A NEW CONVERGENCE THEOREM FOR THE METHOD OF TANGENT HYPERBOLAS IN BANACH SPACE

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Abstract

In this study we approximate a locally unique solution of a non-linear operator equation in Banach space using the method of tangent hyperbolas. A new semilocal convergence theorem is provided using Lipschitz conditions on the second Fréchet-derivative. Our conditions are different than earlier ones. Hence, they have theoretical and practical value. Numerical examples are also provided.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \quad (1)$$

where F is a twice Fréchet-differentiable operator defined on a convex subset D of a Banach space E with values in itself.

The method of tangent hyperbolas (Halley)

$$x_{n+1} = x_n - \left[I - \frac{1}{2} F'(x_n)^{-1} F''(x_n) F'(x_n)^{-1} F(x_n) \right]^{-1} F'(x_n)^{-1} F(x_n) \quad (x_0 \in D) \quad (n \geq 0) \quad (2)$$

is a cubically convergent method, usually employed to approximate a solution x^* . Here $F'(x)$, $F''(x)$ ($x \in D$) denote the first and second Fréchet-derivatives of the operator F at $x \in D$. Note that $F'(x) \in L(E, E)$, the space of bounded linear operators from E into E and $F''(x) \in L(E, L(E, E))$.

Sufficient convergent conditions for this method have been given by many authors [1], [2], [3], [5], [6], [9], [11]. In our study, we use Taylor's theorem for a twice Fréchet-differentiable operator and weaker Lipschitz conditions on the second Fréchet-derivative. This way we provide sufficient conditions different than earlier ones. Hence, our results have theoretical as well as practical value.

Finally, we complete our study by providing a numerical example to show that the method of tangent parabolas starting from x_0 converges to x^* , whereas the same cannot be guaranteed by existing methods [1], [2], [3], [5], [6], [9], [11].

2. Convergence Analysis

We state a theorem whose proof can be found in [5], [6], [9], [11], for comparison:

Theorem 1. *Let $F: D \subseteq E_1 \rightarrow E_2$ be a twice Fréchet-differentiable operator. Assume:*

- (a) $F'(x_0)^{-1} \in L(E, E)$ ($x_0 \in D$);
- (b) $d_1 \geq \|F'(x_0)^{-1} F(x_0)\|$, $b_1 \geq \|F'(x_0)^{-1} F''(x_0)\|$;

(c) $\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq a_1\|x - y\|$ for all $x, y \in D$;

(d) $6d_1b_1^3 + 9d_1^2a_1^2 + 18d_1b_1a_1 - 3b_1^2 - 8a_1 \leq 0$.

Then,

(i) the scalar polynomial

$$\delta(t) = \frac{1}{6}a_1t^3 + \frac{1}{2}b_1t^2 - t + d_1$$

has two positive zeros γ_1 and γ_2 with $\gamma_1 \leq \gamma_2$, and a negative zero γ_3 ;

(ii) the method of tangent hyperbolas is well defined, remains in $\bar{U}(x_0, \gamma_1) = \{x \in E \mid \|x - x_0\| \leq \gamma_1\}$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $U(x_0, \gamma_2)$.

We now provide an example, where the conditions of Theorem 1 are not satisfied.

Example 1. Let $E = \mathbf{R}$, $D = [-9.9, 9.9]$, $x_0 = 0$, and define the real function on D by

$$F(x) = \frac{1}{2400}x^4 + x - 3.$$

Using (a)–(c) of Theorem 1, we can easily obtain, $b_1 = 0$, $d_1 = 3$, and $a_1 = .099$. Moreover, condition (d) is not satisfied since it gives $.019 < 0$, which is not true. Hence, Theorem 1 cannot guarantee that the method of tangent parabolas starting from $x_0 = 0$ converges to a solution x^* of equation $F(x) = 0$ in this case. However we will return back to this example in Remark 2 after our Theorem 2 to show that the method of tangent hyperbolas indeed converges to some $x^* \in D$.

