ON THE HYPERBOLIC DIRICHLET TO NEUMANN FUNCTIONAL IN H^n AND S^n . *

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Abstract

We prove the injectivity of the linearization of the hyperbolic Dirichlet to Neumann functional in a "small" bounded domain of the hyperbolic space H^n (resp., the sphere S^n) under some suitable transversality condition.

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1. Introduction and Statement of the Results

Let $\bar{\Omega}$ be a compact manifold of dimension n with smooth boundary $\partial\Omega$ and $\text{Met}(\bar{\Omega})$ denote the set of all Riemannian metrics g on $\bar{\Omega}$.

We consider the anisotropic wave equation

$$(1.1) \quad \begin{aligned} \square_g u &= \frac{\partial^2 u}{\partial t^2} - \Delta_g u = 0 \quad \text{in } \Omega \times (0, T), \\ [0.5cm] u &= f \quad \text{on } \Gamma = \partial\Omega \times (0, T), \quad f \in C_0^\infty(\Gamma), \\ [0.5cm] u &= \frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega \times \{0\}. \end{aligned}$$

There is a unique solution to (1.1); hence we may define the hyperbolic Dirichlet to Neumann map as the linear operator $\Lambda_g : C_0^\infty(\Gamma) \longrightarrow C^\infty(\Gamma)$, given by

$$(1.2) \quad \Lambda_g f = du \cdot \nu_g \Big|_\Gamma = \frac{\partial u}{\partial \nu_g} \Big|_\Gamma,$$

where u is the unique solution to (1.1) and ν_g is the g -interior unit normal to $\partial\Omega$. The hyperbolic Dirichlet to Neumann Functional:

$$(1.3) \quad \begin{aligned} \Lambda : \text{Met}(\bar{\Omega}) &\longrightarrow O_p(\Gamma), \\ [0.3cm] g &\longmapsto \Lambda_g, \end{aligned}$$

where $O_p(\Gamma)$ denotes the space of all linear operators from $C_0^\infty(\Gamma)$ into $C^\infty(\Gamma)$, is known to be invariantly defined on the orbit obtained by the action over $\text{Met}(\bar{\Omega})$ of the group $\mathcal{D}$ of all diffeomorphism ψ of $\bar{\Omega}$, each of which restricts to the identity on $\partial\Omega$. In fact, any such ψ can be used to construct a new metric, the pull-back metric, ψ^*g , such that $\Lambda_{\psi^*g} = \Lambda_g$. A natural conjecture is that this is the only obstruction to the uniqueness of Λ .

The elliptic Dirichlet to Neumann map $h \longmapsto \frac{\partial v}{\partial \nu_g} \Big|_{\partial\Omega}$, where v solves the Dirichlet problem $\Delta_g v = 0$ in Ω , $v|_{\partial\Omega} = h$, was treated by several authors and is closely related to the physical problem referred as Electrical Impedance Tomography, of determining the conductivity of a body from measurement of voltage potentials and corresponding current fluxes at the boundary, this problem arose in geophysics from attempts to determine the composition of the earth (see [4]). The hyperbolic Dirichlet to Neumann map was considered in the articles [1], [2] and [4].

We follow [1] and define, for fixed g , the map:

$$(1.4) \quad \psi \in \mathcal{D} \xrightarrow{\Lambda_g} \psi^*g \in \text{Met}(\bar{\Omega}).$$

The formal linearizations $A'_g := A'_g[I] : \Gamma_0(T\bar{\Omega}) \rightarrow \Gamma(S^2\bar{\Omega})$ of A_g at I and $\Lambda'_g : \Gamma(S^2\bar{\Omega}) \rightarrow O_p(\Gamma)$ of Λ at g , where $\Gamma_0(T\bar{\Omega})$ (resp. $\Gamma(S^2\bar{\Omega})$) is the space of all vector fields on $\bar{\Omega}$ which vanish on $\partial\Omega$ (resp. the space of all smooth sections of symmetric 2-tensors on $\bar{\Omega}$) were considered in [1], together with $(A'_g)^*$, the formal adjoint of A'_g , with respect to the inner products (also introduced there):

