ARISING OF BERGMAN REPRODUCING KERNEL IN BOSE-FOCK SPACE

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Abstract

We introduce a Bose-Fock space $\mathcal{A}(D^d)$ of anti-holomorphic complex-valued functions on the unit polydisc D^d in C^d , with $d \in N \cup \{\infty\}$. A unitary isomorphism U from the abstract Bose-Fock space $\Gamma\mathcal{H}$ to $\mathcal{A}(D^d)$ is produced, and it is shown that U transforms twisted coherent vectors into Bergman kernels in D^d .

1. Introduction

The traditional Fock space representations as e.g. the Schrödinger representation or the Segal-Bargmann analytic function representation exist as subspaces of L^2 -spaces over infinite measure spaces. On the other hand, spaces of integrable functions over finite measure regions were traditionally examined in mathematics, e.g. Hardy and Bergman spaces. It seems natural to find out, how Bose-Fock spaces can be represented as spaces of functions on domains of finite measure.

Some aspects of this question are exposed in [1] and [4]. The approach of this paper is different. The model is inspired by the complex wave-representation (cf. [3]). We introduce a Bose-Fock algebra $\mathcal{A}_0(D^d)$ of antiholomorphic complex-valued functions on the unit polydisc in C^d . Then we produce a isometric map U from the abstract Bose-Fock algebra $\Gamma_0\mathcal{H}$ to $\mathcal{A}_0(D^d)$. This map is extended to a unitary isomorphism U from the Bose-Fock space $\Gamma\mathcal{H}$ to $\mathcal{A}(D^d)$. Eventually we see that U transforms twisted coherent vectors into Bergman kernels in D^d .

2. Operators on the algebra $\mathcal{A}_0(D^d)$

In this paper most results are valid in both finite and infinite dimensions. We shall thus let d denote the dimension, where $d \in N \cup \{\infty\}$. Unless the opposite is explicitly noticed, we shall always allow d to be infinite.

Let

$$C_0^d = \left\{ (x_1, x_2, \dots) \in C^d, x_n = 0 \text{ for almost every } n \in N_d \right\},$$

where $N_d = \{n \in N \mid n < d + 1\}$, and consider the sesquilinear scalar product $\langle x, y \rangle = \sum_{n=1}^d \overline{x_n} y_n$, where $x \in C_0^d$ and $y \in C^d$. Let, in the sequel, $\{e_n\}_{n=1}^d$ be the orthonormal basis in C_0^d with $e_n = (0, \dots, 0, 1, 0, \dots) \in C_0^d$, where 1 occurs on the n 'th place. Consider the polydiscs

$$D^d = \left\{ z \in C^d \mid |z_n| < 1, \text{ where } z_n = \langle e_n, z \rangle \text{ and } n \in N_d \right\}$$

and

$$\overline{D^d} = \left\{ z \in C^d \mid |z_n| \leq 1, \text{ where } z_n = \langle e_n, z \rangle \text{ and } n \in N_d \right\}.$$

We provide $\overline{D^d}$ with a probability measure, which is the (finite or infinite) product of the normalized Lebesgue measure on $\overline{D^1}$. Notice that

$\overline{D^d} \setminus D^d$ is a set of measure zero. We write $L^2(D^d), \langle \cdot, \cdot \rangle_2$ for the corresponding L^2 -space.

Let $\mathcal{A}_0(D^d) \subset L^2(D^d)$ be the linear space of anti-holomorphic tame polynomials (i.e. each dependent on only finitely many variables) on D^d , and denote by $\mathcal{A}(D^d)$ the closure of $\mathcal{A}_0(D^d)$ in $L^2(D^d)$.

Set

$$(\partial_x f)(\cdot) = \frac{d}{dt} f(\cdot + tx) |_{t=0},$$

where $x \in C_0^d$, $t \in \mathbb{R}$ and $f(\cdot) \in \mathcal{A}_0(D^d)$. We will use the operator of multiplication μ_x , where

$$(\mu_x f)(\cdot) = \langle \cdot, x \rangle f(\cdot)$$

for $x \in C_0^d$ and $f \in \mathcal{A}_0(D^d)$.