Let a, b, c, d, R be given nonnegative constants with $c \in [0, 1)$, $x_0 \in D$ such that $A_0^{-1} = A(x_0)^{-1} \in L(E, E)$, f be an increasing real function, which is continuous and nonvanishing on $[0, R]$. Define the polynomial p by

$$p(t) = \frac{1}{6}at^3 + \frac{1}{2}bt^2 - (1 - c)t + d, \quad (3)$$

the constants α, β by

$$\alpha = \frac{2(1 - c)}{b + \sqrt{b^2 + 2a(1 - c)}}, \quad (4)$$

$$\beta = (1 - c)\alpha - \frac{1}{6}a\alpha^3 - \frac{1}{2}b\alpha^2, \quad (5)$$

and the iteration $\{t_n\}$ ($n \geq 0$), by

$$t_{n+1} = t_n - \frac{p(t_n)}{f(t_n)}, \quad t_0 = 0 \quad (n \geq 0). \quad (6)$$

We need the lemmas:

Lemma 1. *The real polynomial p has two positive zeros r_1, r_2 with $r_1 \leq r_2$ and a negative zero $-r_3$ ($r_3 > 0$) if and only if*

$$d \leq \beta. \quad (7)$$

Proof. Polynomial p has a negative zero $-r_3$, since $p(0) = d > 0$, and $p(t) < 0$ as $t \rightarrow -\infty$. Moreover, $p'(0) = -(1 - q) < 0$, and $p'(t) > 0$ as $t \rightarrow +\infty$. Hence, there exists a zero of p' in $(0, \infty)$, which by the form of p is given by (4). Thus, p has two positive zeros if and only if

$$p(\alpha) \leq 0, \quad (8)$$

which is equivalent to condition (7).

That completes the proof of Lemma 1.

Lemma 2. *Assume condition (7) holds, and*

$$f(t) \neq 0, \quad f(t) \leq p'(t) \quad \text{for all} \quad t \in [0, r_1]. \quad (9)$$

Then iteration $\{t_n\}$ ($n \geq 0$) given by (10) is monotonically increasing and converges to r_1 .

Proof. Define function g by

$$g(t) = t - \frac{p(t)}{f(t)}. \quad (10)$$

Then by differentiating function g we get

$$g'(t) = \frac{f(t)(f(t) - p'(t)) + f'(t)p(t)}{f(t)^2}. \quad (11)$$

It follows from the proof of Lemma 1, (3) and (9) that $p'(t) < 0$, $p(t) > 0$, $f(t) < 0$, and $f'(t) > 0$ for all $t \in [0, r_1]$. Hence, by (11) function g increases on $[0, r_1]$. So, if $t_k \in [0, r_1]$ for some k , then

$$t_k \leq t_k - \frac{p(t_k)}{f(t_k)} = t_{k+1}, \quad \text{and} \quad t_{k+1} \leq t_k - \frac{p(t_k)}{f(t_k)} \leq r_1 - \frac{p(r_1)}{f(r_1)} = r_1.$$

That completes the proof of Lemma 2.

Remark 1. It can easily be seen by (11) that condition (9) can be replaced by the weaker

$$f(t) \neq 0, \quad f(t)(f(t) - p'(t)) + f'(t)p(t) \geq 0 \quad \text{for all } t \in [0, r_1].$$

From now on, we set

$$A^{-1}(x) = F'(x) - \frac{1}{2}F''(x)F'(x)^{-1}F(x) \quad (x \in D),$$

and $A(x_n) = A_n$ ($n \geq 0$).

We can now prove the semilocal convergence theorem for the method of tangent hyperbolas (2).

Theorem 2. *Let $F: D \subseteq E \rightarrow E$ be a twice Fréchet-differentiable operator. Assume:*