$$(1.5) \quad \langle X, Y \rangle = \int_{\bar{\Omega}} g(X, Y) v_g, \quad X, Y \in \Gamma_0(T\bar{\Omega}),$$

and

$$(1.6) \quad \langle \langle m, l \rangle \rangle = \frac{1}{n} \int_{\bar{\Omega}} \text{tr}(\hat{m} \circ \hat{l}) v_g, \quad m, l \in \Gamma(S^2\bar{\Omega}),$$

v_g (resp. tr) denoting the g -volume element (resp. the g -trace) and $\hat{m} \in \text{End}(T\bar{\Omega})$ being the unique linear map corresponding to m via g . In [1], the authors stated the following

Conjecture 1.1. *Let $\Omega_0 \subset \Omega$ be a submanifold, $m \in \Gamma(S^2\bar{\Omega})$ have support in $\bar{\Omega}_0$ i.e., $m \in \Gamma_0(S^2\bar{\Omega})$ and assume that a) $\Lambda'_g(m) = 0$, b) $(A'_g)^*(m) = 0$ and c) $\text{diam}_g(\Omega_0) < T$ is sufficiently small that the exponential map for g is a global diffeomorphism. Then m is identically zero.*

As in [1] we refer to Condition b) as the Transversality Condition and we remind that Condition c) is necessary to avoid the appearance of caustics.

The main results of this paper are:

Theorem 1.1. *Conjecture 1.1 holds if $\bar{\Omega}$ is a bounded domain of the hyperbolic space H^n , g is the canonical metric on H^n and $m \in \Gamma_0(S^2\bar{\Omega})$ is orthogonal to g .*

Theorem 1.2. *Conjecture 1.1 holds if $\bar{\Omega}$ is a bounded domain of S^n , g is the metric on S^n induced by the euclidean metric on $\mathbb{R}^{n+1}$ and $m \in \Gamma_0(S^2\bar{\Omega})$ is orthogonal to g .*

Remark 1.1. *Cardoso and Mendoza, [1], proved that Conjecture 1.1 holds if $\bar{\Omega}$ is a bounded domain of $\mathbb{R}^n$, $n \geq 2$ and g is the euclidean metric. They also prove Conjecture 1.1 when $n = 2$ and g is near the euclidean metric in the C^3 topology. We shall make use of the invariant formulas for A'_g , $(A'_g)^*$ and Λ'_g proved in Section 2 of [1].*

2. Proof of Theorem 1.1

Given $(r, u), (s, v) \in \mathbb{R} \times \mathbb{R}^n$, define $\beta((r, u), (s, v)) := -rs + \langle u, v \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$ and β is a Lorentz inner product on $\mathbb{R}^{n+1}$. For each nonzero vector $v \in \mathbb{R}^{n+1}$, $\beta(v, v)$ is negative, zero, or positive, depending on whether the usual euclidean angle between v and the first coordinate axis is less than, equal to, or greater than $\pi/4$. Extend β to a Lorentz metric on the manifold $\mathbb{R}^{n+1}$ by setting $\beta((p, u), (p, v)) := \beta(u, v)$ for $(p, u), (p, v) \in T\mathbb{R}^{n+1}$. The Levi-Civita connection of β is just the canonical connection on $\mathbb{R}^{n+1}$.

Let H^n be the component $\{(r, u) \in \mathbb{R} \times \mathbb{R}^n / r^2 = \|u\|^2 + 1, r > 0\}$ of the hyperboloid $\beta^{-1}(-1)$ and denote the inclusion map of H^n into $\mathbb{R}^{n+1}$ by i . If $\xi \in T\mathbb{R}^{n+1}$ is tangent to H^n and nonzero, then $\beta(\xi, \xi) > 0$, which implies that $g := i^*\beta$ is a riemannian metric on H^n called the canonical metric of the hyperbolic space H^n (see [3]).

Let $\bar{\Omega}$ be a compact domain of H^n with smooth boundary $\partial\Omega$ and let $\Gamma_0(S^2\bar{\Omega})$ denote the vector space of all smooth sections of symmetric 2-tensors on H^n which are supported on $\bar{\Omega}$.

