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ON LAGRANGES POLYNOMIALS OF THREE VARIABLES

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Abstract

The present paper introduces a three variables analogue of Lagranges Polynomials and gives certain results involving these polynomials and the four Appell's functions F_1 , F_2 , F_3 and F_4 .

1. Introduction

Lagrange Polynomials arise in certain problems in statistics. In literature they are denoted by the symbol $g_n^{(\alpha, \beta)}(x, y)$ and are defined by means of the following generating relation (see [2 – 3]) :

$$(1.1) \quad (1 - xt)^{-\alpha} (1 - yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha, \beta)}(x, y) t^n$$

Before we introduce a three variable generalization of $g_n^{(\alpha, \beta)}(x, y)$, we first prove the following lemma :

Lemma :

$$(1.2) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A(k, m, n) = \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m A(k, m - k, n - m)$$

and

$$(1.3) \quad \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m A(k, m, n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A(k, m + k, n + m)$$

Proof : Consider the series

$$(1.4) \quad \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m A(k, m, n) t^{n+m+k}$$

in which t^{n+m+k} has been inserted for convenience and will be removed later by placing $t = 1$. Let us collect the powers of t in (1.4). We introduce new indices of summation j, s and r by

$$(1.5) \quad k = j, \quad m = s - j, \quad n = r - s$$

so that the exponent $(n + m + k)$ in (1.4) becomes r . The old indices n, m and k in (1.4) are restricted, as indicated in the summation symbol, by the inequalities

$$(1.6) \quad n \geq 0, \quad m \geq 0, \quad k \geq 0$$

Because of (1.5) the inequalities (1.6) become

$$r - s \geq 0, \quad s - j \geq 0, \quad j \geq 0$$

or $0 \leq s \leq r$, $0 \leq j \leq s$ with r , the exponent on t , restricted only in that it must be a non-negative integer. Thus we arrive at

$$(1.7) \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m A(k, m, n) t^{n+m+k} = \sum_{r=0}^{\infty} \sum_{s=0}^r \sum_{j=0}^s A(j, s-j, r-s) t^r$$

and identity (1.2) of the Lemma follows by putting $t = 1$ and replacing the dummy indices j, s and r on the right by dummy indices k, m and n .

In the Lemma, equation (1.3) is merely (1.2) written in reverse ; it needs no separate derivation.

Some of the definitions and notations used in this paper are as follows (see [6]) :

The binomial coefficient is expressed as

$$(1.8) \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\dots(\lambda-n+1)}{n!} = \frac{(-1)^n (-\lambda)_n}{n!}$$

$$(1.9) \quad (1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!}$$

Appell's four functions of two variables are given by (see [4])

$$(1.10) \quad F_1(a; b, b'; c; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k} (b)_n (b')_k x^n y^k}{n! k! (c)_{n+k}}$$

$$(1.11) \quad F_2(a; b, b'; c, c'; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k} (b)_n (b')_k x^n y^k}{n! k! (c)_n (c')_k}$$

$$(1.12) \quad F_3(a, a'; b, b'; c; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_n (a')_k (b)_n (b')_k x^n y^k}{n! k! (c)_{n+k}}$$

$$(1.13) \quad F_4(a, b; c, c'; c; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k} (b)_{n+k} x^n y^k}{n! k! (c')_k}$$

For definitions of various quadruple hypergeometric functions see H. Exton [1]. However, the Lauricella function $F_D^{(4)}$ of four variables is defined as (see [5])

$$F_D^{(4)}(a, b_1, b_2, b_3, b_4; c; x_1, x_2, x_3, x_4$$

$$(1.14) \quad = \sum_{k,m,n,p=0}^{\infty} \frac{(a)_{k+m+n+p}^{(b_1)_k (b_2)_m (b_3)_n (b_4)_p}}{(c)_{k+m+n+p}} \frac{x_1^k x_2^m x_3^n x_4^p}{k! m! n! p!}$$

The quadruple hypergeometric function K_{11} and K_{15} are defined as (see [1])

$$(1.15) \quad K_{11}(a, a, a, a; b_1, b_2, b_3, b_4; c, c, c, d; x, y, z, t) \\ = \sum_{k,m,n,p=0}^{\infty} \frac{(a)_{k+m+n+p} (b_1)_k (b_2)_m (b_3)_n (b_4)_p}{k! m! n! p! (c)_{k+m+n} (d)_p} x^k y^m z^n t^p$$

$$(1.16) \quad K_{15}(a, a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, t) \\ = \sum_{k,m,n,p=0}^{\infty} \frac{(a)_{k+m+n+p} (b_1)_k (b_2)_m (b_3)_n (b_4)_p (b_5)_p}{k! m! n! p! (c)_{k+m+n+p}} x^k y^m z^n t^p$$