- (a) *there exists $x_0 \in D$ such that $A_0^{-1} = A(x_0)^{-1} \in L(E, E)$;*
- (b) *for all $x \in \bar{U}(x_0, R)$ there exist constants a, b, c such that*

$$\|A_0^{-1}(F''(x) - F''(x_0))\| \leq \|x - x_0\|, \quad (12)$$

$$\|A_0^{-1}F''(x_0)\| \leq b, \quad (13)$$

$$\|A_0^{-1}(F'(x) - A(x))\| \leq c; \quad (14)$$

- (c) *conditions (7) and (9) are satisfied for some continuous monotonically increasing and nonvanishing function f on $[0, r_1]$ such that*

$$\|A_0^{-1}(A(x) - A_0)\| \leq f(\|x - x_0\|) + 1 < 1, \quad \text{for all } x \in \bar{U}(x_0, r_1) \quad (15)$$

and

$$d \geq \|A_0^{-1}F(x_0)\|; \quad (16)$$

- (d) *the following hold:*

$$c \in [0, 1), \quad (17)$$

$$r_2 \leq R, \quad (18)$$

$$\bar{U}(x_0, R) \subseteq D, \quad (19)$$

where r_1 and r_2 are the positive zeros of equation $p(t) = 0$, and polynomial p is given by (3).

Then, the method of tangent hyperbolas $\{x_n\}$ ($n \geq 0$) generated by (2) is well defined, remains in $\bar{U}(x_0, r_1)$ for all $n \geq 0$, and converges to a solution $x^* \in \bar{U}(x_0, r_1)$ of equation $F(x) = 0$. If $r_1 < r_2$ the solution x^* is unique in $U(x_0, r_2)$, whereas if $r_1 = r_2$, x^* is unique in $\bar{U}(x_0, r_1)$.

Moreover, the following error bounds hold for all $n \geq 0$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \quad (20)$$

and

$$\|x_n - x^*\| \leq r_1 - t_n. \quad (21)$$

Proof. We first show linear operator $A(x)$ is invertible for all $x \in \bar{U}(x_0, \alpha)$, where α is given by (4). It follows from (15), the Banach lemma on invertible operators [4], [8], the estimate

$$\|A_0^{-1}(A(x) - A_0)\| \leq f(\|x - x_0\|) + 1 < 1,$$

that $A(x)^{-1} \in L(E, E)$, and

$$\|A(x)^{-1}A_0\| \leq -f(\|x - x_0\|)^{-1} \leq -f(\alpha)^{-1}. \quad (22)$$

We must show that estimate (20) holds for all $n \geq 0$. First, note that x_1 is defined, and by using (2), (6), and (16) we get $\|x_1 - x_0\| = \| -A_0^{-1}F(x_0) \| \leq d = t_1 - t_0$, which shows (20) for $n = 0$. It follows from (22) that linear operator $A(x_1)^{-1} \in L(E, E)$, and hence x_2 can then be defined by (2). Let $x \in [x_0, x_1] = \{x: x = \lambda x_1 + (1 - \lambda)x_0, 0 \leq \lambda \leq 1\}$. By Taylor's formula [5], [7], [10] for a twice Fréchet-differentiable operator G on D , we can write

$$G(x) = G(x_0) + G'(x_0)(x - x_0) + \frac{1}{2}G''(x_0)(x - x_0)^2 + \int_{x_0}^x [G''(y) - G'(x_0)](x - y) dy. \quad (23)$$

Using approximation (23) for $G(x) = A_0^{-1}F(x)$ ($x \in D$), we can get

$$A_0^{-1}F(x) = A_0^{-1}F(x_0) + (x - x_0) + A_0^{-1}(F'(x_0) - A_0) + \frac{1}{2}A_0^{-1}F''(x_0)(x - x_0)^2 + \int_{x_0}^x A_0^{-1}(F''(y) - F''(x_0))(x - y) dy. \quad (24)$$

Let $\lambda d = s$, then by using (3), (12)–(14), (16), approximation (24) gives

$$\|A_0^{-1}F(x)\| \leq (1 - \lambda)d + c\lambda d + \frac{1}{2}b\lambda^2d^2 + \frac{1}{6}a\lambda^3d^3 = p(s) \quad (25)$$

(since $x - x_0 = \lambda(x_1 - x_0) = -\lambda A_0^{-1}F(x_0)$).