We have the following straightforward lemma.

Lemma 1 : Given $x, y, u, v \in C_0^d$, we have on $\mathcal{A}_0(D^d)$ the identities

$$[\partial_x, \mu_y] = \langle x, y \rangle I,$$

$$[\partial_x, \partial_y] = [\mu_x, \mu_y] = 0$$

and

$$[\partial_x, \mu_f] = (\partial_x f),$$

where $\mu_f g = fg$ for $f, g \in \mathcal{A}_0(D^d)$. $\square$

The partial number operator $N_k : \mathcal{A}_0(D^d) \rightarrow \mathcal{A}_0(D^d)$ assumes the form

$$N_k = \mu_{e_k} \partial_{e_k}.$$

We consider on $\mathcal{A}_0(D^d)$ the number operator $N = \sum_{p=1}^d N_p$, with the property of $N \langle \cdot, x_1 \rangle \cdots \langle \cdot, x_m \rangle = m \langle \cdot, x_1 \rangle \cdots \langle \cdot, x_m \rangle$, for $x_1, \dots, x_m \in C_0^d$, and get the following trivial proposition.

Proposition 2 : Given $x, y \in C_0^d$, we have $N_k \partial_{e_k} = \partial_{e_k} (N_k - I)$ and $N_k \mu_{e_k} = \mu_{e_k} (N_k + I)$ on $\mathcal{A}_0(D^d)$. $\square$

For $x \in C_0^d$ we define the operators of creation $\mathbf{a}_D^+(x)$ and annihilation $\mathbf{a}_D(x)$ on $\mathcal{A}_0(D^d)$ setting

$$\mathbf{a}_D^+(e_k) = (N_k + I)^{\frac{1}{2}} \mu_{e_k}$$

and

$$\mathbf{a}_D(e_k) = \partial_{e_k} (N_k + I)^{-\frac{1}{2}}.$$

Proposition 3 : For $x, y \in C_0^d$ the identities

$$[\mathbf{a}_D(x), \mathbf{a}_D^+(y)] = \langle x, y \rangle I$$

and

$$[\mathbf{a}_D^+(x), \mathbf{a}_D^+(y)] = [\mathbf{a}_D(x), \mathbf{a}_D(y)] = 0$$

hold on $\mathcal{A}_0(D^d)$.

Proof. Follows trivially from Proposition ?? and Lemma ??. $\square$

We write $\varphi_D = 1$ for the vacuum in $\mathcal{A}_0(D^d)$ and notice that $\mathbf{a}_D(x) \varphi_D = 0$ for all $x \in C_0^d$. In section 3 it will be clear, that $\mathcal{A}_0(D^d)$ now fulfils all the requirements for a Bose-Fock algebra, with the exception of $\mathbf{a}_D(x)$ and $\mathbf{a}_D^+(x)$ being dual operators. The proof of this is a little intricate and will be postponed to proposition ??.

Define

$$\begin{aligned} \varepsilon_{\underline{n}} &= \frac{1}{\sqrt{n_1!}} \left(\mathbf{a}_D^+(e_1) \right)^{n_1} \frac{1}{\sqrt{n_2!}} \left(\mathbf{a}_D^+(e_2) \right)^{n_2} \cdots \varphi_D \\ &= \prod_{k=1}^d \left(\sqrt{(n_k + 1)} \langle \cdot, e_k \rangle^{n_k} \right). \end{aligned}$$

It is obvious that $\{\varepsilon_{\underline{n}}\}_{\underline{n} \in I}$ spans $\mathcal{A}_0(D^d)$, where

$$I = \left\{ (n_1, n_2, \dots) \in \mathbb{R}^d \mid n_i \in \mathbb{N}_0, n_i = 0 \text{ for almost all } i \in \mathbb{N}_d \right\}.$$

Proposition 4 : $\{\varepsilon_{\underline{n}}\}_{\underline{n} \in I}$ is an orthonormal basis in $\mathcal{A}_0(D^d)$.

