle A, B \rangle$ and $D_u = \langle B, C \rangle$, amalgamated along the cyclic group of order two $\langle B \rangle$.

Let us consider the surjective homomorphism $\phi : K \rightarrow D_n$ defined by $\phi(A) = x^{n/t}$, $\phi(B) = y$ and $\phi(C) = x^{n/u}$. Denote by G the kernel of the above homomorphism. We have that G is a normal subgroup of index $2n$ in K . Moreover, from the Klein-Maskit's combination theorems, we have that each elliptic transformation in K is conjugated to either B , AB , BC and non-trivial powers of A and C . In particular, the group G is torsion free. Since K has no parabolic transformations, we have that neither G does. Since G is a finite index torsion free subgroup of K , we have that it is a free group of finite rank. Riemann-Hurwitz's formula implies that the genus of G is $1 + n(1 - \frac{1}{t} - \frac{1}{u})$.

Remark. Observe that if we use $t = 2$ (then $u > 2$), then we obtain that the genus of G is then $1 + n(\frac{1}{2} - \frac{1}{u})$. If $n > 2$ is a prime, then necessarily $t = u = n$ and the genus of G is $n - 1$.

3. Computation of the Schottky genus for Cyclic groups

In the above section, we have shown an upper bound for the Schottky genus of a cyclic group. In this section, we proceed to show that this bound is exactly the Schottky genus. Let us assume that $H \cong \mathbf{Z}_n$ is a group of conformal automorphisms of Schottky type of a closed Riemann surface S of genus $g \geq 2$.

Case $n = 2$. In this case, the cyclic group of order two acts on any Riemann surface of genus two as the hyperelliptic involution. Since a group of order two always is Schottky type [5], it follows that $\mathcal{S}(\mathbf{Z}_2) = 2$.

Case $n \geq 3$. As a consequence of the condition (A), there are non-negative integers γ, k, r , $2 \leq l_1 \leq l_2 \leq \dots \leq l_r < n$, with l_i a divisor of n , so that $R = S/H$ has signature

$$(\gamma, 2k + 2r; n, \dots, n, l_1, l_1, l_2, l_2, \dots, l_r, l_r)$$

The Riemann-Hurwitz formula then gives the equality

$$(*) \quad g = n(\gamma - 1) + 1 + k(n - 1) + n(r - \sum_{i=1}^r \frac{1}{l_i})$$

We want to minimize the value of $g \geq 2$ in function of $n \geq 3$. Since $l_i \geq 2$, then we have that $(r - \sum_{i=1}^r \frac{1}{l_i}) \geq \frac{r}{2}$.

Subcase $\gamma \geq 2$. Formula $(*)$ asserts that $g > n$.

Subcase $\gamma = 1$. In this case, $k + r > 0$. If $k \geq 1$, then $(*)$ asserts that $g = 1 + k(n - 1) + n(r - \sum_{i=1}^r \frac{1}{l_i}) \geq n$.

If $k = 0$, then we must have $r \geq 1$ and, in particular, n cannot be a prime. If $r \geq 2$, then $(*)$ asserts that $g \geq n + 1$. If $r = 1$, then $g = 1 + n(1 - \frac{1}{l_1})$.

Subcase $\gamma = 0$. In this case, $k + r \geq 2$. If we have that $k \geq 2$, then $g \geq n - 1$.

If $k = 1$, then $g = n(r - \sum_{i=1}^r \frac{1}{l_i}) \geq \frac{rn}{2}$. For $r \geq 2$, we have then $g \geq n$, and for $r = 1$, we get $g = n(1 - \frac{1}{l_1})$.

If $k = 0$, then $r \geq 2$, n is not a prime, and $g = 1 - n + n(r - \sum_{i=1}^r \frac{1}{l_i})$. If $r \geq 4$, we have $g \geq 1 - n + n\frac{r}{2} \geq n - 1$. If $r = 3$, then $g = 1 - n + n(3 - \sum_{i=1}^3 \frac{1}{l_i}) \geq 1 + \frac{n}{2}$. In this situation, for n even (then $n \geq 4$), we must have $g \geq n(1 - \frac{1}{2})$. For n odd (then $n \geq 9$), we have $l_i \geq 3$. In particular, $g = 1 - n + n(3 - \sum_{i=1}^3 \frac{1}{l_i}) \geq n + 1$. Finally, if $r = 2$, then $(*)$ asserts that $g = 1 - n + n(2 - \sum_{i=1}^2 \frac{1}{l_i}) = 1 + n(1 - \sum_{i=1}^2 \frac{1}{l_i})$.

If we compute the minimum of all above numbers and we use the construction done in the last section, we obtain the following.

Theorem 1. *Let H be cyclic group of order $n \geq 2$. If $n > 2$, let $2 \leq t, u \leq n$ be divisors of n so that $\frac{n}{t}$ and $\frac{n}{u}$ are relatively prime, maximizing the value $\frac{1}{t} + \frac{1}{u} < 1$. Then the Schottky genus of H is given by*

$$S(H) = \begin{cases} 2, & \text{if } n = 2 \\ 1 + n(1 - \frac{1}{t} - \frac{1}{u}), & \text{if } n > 2 \end{cases}$$

Proof. The construction done in section 2 gives us that the Schottky genus of H is less or equal to the given in the hypothesis of theorem 1. The computations done above asserts that the Schottky genus is given only in the case $\gamma = 0$, $k = 0$ and $r = 2$. If the group H is of Schottky type, then we must have a surjective homomorphism between $\langle a_1, a_2, b_1, b_2 : a_i^{l_1} = b_i^{l_2} = 1 \rangle$ and $\langle x : x^n = 1 \rangle$, so that a_i goes onto an element of order l_1 , b_i goes onto an element of order l_2 . Moreover, the Schottky condition (A) asserts that a_1 and a_2 (respectively, b_1 and b_2) go to inverse elements. The surjectivity obligates that $\frac{n}{l_1}$ and $\frac{n}{l_2}$ must be relatively prime. $\square$

Theorem 2. *The Schottky genus of the Dihedral group H of order $2n$ is the Schottky genus of the cyclic group of order n . Moreover, this is the same as the minimum genus of H .*

Proof. Assume we have a group H , isomorphic to a dihedral group of order $2n$, of conformal automorphisms of a closed Riemann surface S of genus $g \geq 2$. We write $H = \langle x, y : x^n = y^2 = (xy)^2 = 1 \rangle$.

It is not hard to see that the cyclic subgroup of order n , generated by the automorphism x , satisfies the condition (A) and, in particular, it is of Schottky type.

It follows from the above theorem that the Schottky genus of H is at least the Schottky genus of a cyclic group of order n . The construction done in section 2 completes the proof. $\square$

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