Proof of Theorem 1.1 Let $x \in \Omega$, $v \in T_x H^n$ such that $\beta(v, v) = 1$. The geodesic with initial tangent vector $(x, v) \in TH^n$ is given by

$$\gamma(t) = \cosh(t)x + \sinh(t)v.$$

Let $v_2 := \dot{\gamma}, v_3, \dots, v_{n+1}$ be an orthonormal basis for $T_{\gamma(t)}H^n$; then $\gamma, v_2, v_3, \dots, v_{n+1}$ is a basis for $\mathbb{R}^{n+1}$. We extend m to the whole space $\mathbb{R}^{n+1}$ by defining

$$m(\gamma, \gamma) = m(\gamma, v_i) = 0, \quad i = 2, \dots, n+1.$$

We need the following

Lemma 2.1. *The following identity is true:*

$$(2.1) \quad -A_1 + \sum_{\mu=2}^{n+1} A_\mu = \Pi_{\dot{\gamma}}((A'_g)^*(m)(\gamma)),$$

where $\Pi_{\dot{\gamma}}(\bullet)$ denotes the $\dot{\gamma}$ -component of $(A'_g)^*(m)(\gamma)$ in the g -orthonormal basis

$\dot{\gamma}, v_3, \dots, v_{n+1}$ and

$$(2.2) \quad A_\mu = \sum_{\beta=1}^{n+1} a_\beta \frac{\partial}{\partial x_\mu} m_{\beta\mu}(\gamma), \quad \mu = 1, \dots, n+1,$$

where $a_\beta, \beta = 1, \dots, n+1$ are the euclidean coordinates of $\dot{\gamma}$.

Proof: Write

$$\begin{aligned}\dot{\gamma} &= a_1 e_1 + \cdots + a_{n+1} e_{n+1}, \\ v_j &= \alpha_1^j e_1 + \cdots + \alpha_{n+1}^j e_{n+1}, \quad j = 3, \dots, n+1.\end{aligned}$$

We note that

$$\begin{aligned}A_1 &= -a_1 e_1 m(\dot{\gamma}, \dot{\gamma}) - \sum_{i=3}^{n+1} \alpha_1^i e_1 m(v_i, \dot{\gamma}), \\ A_\mu &= a_\mu e_\mu m(\dot{\gamma}, \dot{\gamma}) + \sum_{i=3}^{n+1} \alpha_\mu^i e_\mu m(v_i, \dot{\gamma}),\end{aligned}$$

where $\mu = 2, \dots, n+1$. Therefore

$$\begin{aligned}-A_1 + \sum_{\mu=2}^{n+1} A_\mu &= (a_1 e_1 + \cdots + a_{n+1} e_{n+1}) m(\dot{\gamma}, \dot{\gamma}) \\ &\quad + \sum_{j=3}^{n+1} (\alpha_1^j e_1 + \cdots + \alpha_{n+1}^j e_{n+1}) m(v_j, \dot{\gamma}) \\ &= \nabla_{\dot{\gamma}}^g m(\dot{\gamma}, \dot{\gamma}) + \sum_{j=3}^{n+1} \nabla_{v_j}^g m(v_j, \dot{\gamma}) \\ &= \Pi_{\dot{\gamma}}((A'_g)^*(m)(\gamma)),\end{aligned}$$

which completes the proof of Lemma 2.1. ■

We now turn to the proof of Theorem 1.1. Consider the function $F : \mathbb{R} \rightarrow \mathbb{R}$, defined by $F(t) = m(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))$. Let

$$\mathcal{L} = -\frac{\partial}{\partial x_1} \frac{\partial}{\partial v_1} + \sum_{\alpha=2}^{n+1} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial v_\alpha}, \quad \text{with } v = (v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1} / \{0\}.$$

We obtain (taking into account (2.1)) that

$$\begin{aligned}\mathcal{L}F &= \tanh(t) \Delta_g F + 2 \tanh(t) \left(-m_{11} + \sum_{i=2}^{n+1} m_{ii} \right) \\ &\quad + 2 \left(-A_1 + \sum_{i=2}^{n+1} A_i \right) \\ &= \tanh(t) \Delta_g F + 2 \tanh(t) \operatorname{tr} \dot{m}(\gamma) + \Pi_{\dot{\gamma}}((A'_g)^*(m)(\gamma)),\end{aligned}$$

where Δ_g denotes the Laplace-Beltrami operator associated to the metric g .