Proof. We use the notation $\delta_{\underline{m}, \underline{n}} = \prod_{p=1}^d \delta_{m_p, n_p}$. Let $z_p = r_p e^{i\phi_p}$, $R_p = r_p^2$ and $z = \sum_{p=1}^d z_p e_p$, and write briefly

$$\int_D f(R_p, \phi_p) \equiv \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} f(R_p, \phi_p) d\phi_p dR_p.$$

Then

$$\langle z, e_p \rangle^{n_p} = \langle z_p e_p, e_p \rangle^{n_p} = \overline{z_p}^{n_p} = r_p^{n_p} e^{-in_p \phi_p},$$

such that

$$\begin{aligned} \left\langle \prod_{i=1}^d \langle \cdot, e_i \rangle^{m_i}, \prod_{j=1}^d \langle \cdot, e_j \rangle^{n_j} \right\rangle_2 &= \int_{D^d} \overline{\prod_{i=1}^d \langle \cdot, e_i \rangle^{m_i}} \prod_{j=1}^d \langle \cdot, e_j \rangle^{n_j} \\ &= \prod_{k=1}^d \int_D \overline{\langle \cdot, e_k \rangle^{m_k}} \langle \cdot, e_k \rangle^{n_k} \\ &= \prod_{p=1}^d \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} r_p^{m_p+n_p} e^{i(m_p-n_p)\phi_p} d\phi_p dR_p \\ &= \delta_{\underline{n}, \underline{m}} \left(\prod_{p=1}^d \frac{1}{n_p + 1} \right), \end{aligned}$$

and hence

$$\langle \varepsilon_{\underline{m}}, \varepsilon_{\underline{n}} \rangle_2 = \delta_{\underline{n}, \underline{m}}$$

follows. $\square$

3. Introduction to Bose-Fock space

To a given separable complex Hilbert space $\mathcal{H}$, $\langle \cdot, \cdot \rangle$ attach a commutative algebra $\Gamma_0 \mathcal{H}$ generated by $\mathcal{H}$ (*one-particle space*) and a unity which will be denoted by φ and called the *vacuum*. We call $\Gamma_0 \mathcal{H}$ a *Bose-Fock algebra* (cf. [2], [3] and [5]) if

1. The scalar product from $\mathcal{H}$ extends over $\Gamma_0 \mathcal{H}$ in such a way that for every $x \in \mathcal{H}$ the operator $\mathbf{a}^+(x)$ of multiplication by x admits the adjoint $\mathbf{a}(x)$ defined on the whole $\Gamma_0 \mathcal{H}$.
2. The adjoint fulfils the Leibniz rule, i.e. for every $x, y \in \mathcal{H}$ and $f, g \in \Gamma_0 \mathcal{H}$,

$$\langle xf, g \rangle = \langle f, \mathbf{a}(x)g \rangle,$$

$$[\mathbf{a}(x), \mathbf{a}^+(y)] = \langle x, y \rangle I$$

on $\Gamma_0\mathcal{H}$, and

$$\mathbf{a}(x)\emptyset = 0.$$

The operators $\mathbf{a}^+(x)$ (respectively $\mathbf{a}(x)$) will be called creations (respectively annihilations).

We define $\Gamma\mathcal{H}$ as the limits with respect to $|\cdot|$ (where $|x| = \langle x, x \rangle^{\frac{1}{2}}$) of all Cauchy sequences $\{g_n\} \subset \Gamma_0\mathcal{H}$. The Hilbert space $\Gamma\mathcal{H}$ will be called the *Bose-Fock space*.

Let $\{e_i\}_{i=1}^d$ be an orthonormal basis in $\mathcal{H}$ and $\mathcal{D} = \text{span}_{n \in N_d} \{e_n\}$. Though $\mathcal{D}$ is not a Hilbert space, we can still consider $\Gamma_0\mathcal{D}$, and will call $\Gamma_0\mathcal{D}$ a Bose-Fock algebra. If $\mathcal{H}$ is finite dimensional, then $\Gamma_0\mathcal{D}$ obviously equals $\Gamma_0\mathcal{H}$. In any case the completion $\Gamma\mathcal{D}$ equals $\Gamma\mathcal{H}$.