2. Lagrange Polynomials of Three Variables

We define the three variable analogue of Lagrange Polynomials $g_n^{(\alpha, \beta, \gamma)}(x, y, z)$ by means of the following generating relation :

$$(2.1) \quad (1 - xt)^{-\alpha} (1 - yt)^{-\beta} (1 - zt)^{-\gamma} = \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) t^n$$

Expanding the L. H. S. of (2.1) by (1.9), using the result of the lemma proved and finally equating the coefficient of t^n , one gets

$$(2.2) \quad g_n^{(\alpha, \beta, \gamma)}(x, y, z) = \sum_{k=0}^n \sum_{p=0}^k \frac{(\alpha)_{n-k} (\beta)_{k-p} (\gamma)_p}{(n-k)! (k-p)! p!} x^{n-k} y^{k-p} z^p$$

Polynomials $g_n^{(\alpha, \beta, \gamma)}(x, y, z)$ can be regarded as a generalization of Lagrange Polynomials from two to three variables as it can easily be seen that

$$(2.3) \quad g_n^{(\alpha, \beta, \gamma)}(x, y, 0) = g_n^{(\alpha, \beta)}(x, y)$$

Also,

$$(2.4) \quad g_n^{(\alpha, \beta, \gamma)}(x, y, y) = g_n^{(\alpha, \beta + \gamma)}(x, y)$$

$$(2.5) \quad g_n^{(\alpha, \beta, \gamma)}(x, y, z) = g_n^{(\alpha + \gamma, \beta)}(x, y)$$

$$(2.6) \quad g_n^{(\alpha, \beta, \gamma)}(x, y, z) = g_n^{(\alpha + \beta, \gamma)}(x, y)$$

$$(2.7) \quad g_n^{(\alpha, \beta, 0)}(x, y, z) = g_n^{(\alpha, \beta)}(x, y)$$

Now consider

$$\begin{aligned}
 & \sum_{n=0}^{\infty} g_n^{(a+\alpha+\lambda, b+\beta+\mu, c+\gamma+\nu)}(x, y, z) t^n \\
 &= (1 - xt)^{-(a+\alpha+\lambda)} (1 - yt)^{-(b+\beta+\mu)} (1 - zt)^{-(c+\gamma+\nu)} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} g_n^{(a,b,c)}(x, y, z) g_m^{(\alpha,\beta,\gamma)}(x, y, z) g_k^{(\lambda,\mu,\nu)}(x, y, z) t^{n+m+k} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m g_{n-m}^{(a,b,c)}(x, y, z) g_{m-k}^{(\alpha,\beta,\gamma)}(x, y, z) g_k^{(\lambda,\mu,\nu)}(x, y, z) t^n
 \end{aligned}$$

using (1.2).

Now equating the coefficient of t^n , we get $g_n^{(a+\alpha+\lambda, b+\beta+\mu, c+\gamma+\nu)}(x, y, z)$

$$(2.8) \quad = \sum_{m=0}^n \sum_{k=0}^m g_{n-m}^{(a,b,c)}(x, y, z) g_{m-k}^{(\alpha,\beta,\gamma)}(x, y, z) g_k^{(\lambda,\mu,\nu)}(x, y, z)$$

As a particular case of (2.8) it can easily be verified that

$$(2.9) \quad g_n^{(a+\alpha', b+\beta', c+\gamma')}(x, y, z) = \sum_{r=0}^n g_{n-r}^{(\alpha,\beta,\gamma)}(x, y, z) g_r^{(\alpha',\beta',\gamma')}(x, y, z)$$