Moreover, by (2), (22) and (25), we get

$$\|x_2 - x_1\| \leq \|A(x_1)^{-1}A_0\| \cdot \|A_0^{-1}F(x_1)\| \leq -\frac{p(t_2)}{f(t_1)} = t_2 - t_1.$$

Similarly, we can show (20) for all $n \geq 0$. Estimate (20) and Lemma 2 imply that the method of tangent hyperbolas $\{x_n\}$ ($n \geq 0$) is Cauchy in a Banach space E , and as such it converges to some $x^* \in \bar{U}(x_0, s)$ (since $\bar{U}(x_0, s)$ is a closed set). From (25) and the continuity of F we get $F(x^*) = 0$. Furthermore, estimate (21) follows immediately from (20) by using standard majorization techniques [2], [3], [4], [5], [6], [8], [10]. To show uniqueness, let $z \in U(x_0, r_2)$ with $F(z) = 0$. Using (24) for $x = z$, we obtain,

$$x_1 - z = A_0^{-1}(F(x_0) - A_0)(z - x_0) + \frac{1}{2}A_0^{-1}F''(x_0)(z - x_0)^2 + \int_{x_0}^z A_0^{-1}[F''(y) - F''(x_0)](z - y) dy. \quad (26)$$

We get $\|z - x_0\| \leq r_1 - t_0$ if $z \in \bar{U}(x_0, r_1)$, and $\|z - x_0\| = \mu(r_2 - t_0)$, $0 < \mu < 1$, if $z \in U(x_0, r_2)$. Hence, by (21), and (26), we get for all $n \geq 0$: $\|z - x_n\| \leq r_1 - t_n$, if $z \in \bar{U}(x_0, r_1)$, and $\|z - x_n\| \leq \mu^n(r_2 - t_n)$, if $z \in U(x_0, r_2)$. In either case, we get $\lim_{n \rightarrow \infty} x_n = z$, which yields $x^* = z$.

That completes the proof of Theorem 2.

Remark 2. Returning back to Example 1, and using (4), (5), (12)–(14), (16), we get $a = .0495$, $b = 0$, $c = .0049906$, $d = 3$, $\alpha = 6.1453959$, and $\beta = 4.20000143$. Moreover, set

$$f(t) = \left| F'(t) - \frac{1}{2}F''(t)F'(t)^{-1}F(t) - 1 \right| - 1, \quad t \in [0, r_1].$$

It is simple algebra to show that conditions (9) and (15) are satisfied for this choice of f . Condition (7) is also satisfied, since $d < \beta$. That is the hypotheses of Theorem 2 are satisfied. Hence, the method of tangent hyperbolas $\{x_n\}$ ($n \geq 0$) given by (2) converges to a solution $x^* \in \bar{U}(x_0, \alpha)$ of equation $F(x) = 0$. We found $x^* = 2.967681021$.

Remark 3. Condition (7) can be replaced by a stronger, but easier to check Newton–Kantorovich-type hypothesis [2], [5], [10] as follows: Define polynomials p_1, f_1 by

$$p_1(t) = \frac{1}{2}q_1t^2 - (1 - c)t + d, \quad f_1(t) = q_1t - q_2,$$

where

$$q_1 = \frac{1}{3}ab + b, \quad \text{and} \quad q_2 = 1 - c.$$

It can easily be seen that the conclusions of Theorem 2 hold in the balls $\bar{U}(x_0, r_4)$, $U(x_0, r_5)$, provided that the Newton–Kantorovich hypothesis

$$2q_1d \leq (1 - c)^2,$$

holds, and r_4, r_5 ($r_4 \leq r_5$) are the nonnegative zeros of the equation $p_1(t) = 0$. Note also that since $p(t) \leq p_1(t)$ for all $t \in [0, r_2]$, we have $r_1 \leq r_4 \leq r_5 \leq r_2$.

Remark 4. Condition (14) can be replaced by

$$\|A_0^{-1}(F'(x) - A(x))\| \leq c_0 + c_1\|x - x_0\| \quad \text{for some } c_0 \geq 0, \quad c_1 \geq 0$$

and all $x \in \bar{U}(x_0, R)$. We can also set $c = c_0 + c_1R$. Condition (18) can be replaced by $\alpha \leq R$, but uniqueness is then guaranteed only in $\bar{U}(x_0, \alpha)$.