We shall identify $\mathcal{D}$ with C_0^d by use of the transformation

$$\mathcal{D} \ni x \rightarrow \underline{x} = (x_1, x_2, \dots) \in C_0^d,$$

where $x_n = \langle e_n, x \rangle$, $n \in N_d$. This identification will also be used for extensions of $\mathcal{D}$ and C_0^d .

We consider on $\Gamma_0\mathcal{H}$ the number operator

$$\mathcal{N} = \sum_{p=1}^{\infty} \mathbf{a}^+(e_p) \mathbf{a}(e_p),$$

characterised by the property that $\mathcal{N}x_1 \cdots x_m = mx_1 \cdots x_m$, for $x_1, \dots, x_m \in \mathcal{H}$. Furthermore the set $\{e_{\underline{n}}\}_{\underline{n} \in I}$ is an orthonormal basis for $\Gamma_0\mathcal{H}$, where

$$e_{\underline{n}} = \frac{1}{\sqrt{n_1!}} (\mathbf{a}^+(e_1))^{n_1} \cdots \frac{1}{\sqrt{n_d!}} (\mathbf{a}^+(e_d))^{n_d} \emptyset.$$

Following [3], we introduce the complex wave-representation assigning to $f \in \Gamma\mathcal{H}$ the function $f[\cdot]$, where

$$f[z] = \langle e^z, f \rangle, \quad z \in \mathcal{H}.$$

For $f, g \in \Gamma_0\mathcal{H}$ and $z \in \mathcal{H}$ we have

$$(f \cdot g)[z] = f[z] \cdot g[z].$$

4. Bose-Fock space on the polydisc

In this section we shall produce the unitary isomorphism U from $\Gamma_0\mathcal{H}$ onto $\mathcal{A}_0(D^d)$. Notice that

$$\left(\prod_{k=1}^{\infty}(\mathcal{N}_k + I)!\right)^{\frac{1}{2}} e_1^{n_1} \cdots e_m^{n_m} \cdots = \left(\prod_{k=1}^{\infty}(n_k + 1)!\right)^{\frac{1}{2}} e_1^{n_1} \cdots e_m^{n_m} \cdots$$

holds, where

$$\mathcal{N}_k = \mathbf{a}^+(e_k) \mathbf{a}(e_k).$$

Setting $k_z = (\prod_{k=1}^{\infty}(\mathcal{N}_k + I)!)^{\frac{1}{2}} e^z$ for $z \in \ell_2 \cap D^d$, we get

$$\begin{aligned} |k_z|^2 &= \left| \left(\prod_{k=1}^d (\mathcal{N}_k + I) ! \right)^{\frac{1}{2}} e^z \right|^2 \\ &= \left| \prod_{k=1}^d \left((\mathcal{N}_k + I) !^{\frac{1}{2}} e^{z_k} \right) \right|^2 \\ &= \prod_{k=1}^d \left(\sum_{n=0}^{\infty} (n+1) \langle z_k, z_k \rangle^n \right) \\ &= \prod_{k=1}^d \left(1 - |z_k|^2 \right)^{-2}, \end{aligned}$$

where the last equation follows from the Taylor expansion. Hence $k_z \in \Gamma\mathcal{H}$, for $z \in \ell_2 \cap D^d$.

If $\mathcal{H}$ is finite dimensional, we have $D^d \subset \ell_2$ and thus $k_z \in \Gamma\mathcal{H}$ for $z \in D^d$.

For pairs (f, z) where either $f \in \Gamma\mathcal{H}$ and $z \in \ell_2 \cap D^d$ or $f \in \Gamma_0\mathcal{D}$ and $z \in D^d$, we define the linear operator U setting

$$(Uf)(z) = \langle k_z, f \rangle.$$

It is clear that in the first case $(Uf) : \ell_2 \cap D^d \rightarrow C$, and in the second case $(Uf) : D^d \rightarrow C$. We have $U : \Gamma_0\mathcal{D} \rightarrow \mathcal{A}_0(D^d)$.