3. Main Results

First consider the series

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+m+k)!}{n! m! k!} g_{n+m+k}^{(\alpha,\beta,\gamma)}(x, y, z) t^n v^m w^k \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m \frac{(-n)_m (-m)_k}{m! k!} g_n^{(\alpha,\beta,\gamma)}(x, y, z) t^n \left(-\frac{v}{t}\right)^m \left(-\frac{w}{v}\right)^k \\
 &\text{by (1.2)} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(-n)_m}{m!} g_n^{(\alpha,\beta,\gamma)}(x, y, z) t^n \left(-\frac{v}{t}\right)^m \left(1 + \frac{w}{v}\right)^m \quad \text{by (1.9)} \\
 &= \sum_{n=0}^{\infty} g_n^{(\alpha,\beta,\gamma)}(x, y, z) (t + v + w)^n \\
 &= (1 - xt - xv)^{-\alpha} (1 - yt - yv)^{-\beta} (1 - zt - zv)^{-\gamma} \\
 &\sum_{k=0}^{\infty} g_k^{(\alpha,\beta,\gamma)} \left(\frac{x}{1 - xt - xv}, \frac{y}{1 - yt - yv}, \frac{z}{1 - zt - zv} \right) w^k
 \end{aligned}$$

Equating the coefficient of w^k , we arrive at

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(n+m+k)!}{n! m! k!} g_{n+m+k}^{(\alpha, \beta, \gamma)}(x, y, z) t^n v^m \\ = (1 - xt - xv)^{-\alpha} (1 - yt - yv)^{-\beta} (1 - zt - zv)^{-\gamma} \\ (3.1) \quad g_k^{(\alpha, \beta, \gamma)} \left(\frac{x}{1 - xt - xv}, \frac{y}{1 - yt - yv}, \frac{z}{1 - zt - zv} \right)$$

As a simpler version of (3.1), we have

$$\sum_{n=0}^{\infty} \binom{m+n}{n} g_{m+n}^{(\alpha, \beta, \gamma)}(x, y, z) t^n \\ = (1 - xt)^{-\alpha} (1 - yt)^{-\beta} (1 - zt)^{-\gamma} g_m^{(\alpha, \beta, \gamma)} \left(\frac{x}{1 - xt}, \frac{y}{1 - yt}, \frac{z}{1 - zt} \right) \\ (3.2)$$

Next, consider

$$\sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) F_2[\lambda; -n, \delta; \mu, \nu; u, uw] t^n \\ \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\lambda)_{p+q} (n+p)! (\delta)_q}{n! p! q! (\mu)_p (\nu)_q} g_{n+p}^{(\alpha, \beta, \gamma)}(x, y, z) (-u)^p (-uw)^q t^{n+p} \\ \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\lambda)_{p+q} (\delta)_q}{q! (\mu)_p (\nu)_q} (-ut)^p (uw)^q (1 - xt)^{-\alpha} (1 - vt)^{-\beta} (1 - zt)^{-\gamma} \\ g_{n+p}^{(\alpha, \beta, \gamma)} \left(\frac{x}{1 - xt}, \frac{y}{1 - yt}, \frac{z}{1 - zt} \right) \text{ by (3.2)} \\ = (1 - xt)^{-\alpha} (1 - yt)^{-\beta} (1 - zt)^{-\gamma} \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\lambda)_{p+q+k+j}}{(\mu)_{p+k+j} (\nu)_q} \\ \frac{(\alpha)_p (\beta)_k (\gamma)_j (\delta)_q}{p! q! k! j!} \left(\frac{xut}{xt - 1} \right)^p \left(\frac{yut}{yt - 1} \right)^k \left(\frac{zut}{zt - 1} \right)^j (uw)^q$$

Thus we arrive at the result

$$\sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) F_2[\lambda; -n, \delta; \mu, \nu; u, uw] t^n \\ = (1 - xt)^{-\alpha} (1 - yt)^{-\beta} (1 - zt)^{-\gamma} \\ (3.3) \quad K_{11} \left(\lambda, \lambda, \lambda, \lambda; \alpha, \beta, \gamma, \delta; \mu, \mu, \mu, \nu; \frac{xut}{xt - 1}, \frac{yut}{yt - 1}, \frac{zut}{zt - 1}, uw \right)$$