Since m is orthogonal to g (see (1.6)), we obtain that

$$\int_{\bar{\Omega}} \varphi \operatorname{tr} \hat{m} v_g = 0, \quad \text{for all } \varphi \in C^\infty(\bar{\Omega});$$

consequently $\operatorname{tr} \hat{m} = 0$. On the other hand using the Transversality Condition, we conclude that

$$(2.3) \quad \mathcal{L}F = \tanh(t) \Delta_g F.$$

We define

$$(2.4) \quad F^k(x, v) = \int_{-\infty}^{\infty} (\tanh(t))^k F(t) dt,$$

for all $k = 0, 1, \dots$

Using (2.3), it is easy to show that

$$(2.5) \quad \mathcal{L}F^k = \Delta_g F^{k+1}, \quad k = 0, 1, \dots$$

If we fix v , since m is supported in $\bar{\Omega}$, it follows that $F^k(\bullet, v)$ for all $k = 1, 2, \dots$ vanish on a non-empty open subset of H^n . From Proposition 3.1 in [1], $F^0 = 0$ and hence F^1 and in fact all F^k are harmonic functions. Therefore F^k is identically zero for all $k = 0, 1, 2, \dots$

Observe that F is supported in some closed interval $[a, b]$ containing zero, and since the subalgebra generated by the family of functions $\{(\tanh(t))^k\}_{k=0,1,\dots}$ is dense in $C([a, b]; \mathbb{R})$, we obtain that F is identically zero. If we take $t = 0$ and recall that m is symmetric, we conclude that m is identically zero. ■

3. Proof of Theorem 1.2

Let $x \in \Omega$, $v \in T_x S^n$ with $|v|_e = 1$, e being the euclidean metric. The geodesic γ through the point x , with $\dot{\gamma}(0) = v$, is given by $\gamma(t) = \cos(t)x + \sin(t)v$.

We extend m in a similar way as in Section 2. We need the following

Lemma 3.1. *The following identity is true:*

$$(3.1) \quad e((A'_e)^*(m)(\gamma), \dot{\gamma}) = \Pi_{\dot{\gamma}}((A'_e)^*(m)(\gamma)).$$

Proof: Similar to Lemma 2.1. ■

We now begin the proof of the theorem. Consider the function $F : \mathbb{R} \rightarrow \mathbb{R}$, defined by $F(t) = m(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))$. Let

$$\mathcal{L} = \sum_{\alpha=1}^{n+1} \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial v_{\alpha}},$$

with $v = (v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$. We obtain (taking into account (3.1)) that

$$\begin{aligned} \mathcal{L}F &= \text{tang}(t)\Delta_x F + 2e((A'_e)^*(m)(\gamma), \dot{\gamma}) - 2\text{tang}(t)\text{tr } \hat{m}(\gamma) \\ &= \text{tang}(t)\Delta_x F + 2\Pi_{\dot{\gamma}}((A'_e)^*(m)(\gamma)) - 2\text{tang}(t)\text{tr } \hat{m}(\gamma), \end{aligned}$$

where Δ_x denotes the Laplacian in the x -variables.

It follows from the fact that m is orthogonal to e and from the Transversality Condition that

$$(3.2) \quad \mathcal{L}F = \text{tang}(t)\Delta_x F.$$

We define

$$(3.3) \quad F^k(x, v) = \int_{-\infty}^{\infty} (\text{tang}(t))^k F(t) dt.$$

for all $k = 0, 1, \dots$. We note that $F^0 = 0$ (see Proposition 3.1, [1]). The remainder of the proof is similar to that of Theorem 1.1. ■

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