Moreover, for $k \in N_d$

$$Ue_k^n = (n+1)!^{\frac{1}{2}} (e_k^n)[\cdot],$$

and

$$U\emptyset = 1 = \emptyset_D.$$

Lemma 5 : The operator $U : \Gamma_0 D \rightarrow \mathcal{A}_0(\mathbf{D}^d)$ is an isometry.

Proof. Take the orthonormal basis $\{e_n\}$ in $\mathcal{D}$ and let $e_{\underline{n}}$ be as defined in section 3. Then

$$\begin{aligned} (Ue_{\underline{n}})(\underline{z}) &= \left\langle \left(\prod_{k=1}^{\infty} (\mathcal{N}_k + I)! \right)^{\frac{1}{2}} e^{\underline{z}}, e_{\underline{n}} \right\rangle \\ &= \left(\prod_{k=1}^{\infty} \sqrt{(n_k + 1)!} \frac{1}{\sqrt{n_k!}} \langle \underline{z}, e_k \rangle^{n_k} \right) \\ &= \left(\prod_{k=1}^{\infty} \sqrt{(n_k + 1)} \langle \underline{z}, e_k \rangle^{n_k} \right) \\ &= \varepsilon_{\underline{n}}(\underline{z}). \end{aligned}$$

and thus $U : \Gamma_0 \mathcal{D} \rightarrow \mathcal{A}_0(D^d)$ takes an orthonormal basis into an orthonormal basis, and hence is isometric on $\Gamma_0 \mathcal{D}$. $\square$

Lemma 6 : Let $\mathcal{H}$ be finite dimensional and take $\{f_n\} \subset \Gamma \mathcal{H}$. If $f_n \rightarrow 0$, then $(Uf_n)(\underline{z}) \rightarrow 0$ for every $\underline{z} \in \mathcal{D}^d$.

Proof. Let $\underline{z} \in D^d$ be given. If $\mathcal{H}$ is finite dimensional, then $k_z \in \Gamma \mathcal{H}$. For $f \in \Gamma \mathcal{H}$ we have $|(Uf)(\underline{z})| = |\langle k_z, f \rangle| \leq |k_z| |f|$, and hence $(Uf_n)(\underline{z}) \rightarrow 0$ for $f_n \rightarrow 0$. $\square$

Theorem 7 : Let $\mathcal{H}$ be finite dimensional. Then the operator $U : \Gamma \mathcal{H} \rightarrow \mathcal{A}(D^d)$ is unitary.

Proof. It follows from Lemma ??, that U is an isometric mapping of $\Gamma_0 \mathcal{H}$ onto $\mathcal{A}_0(D^d)$. Let $\{f_n\} \subset \Gamma_0 \mathcal{H}$ be a sequence converging to $f \in \Gamma \mathcal{H}$. Then $g_n = Uf_n$ is a Cauchy sequence and $g_n \rightarrow g$ in $\mathcal{A}(D^d)$. Since $\mathcal{H}$ is

finite dimensional it follows from Lemma ??, that g_n converges pointwise to Uf . Hence $g = Uf$ a.e. and $Uf \in \mathcal{A}(D^d)$. $\square$

If $\mathcal{H}$ is infinite dimensional, we write $U : \Gamma\mathcal{H} \rightarrow \mathcal{A}(D^d)$ for the unitary extension of $U : \Gamma_0\mathcal{D} \rightarrow \mathcal{A}_0(D^d)$.