We then consider

$$\begin{aligned}
 & \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) F_1[\lambda; -n, \delta; \mu, u, uw] t^n \\
 &= \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\lambda)_{p+q} (\delta)_q}{q! (\mu)_{p+q}} (-ut)^p (uw)^q \sum_{n=0}^{\infty} \binom{n+p}{n} g_{n+p}^{(\alpha, \beta, \gamma)}(x, y, z) t^n \\
 &= (1-xt)^{-\alpha} (1-yt)^{-\beta} (1-zt)^{-\gamma} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\lambda)_{p+q+k+j}}{(\mu)_{p+q+k+j}} \\
 & \quad \frac{(\alpha)_p (\beta)_k (\gamma)_j (\delta)_q}{p! q! k! j! q!} \left(\frac{xut}{xt-1} \right)^p \left(\frac{yut}{yt-1} \right)^k \left(\frac{zut}{zt-1} \right)^j (uw)^q
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 & \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) F_1[\lambda; -n, \delta; \mu, u, uw] t^n \\
 &= (1-xt)^{-\alpha} (1-yt)^{-\beta} (1-zt)^{-\gamma} F_D^{(4)} \left(\lambda, \alpha, \beta, \gamma, \delta; \mu \frac{xut}{xt-1}, \frac{yut}{yt-1}, \frac{zut}{zt-1}, uw \right)
 \end{aligned} \tag{3.4}$$

Next, we consider

$$\begin{aligned}
 & \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) F_3[\lambda, -n; \delta, \mu; \rho; u, uw] t^n \\
 &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda)_q (\delta)_q (\mu)_p}{q! (\rho)_{p+q}} (-ut)^p (uw)^q \sum_{n=0}^{\infty} \binom{n+p}{n} g_{n+p}^{(\alpha, \beta, \gamma)}(x, y, z) t^n \\
 &= (1-xt)^{-\alpha} (1-yt)^{-\beta} (1-zt)^{-\gamma} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda)_q (\delta)_q (\mu)_p}{q! (\rho)_{p+q}} (-ut)^p (uw)^q \\
 & \quad \sum_{k=0}^p \sum_{j=0}^k \frac{(\alpha)_{p-k} (\beta)_{k-j} (\gamma)_j}{(p-k)! (k-j)! j!} \left(\frac{x}{1-xt} \right)^{p-k} \left(\frac{y}{1-yt} \right)^{k-j} \left(\frac{z}{1-zt} \right)^j \text{ by}
 \end{aligned} \tag{3.2}$$

Thus, we obtain

$$\begin{aligned}
 & \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) F_3[\lambda, -n; \delta, \mu; \rho; u, uw] t^n \\
 &= (1-xt)^{-\alpha} (1-yt)^{-\beta} (1-zt)^{-\gamma} \\
 & (3.5) \quad K_{15} \left(\mu, \mu, \mu, \lambda; \alpha, \beta, \gamma, \delta; \rho, \rho, \rho; \frac{xut}{xt-1}, \frac{yut}{yt-1}, \frac{zut}{zt-1}, uw \right)
 \end{aligned}$$

Finally, we consider

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\lambda)_n (\delta)_n}{(1-\alpha)_n (1-\beta)_n (1-\gamma)_n} g_n^{(\alpha-n, \beta-n, \gamma-n)} (x, y, z) \\
 & F_4 [\lambda + n; \delta + n; \mu, \rho; u, w] t^n \\
 &= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda)_{n+p+q} (\delta)_{n+p+q}}{p! q! (\mu)_p (\rho)_q} u^p w^q \sum_{k=0}^n \sum_{j=0}^k \frac{(\alpha)_{-k} (\beta)_{-n+k-j}}{(n-k)!} \\
 & \frac{(\gamma)_{-n+j}}{(k-j)! j!} (-1)^n x^{n-k} y^{k-j} z^j t^n \\
 &= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{k=0}^p \sum_{j=0}^k \\
 & \frac{(\lambda)_{n+k+j+p+q} (\delta)_{n+k+j+p+q} (-xt)^n (-yt)^k (-zt)^j u^p w^q}{p! q! n! k! j! (\mu)_p (\rho)_q (1-\alpha)_{k+j} (1-\beta)_{n+j} (1-\gamma)_{n+k}}
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\lambda)_n (\delta)_n}{(1-\alpha)_n (1-\beta)_n (1-\gamma)_n} g_n^{(\alpha-n, \beta-n, \gamma-n)} (x, y, z) \\
 & F_4 [\lambda + n; \delta + n; \mu, \rho, u, w] t^n
 \end{aligned}$$

$$(3.6) = F \left(\begin{matrix} \lambda, \lambda, \lambda, \lambda, \lambda, \delta, \delta, \delta, \delta, \delta; \mu, \rho, 1-\alpha, 1-\alpha, 1-\beta, 1-\beta, \\ 1-\gamma, 1-\gamma; -xt, -yt, -zt, u, w \end{matrix} \right)$$

where F is a quintuple hypergeometric function.

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