Remark 5. The results obtained in Theorem 2 can be extended so as to hold a more general setting as follows:

- (a) Let c_0, c_1 be nonnegative constants; v_1, v_2 be positive monotonically increasing functions of one variable on $[0, R]$ with $\lim_{t \rightarrow 0} v_1(t) = \lim_{t \rightarrow 0} v_2(t) = 0$ such that

$$\|A_0^{-1}(F''(x) - F''(x_0))\| \leq v_1(\|x - x_0\|),$$

and

$$\|A_0^{-1}(F'(x) - A(x))\| \leq c_0 + c_1v_2(\|x - x_0\|)$$

for all $x \in U(x_0, R)$.

- (b) Function $\bar{p}$ given by

$$\bar{p}(t) = \int_0^t (t - r)v_1(r)dr + \frac{1}{2}bt^2 - (1 - c_0 - c_1v_2(t))t + d \quad \text{on } [0, R],$$

has a unique zero $\varepsilon_0 \in [0, R]$, and $\bar{p}(R) \leq 0$.

Moreover, set $\varepsilon_0 = r_1$, and $R = r_2$. Furthermore, replace conditions (12), (14) by (a), (7) by (b), and polynomial p by function $\bar{p}$ above. Then, under the rest of the hypotheses, as it can easily be seen from the proof,

the conclusions of Theorem 2 hold in this more general setting. Call such a result Theorem 2'.

Finally, note that for $v_1(t) = at$, $c_0 = c$, and $v_1(t) = 0$ (or $c_1 = 0$), $t \in [0, R]$ function $\bar{p}$ reduces to polynomial p and Theorem 2' to Theorem 2.

Finally, we complete this study with the example.

Example 2. We use Theorem 2 to solve quadratic integral equations of the form

$$x(s) = y(s) + \lambda x(s) \int_0^1 q(s, t)x(t) dt \quad (27)$$

in the space $E_1 = C[0, 1]$, of all functions continuous on the interval $[0, 1]$, with norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Here we assume that λ is a real number called the “albedo” for scattering and the kernel $q(s, t)$ is a continuous function of two variables s, t with $0 \leq s, t \leq 1$ and satisfying

- (i) $0 < q(s, t) < 1$, $0 \leq s, t \leq 1$ with $q(0, 0) = 1$;
- (ii) $q(s, t) + q(t, s) = 1$, $0 \leq s, t \leq 1$.

The function $y(s)$ is a given continuous function defined on $[0, 1]$, and finally $x(s)$ is the unknown function sought in $[0, 1]$.

Equations of this type are closely related with the work of S. Chandrasekhar [5], (Nobel Prize of Physics 1983), and arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gasses, [4], [5], [10]. There exists an extensive literature on equations like (27) under various assumptions on the kernel $q(s, t)$ and λ is a real or complex number. One can refer to the recent work in [4], [5] and the references there.

For simplicity (without loss of generality) we will assume that

$$q(s, t) = \frac{s}{s+t} \quad \text{for all } 0 \leq s, t \leq 1 \text{ with } q(0, 0) = 1.$$

Note that q so defined satisfies (i) and (ii) above.

Let us now choose $\lambda = .25$, $y(s) = 1$ for all $s \in [0, 1]$; and define the operator H on E_1 by

$$H(x) = \lambda x(s) \int_0^1 \frac{s}{s+t} x(t) dt - x(s) + 1.$$

Note that every zero of equation $H(x) = 0$ satisfies equation (27).

Set $x_0(s) = 1$, $A_0 = F'(x_0)$, use the definition of the first and second Fréchet-derivatives of the operator H to obtain using Theorem 2,

$$a = c = 0, \quad b = 2|\lambda| \max_{0 \leq s \leq 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = 2|\lambda| \ln 2 = .34657359, \quad f(t) = bt,$$

$$\bar{\beta} = \|H'(1)^{-1}\| = 1.53039421,$$

$$d \geq \|H'(1)^{-1}H(t)\| \geq \bar{\beta}\lambda \ln 2 = .265197107, \quad R = r_2 = 5.492127839 \text{ and} \\ r_1 = .278652329.$$

(For detailed computations see also [4], [5]).

The conclusions of Theorem 2 for equation (27) follow since all hypotheses are satisfied. That is there exists a solution $x^* \in \bar{U}(x_0, r_1)$ of equation (27) which is unique in $U(x_0, r_2)$.

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