We have thus produced an isomorphism $U : \Gamma\mathcal{H} \rightarrow \mathcal{A}(D^d)$, such that the Bergman Bose-Fock space $\mathcal{A}(D^d)$ and the abstract Bose-Fock space $\Gamma\mathcal{H}$ are isomorphic. Further we have that $\mathcal{A}_0(D^d)$ is the Bergman Bose-Fock algebra generated by the space of anti-holomorphic functions $\{\langle \cdot, \underline{x} \rangle \mid \underline{x} \in C_0^d\}$, $\langle \cdot, \cdot \rangle_2$. The operators $\mathbf{a}_D^+(\underline{x})$ (respectively $\mathbf{a}_D(\underline{x})$) are the operators of creation (respectively annihilation) and $\phi_D = 1$ is a unit vector.

Proposition 8 : Let $\underline{x} \in C_0^d$. We have $U\mathcal{N}_k = N_k U$, $U\mathbf{a}^+(x) = \mathbf{a}_D^+(\underline{x})U$ and $U\mathbf{a}(x) = \mathbf{a}_D(\underline{x})U$ on $\Gamma_0\mathcal{D}$. If $f, g \in \mathcal{A}_0(D^d)$ then $\langle \mathbf{a}_D(\underline{x})f, g \rangle_2 = \langle f, \mathbf{a}_D^+(\underline{x})g \rangle_2$.

Proof. Let $\underline{z} \in D^d$ and $f = \sum_{n=0}^N f_n \in \Gamma_0\mathcal{D}$, where $\mathcal{N}_k f_n = n f_n$. Then

$$\begin{aligned} (U\mathcal{N}_k f_n)(\underline{x}) &= \langle k_z, \mathcal{N}_k f_n \rangle = n \left\langle \left(\prod_{k=1}^d (\mathcal{N}_k + I) \right)^{\frac{1}{2}} e^z, f_n \right\rangle \\ &= n (Uf_n)(z) = (N_k Uf_n)(z) \end{aligned}$$

and we have $U\mathcal{N}_k = N_k U$ on $\Gamma_0\mathcal{D}$.

Let $f \in \Gamma_0\mathcal{D}$ and $\underline{z} \in D^d$. Then

$$\begin{aligned} (U\mathbf{a}^+(e_n)f)(\underline{z}) &= \langle k_z, \mathbf{a}^+(e_n)f \rangle = \left\langle \mathbf{a}(e_n) \left(\prod_{k=1}^d (\mathcal{N}_k + I) \right)^{\frac{1}{2}} e^z, f \right\rangle \\ &= \left\langle (\mathcal{N}_n + 2 \cdot I)^{\frac{1}{2}} \left(\prod_{k=1, k \neq n}^d (\mathcal{N}_k + I) \right)^{\frac{1}{2}} \mathbf{a}(e_n) e^z, f \right\rangle \\ &= \langle \underline{z}, e_{\underline{n}} \rangle \left\langle \left(\prod_{k=1}^d (\mathcal{N}_k + I) \right)^{\frac{1}{2}} e^z, (\mathcal{N}_n + 2 \cdot I)^{\frac{1}{2}} f \right\rangle \\ &= \mu_{e_{\underline{n}}} \left(U(\mathcal{N}_n + 2 \cdot I)^{\frac{1}{2}} f \right)(\underline{z}) = \mu_{e_{\underline{n}}} (\mathcal{N}_n + 2 \cdot I)^{\frac{1}{2}} (Uf)(\underline{z}) \\ &= (\mathcal{N}_n + I)^{\frac{1}{2}} \mu_{e_{\underline{n}}} (Uf)(\underline{z}) = \left(\mathbf{a}_D^+(e_{\underline{n}}) Uf \right)(\underline{z}), \end{aligned}$$

and thus $U\mathbf{a}^+(x) = \mathbf{a}_D^+(\underline{x})U$ on $\Gamma_0\mathcal{D}$. Moreover

$$\begin{aligned}
(U\mathbf{a}(e_n)f)(z) &= \langle k_z, \mathbf{a}(e_n)f \rangle = \left\langle \mathbf{a}^+(e_n) \left(\prod_{k=1}^d (\mathcal{N}_k + I)! \right)^{\frac{1}{2}} e^z, f \right\rangle \\
&= \left\langle e_n e^z, \left(\prod_{k=1, k \neq n}^d (\mathcal{N}_k + I)! \right)^{\frac{1}{2}} \mathcal{N}_n!^{\frac{1}{2}} f \right\rangle \\
&= \partial_{\underline{e}_n} \left\langle e^z, \left(\prod_{k=1}^d (\mathcal{N}_k + I)! \right)^{\frac{1}{2}} (\mathcal{N}_n + I)^{-\frac{1}{2}} f \right\rangle \\
&= \left(\partial_{\underline{e}_n} U (\mathcal{N}_n + I)^{-\frac{1}{2}} f \right) (\underline{z}) \\
&= \left(\partial_{\underline{e}_n} (N_n + I)^{-\frac{1}{2}} U f \right) (\underline{z}) = (\mathbf{a}_D(\underline{e}_n) U f) (\underline{z}),
\end{aligned}$$

and thus $U\mathbf{a}(x) = \mathbf{a}_D(\underline{x})U$ on $\Gamma_0\mathcal{D}$. Furthermore

$$\begin{aligned}
\langle \mathbf{a}_D(\underline{x})f, g \rangle_2 &= \langle U^{-1}\mathbf{a}_D(\underline{x})f, U^{-1}g \rangle = \langle \mathbf{a}(x)U^{-1}f, U^{-1}g \rangle \\
&= \langle U^{-1}f, \mathbf{a}^+(x)U^{-1}g \rangle = \langle U^{-1}f, U^{-1}\mathbf{a}_D^+(\underline{x})g \rangle \\
&= \langle f, \mathbf{a}_D^+(\underline{x})g \rangle_2.
\end{aligned}$$

for $f, g \in \mathcal{A}_0(D^d)$. $\square$

Notice that on $\Gamma_0\mathcal{D}$ we have that $\mathbf{a}_D(\underline{x})$ is the adjoint operator of $\mathbf{a}_D^+(\underline{x})$.

We conclude that $\mathcal{A}_0(D^d)$ is the Bose-Fock algebra generated by the Hilbert space of L^2 -integrable anti-holomorphic functions $\{ \langle \cdot, \underline{x} \rangle \mid \underline{x} \in C_0^d \}$, $\langle \cdot, \cdot \rangle_2$. The operators $\mathbf{a}_D^+(\underline{x})$ (respectively $\mathbf{a}_D(\underline{x})$) are the operators of creation (respectively annihilation) and $\phi_D = 1$ is a unit vector.

Let $\mathcal{H}$ be finite dimensional and $\underline{\lambda} \in D^d$. Since

$$(Uf)(\underline{\lambda}) = \langle k_{\underline{\lambda}}, f \rangle = \langle Uk_{\underline{\lambda}}, Uf \rangle_2$$

holds for $f \in \Gamma_0\mathcal{H}$, the function $(Uk_{\underline{\lambda}})(\cdot)$ on D^d is the reproducing kernel in $\mathcal{A}(D^d)$, and for $\underline{z} \in D^d$ we have

$$(Uk_{\underline{\lambda}})(\underline{z}) = \left(U \left(\prod_{k=1}^d (\mathcal{N}_k + I)! \right)^{\frac{1}{2}} e^{\mathbf{a}^+(\underline{\lambda})\phi} \right) (\underline{z})$$

$$\begin{aligned}
 &= \left(\left(\prod_{k=1}^d (N_k + I)! \right)^{\frac{1}{2}} e^{\sum_{n=1}^d \mathbf{a}_D^+(\lambda_n e_n) U_{\emptyset}} \right) (\underline{z}) \\
 &= \prod_{k=1}^d \left((N_k + I)! e^{\langle z_k, \lambda_k \rangle} \right) \\
 &= \prod_{k=1}^d \left(\sum_{n=0}^{\infty} (n + I) (\overline{z_k} \lambda_k)^n \right) \\
 &= \prod_{k=1}^d \left((1 - \overline{z_k} \lambda_k)^{-2} \right).
 \end{aligned}$$

Hence $(Uk_{\lambda})(\cdot)$ is the Bergman kernel for $L^2(D^d)$.

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